Comment on "On the ambiguity of solutions of the system of the Maxwell equations"

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ABSTRACT. In this paper we discuss in detail, dismiss and reconstruct the ideas of A. E. Chubykalo & V. Onoochin contained in [1] and the related formulations of J. D. Jackson in [4]. We argue that in both references conceptual mistakes produce inconsistent formalisms for treating gauge invariance in electromagnetic theory. We prove that if gauge invariance is supposed for Maxwell's equations we cannot deduce any contradiction related to the gauge function. And conversely, if we suppose gauge non-invariance, we obtain a consistent theory. Thus, contradicting the results of Chubykalo & Onoochin and of Jackson.

Key words: Gauge transformations, gauge invariance, electromagnetic field, boundary value problems.

PACS numbers: 03.50.-z, 03.50.De

Introduction

In their paper [1] Chubykalo & Onoochin tried to prove two related propositions. The first one asserts that the wave equation for the electric field cannot be solved because, from a physical standpoint, we cannot know the matter fields (charge density ρ and current density $\mathbf{J}$) for some charge configurations involving microparticles (*e.g.,* the electron). Chubykalo & Onoochin stated in their abstract: "It is caused by the physical limitation on our knowledge of the structure of the electron". Certainly, this is the case for the physical system discussed in [1], but it is not a general statement, therefore the scope of its assertion is partially limited. Their second proposition seems more interesting, because they say that the choice of gauge changes the electric field, hence "the gauge condition is a physical condition" and the electromagnetic field is not a gauge invariant quantity. The relation that the authors of [1] establish between these two statements is the following: only with the help of the potentials can be solved the Maxwell's equations, but each choice of them involves an arbitrary gauge condition so "it is reasonable to ask whether any sets of potentials defined by this condition lead to the same expressions for the fields".

We must establish, at this point, our philosophical position to avoid any confusion. The electromagnetic field is a physical reality determined by physical laws and given physical conditions. From this vantage point gauge invariance is the statement that such a physical reality cannot be changed at will, but only through given physical causes. Therefore, if the potentials are not real causes they cannot change any physical reality. The general consensus about the potentials in the classical domain, notwithstanding the Aharonov-Bohm effect, is that the electromagnetic potentials are not real causes because they lack invariance in front of gauge transformations, as any good textbook on electrodynamics explain (see *e.g.* [2] or [3]). This is not an arbitrary position because, in the space of solutions of Maxwell's equations for the potentials ((2.2)-(2.3) of section 2 below), the distance between any two of them can be made as large, or small, at each space-point as the gauge function, which is arbitrary. Then, if we consider two vector potentials related by a gauge transformation $\mathbf{A}$, $\mathbf{A}'$ it is clear that $|\mathbf{A} - \mathbf{A}'| = |\nabla\gamma|$, where γ is the gauge function, which we can choose at will. Naturally, if in the classical domain certain phenomena can be described as influenced by the electromagnetic potentials, then the philosophical interpretation must change. Chubykalo & Onoochin are not discussing new physical phenomena but the hypothesis that the potentials are physically relevant fields because they affect the electromagnetic field. The way in which this can be the case is straightforward: the potentials change the field because Maxwell's equations are not gauge invariant. For proving this assertion, they suppose that Maxwell's equations are gauge invariant, then they show that this is not the case. This result goes against the accepted consensus about the potentials that we have already quoted, and it gets some support because the authors of [1] believe that this non invariance is directly derived from an analysis of Maxwell's equations solutions. Here, we will prove that this is not the case because with the hypothesis of gauge invariance we cannot derive any contradiction from Maxwell's equations

Now we turn to Jackson's treatment in [4]. We find the exact opposite of Chubykalo & Onoochin's reasoning because Jackson starts form the hypothesis that Maxwell's equations are non-gauge invariant. We contend that Jackson's treatment is incomplete and cannot achieve its goal. The organization of the paper is as follows. In section 1 we discuss, in subsection 1.1, Onoochin and Jackson's methodologies, and in subsection 1.2 we unify them in a simple formalism. In sections 2 and 3 we discuss the idea of supposing that Maxwell's equations are gauge invariant; we show that it is possible to obtain a uniform procedure for calculating the gauge function that is logically clearer than that of Jacksons in [4]. As a byproduct of the calculation, it is proved that Chubykalo & Onoochin's goals are not fulfilled. In sections 4 and 5 we follow the supposition of Jackson in [4], *i.e.,* that Maxwell's equations are non-gauge invariant; we show that only under very strong conditions it is possible to prove that, indeed, Maxwell's equations are gauge invariant. In these sections 4 and 5 we reproduce, enlarge and generalize the discussion of [1]. Finally, in section 6, we review and correct another Chubykalo & Onoochin's ideas: that the only way to solve Maxwell's equations are the potentials. This proposition is correct, because the potentials are a general solution for Maxwell's equations but the reasons given in [1] are not right. In section 7 we provide our conclusions.

Comment on notation. Along the paper for each multiple integral only one integration symbol is employed, we skip the range of integration because it will be clear from context. For the differentials of volume, we use dV, dV' for variables $\mathbf{x}$, $\mathbf{x}'$. Equations are independently numerated in each section, hence equation 3 of section 2 is referred to as (2.3) between angular brackets. When referring a section, no bracket is used for the number.

1 Space connectivity, gauge invariance and gauge non-invariance

1.1

According to Chubykalo & Onoochin in [1] "if the gauge condition is introduced arbitrarily, and therefore the potentials are also determined with a certain degree of arbitrariness, it is reasonable to ask whether any set of potentials defined by this condition lead to the same expressions for the fields". They claim even more: "If the expressions for the fields are different when choosing different gauge conditions, we can conclude that the system of the Maxwell equations has several solutions for the

EM fields". Onoochin had treated the question in other papers (*e.g.* [8], [9]) and one of his methodological rules is especially useful for the discussion that follows. This rule tells us: "The simplest way to verify if the gauges are equivalent is to find a connection between the potentials of the Coulomb and the Lorenz gauges". The relation derived by Onoochin in [9] is used again in [1] and is: the electromagnetic field is gauge invariant if the fields calculated in different gauges are equal. To be more precise, and following [9], Onoochin criterion is written for the case of the Coulomb and Lorenz gauges in the form

$$\mathbf{E}_C = \mathbf{E}_L. \tag{1.1}$$

Once written out, and after the use of the field equations for the vector potentials in each gauge, the following relation between the scalar potentials is obtained

$$\nabla\varphi_C - \nabla\varphi_L = \frac{1}{4\pi c^2}\nabla\int dV'dt'G_1\left(\mathbf{x},t;\mathbf{x}',t'\right)\frac{\partial^2}{\partial t^2}\varphi_C(\mathbf{x}',t'). \tag{1.2}$$

Hence, if we introduce

$$\psi = \frac{1}{4\pi c^2}\int dV'dt'G_1\left(\mathbf{x},t;\mathbf{x}',t'\right)\frac{\partial^2}{\partial t'^2}\varphi_C(\mathbf{x}',t'), \tag{1.3}$$

we get

$$\nabla\left(\varphi_C - \varphi_{\mathrm{L.}} - \psi\right) = 0 \tag{1.4}$$

.

Therefore, according to Onoochin there will be gauge invariance if and only if $\varphi_C - \varphi_{\mathrm{L.}} - \psi = 0$. But this is a mistake because from (3) the condition that follows is: $\varphi_C - \varphi_{\mathrm{L.}} - \psi = d(t)$ where $d\left(t\right)$ is a function of time. Then, Onoochin criterion is not in fact necessary, because it can be violated while gauge invariance remains intact. Jackson's criterion of equivalence is quite different. He introduces two gauge functions a_1, a_2 such that each one satisfies

$$\mathbf{A}_L = \mathbf{A}_C + \nabla a_1, \tag{1.5}$$

$$\varphi_L = \varphi_C - \frac{1}{c}\frac{\partial a_2}{\partial t}. \tag{1.6}$$

With this gauge transformation we obtain the result

$$\mathbf{E}_C = \mathbf{E}_L + \frac{1}{c}\frac{\partial}{\partial t}\ \nabla\left(a_1 - a_2\right). \tag{1.7}$$

So, if $a_1 = a_2$ the electromagnetic field is gauge invariant. Onoochin reasoning underlying the use of (1) is quite direct:

(O) If we suppose that gauge invariance is fulfilled then all the solutions obtained through potentials in different gauges are equal. However, if these solutions are not equal, gauge invariance is violated.

Jackson's reasoning is

(J) Let's suppose that gauge invariance is violated then the solutions obtained in different gauges must be different. However, if they are not different, gauge invariance is not violated.

We note a key element in Chubykalo & Onoochin reasoning in [1] and Onoochin in [8] and [9]: they do not introduce the gauge function explicitly while Jackson does. There is a good reason for this that we shall explain right now.

1.2

The procedures (O) and (J) are opposite, and this can be seen more clearly when stated in mathematical terms. In order to discuss the relevance of the existence, or non-existence, of a gauge function for gauge invariance of the electromagnetic field we introduce the 1-form

$$(\mathbf{A}_C - \mathbf{A}_L) \cdot d\mathbf{x} - c\,(\varphi_C - \varphi_L)\,dt = \omega. \tag{1.8}$$

Now, if the 1-form (8) is closed, *i.e.,* $d\omega = 0$ the electromagnetic field is gauge invariant.

Existence of the gauge function implies that (8) is exact $\omega = d\gamma$ therefore, if we have a gauge function along a simply connected region D, then gauge invariance of the electromagnetic field is clear because $d\,(\mathrm{d}\gamma) = 0$ [10]. For this reason, Chubykalo & Onoochin cannot suppose the existence of a gauge function. The relation of the 1-form (8) to Onoochin criterion (1.1.1) is quite simple, just take the exterior derivative of ω to get

$$d\omega = \sum_{i,k} \left(\frac{\partial A^i_C}{\partial x_k} - \frac{\partial A^i_L}{\partial x_k} \right) \mathrm{dx}_k \bigwedge \mathrm{dx}_i + \sum_i \left(E^i_C - E^i_L \right) dt \bigwedge \mathrm{dx}_i. \tag{1.9}$$

Here the symbol $\bigwedge$ is the typical antisymmetric tensor product for p-forms. The terms are easily interpreted: the first involves the magnetic field in two different gauges, while the second show us the electric field. Looking at (9) it is clear that, just like in Onoochin criterion, we only require one of the fields to prove that gauge invariance is violated. However, we can also see that Jackson's gauge transformation (1.1.5)-(1.1.6) allows us to write (8) as

$$\nabla a_1 \cdot d\mathbf{x} + \frac{\partial a_2}{\partial t} dt = \omega^*. \tag{1.10}$$

Now we can be precise about Onoochin and Jackson's aims: for Onoochin the idea is to prove that the 1-form (8) is not exact, while Jackson tries quite the opposite: (8) is exact. Therefore, their criterions are integrability conditions, or, to say it more clearly: conditions for the existence of the gauge function. The relation between closed and exact forms is precisely the relation between gauge invariance and the existence of a gauge function. The difficulty lays on finding conditions to pass from gauge invariance to the existence of the gauge function. On this respect there are some general results of topological nature, the key proposition is:

Proposition 1: If the space D is simply connected, the gauge invariance implies the existence of a gauge function.

Proof: According to Poincare's lemma in any contractible space (a space where any path is homologous to zero) any closed p-form is also exact.

So, in a globally simply connected space a gauge function must exist if we impose gauge invariance from the start. However if the space is multiply connected because some boundaries are imposed, like in spaces with static point charges or in a space traveled by a charged particle in motion, in each connected component a gauge function exists, which is enough to guarantee that gauge invariance is not violated. Then we have:

Proposition 2: If the space is multiple connected, a globally defined gauge function does not exist, but gauge invariance is valid in each connected component.

Proof: In each connected component a gauge function exists, which is enough to prove that gauge invariance is maintained along the whole set of connected components.

Kiskis, in [11], showed that in disconnected spaces charge conservation is violated, hence the deduction of Maxwell equations from charge conservation is only possible in simply connected spaces, a condition that is not explicit in many treatments (see *e. g*, [12]). Here, we have concluded that even if the space is multiple connected, *i.e.,* no globally defined gauge function exists, gauge invariance is possible.

These propositions, however, are quite general. In practice the integrability conditions for the gauge functions are differential equations obtained directly from Maxwell's equations under certain conditions that define the vacuum or any medium with polarization, magnetization and dispersive properties. Therefore, we must be able to express the hypothesis of gauge invariance and gauge non invariance under those conditions. We contend that

- The crux of the question of gauge invariance lies in the existence of the gauge function.
- For the gauge function there are well defined integrability conditions derived from Maxwell's equations.
- If we suppose that the gauge function exists, we cannot derive a contradiction.
- If we suppose that the gauge function does not exist, we cannot derive a contradiction.

Therefore, at least for the case of the Coulomb and Lorenz gauges, the hypothesis of gauge invariance is logically independent of the hypothesis of non-gauge invariance. In order to prove this proposition, in sections 2 and 3 we shall work with the hypothesis of gauge invariance, proving that the integrability conditions can be satisfied. In sections 4 and 5 we shall prove that if we suppose gauge non-invariance that is quite precisely what we obtain, without logical contradiction.

2 Gauge invariance.

We are going to discuss the gauge transformations, from Lorenz to Coulomb gauge, of the potentials of Maxwell's equations for vacuum, *i.e.* for a non-polarizable, non-magnetizable, non-dispersive medium where the Lorentz constitutive relations holds. In this context Maxwell's equations for the fields are

$$\left.\begin{aligned}\nabla\cdot\mathbf{E}&=4\pi\varrho,\\ \nabla\times\mathbf{E}&=-\tfrac{1}{c}\tfrac{\partial\mathbf{B}}{\partial t},\\ \nabla\cdot\mathbf{B}&=0,\\ \nabla\times\mathbf{B}&=\tfrac{1}{c}\tfrac{\partial\mathbf{E}}{\partial t}+\tfrac{4\pi}{c}\mathrm{J}.\end{aligned}\right\}\tag{2.1}$$

We use Gaussian units and $\langle\mathbf{E},\ \mathbf{B}\rangle$ are two N times differentiable (C^N) vector fields representing the electromagnetic field, $\langle\rho,\mathbf{J}\rangle$ are localized BC^N sources, *i.e.* N times differentiable and bounded functions defined on a bounded and path connected region of space-time D. Unless otherwise stated the space-time will be considered globally Euclidean because the symmetry group of the Lorentz constitutive relations is not taken as isometry group of the metric structure. We shall use standard symbols from vector analysis. If we introduce the potentials $\langle\ \mathbf{A}\ ,\ \varphi\ \rangle$, related to the electric and magnetic fields by: $\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A},\quad \mathbf{B}=\nabla\times\mathbf{A}$, Maxwell's equations become

$$\nabla\times\nabla\times\mathbf{A}=-\frac{1}{c}\frac{\partial}{\partial t}\nabla\varphi-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\frac{4\pi}{c}\mathbf{J},\tag{2.2}$$

$$\Delta\varphi+\frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}=-4\pi\rho.\tag{2.3}$$

Certainly, the potentials are C^{N+1} functions. With the help of gauge transformations, it is possible to simplify equations (2)-(3) by choosing a subset of the full space of solutions. Gauge transformations are groups of functional transformations involving arbitrary functions as parameters, for the case of Maxwell's equations these are defined by translations of the potentials involving on arbitrary function, given by

$$\mathbf{A}=\mathbf{A}_g+\nabla\gamma,\tag{2.4}$$

$$\varphi=\varphi_g-\frac{1}{c}\frac{\partial\gamma}{\partial t}.\tag{2.5}$$

Here $\langle\ \mathbf{A}_g\ ,\ \varphi_g\ \rangle$ are new potentials satisfying different field equations if we choose a condition between the potentials of the form $F(\mathbf{A},\ \varphi)=0$. As we explained in the previous section, gauge transformations defined like in (4)-(5) carries along the hypothesis that gauge invariance is respected because when substituted in (1.2.8) the 1-form becomes exact.

What we aim to prove is that from this hypothesis we cannot obtain its converse, *i.e.* that Maxwell's equations are non-gauge invariant. Because the field strengths are gauge invariant the gauge function γ disappears from the calculations, hence it is not explicitly known. Jackson decided to prove that this shortcoming is easy to mitigate and in [4] calculated gauge functions for some typical gauge transformations. His method, as we will show, is not rigorous because some conditions upon the gauge function are omitted. In this section we display the full set of conditions, which are the integrability conditions, to deduce a field equation for the gauge function. We shall suppose that the gauge function is a distribution, not a pointwise function, so

$$\gamma(\mathbf{x},t) = \int G(\mathbf{x},t;\mathbf{x}',t')\rho(\mathbf{x}',t')\,\mathrm{dV'dt'} \quad , \tag{2.6}$$

where the goal will be to provide an equation for the Green's function $G(\mathbf{x},t;\mathbf{x}',t')$. Here $\rho(\mathbf{x},t)$ is a Lebesgue integrable function which represents a localized density of charge in a subset of region D. We shall work the case of gauge transformation from the Coulomb to the Lorenz gauge. The potentials in these gauges satisfies the conditions

$$\nabla\cdot\mathbf{A}_C(\mathbf{x},\ t) = 0, \tag{2.7}$$

$$\nabla\cdot\mathbf{A}_L(\mathbf{x},\ t) = -\frac{1}{c}\frac{\partial\varphi_L(\mathbf{x},\ t)}{\partial t}. \tag{2.8}$$

Using these equations, and the gauge transformation we shall deduce the integrability conditions of (1.2.8), which is equivalent to deduce a pair of differential equations for the gauge function. To do so we start directly from

$$\mathbf{A_C} = \mathbf{A}_L + \nabla\gamma, \tag{2.9}$$

$$\varphi_C = \varphi_L - \frac{1}{c}\frac{\partial\gamma}{\partial t} \tag{2.10}$$

Taking the divergence of (9) and using the Lorenz gauge we see that

$$0 = \nabla\cdot\mathbf{A}_L + \Delta\gamma, \tag{2.11}$$

$$\frac{1}{c}\frac{\partial\varphi_L}{\partial t} = \Delta\gamma. \tag{2.12}$$

Now we take the time derivative of (10) multiplied by $\frac{1}{c}$ to get

$$\frac{1}{c}\frac{\partial}{\partial t}\varphi_C = \frac{1}{c}\frac{\partial}{\partial t}\varphi_L - \frac{1}{c^2}\frac{\partial^2\gamma}{\partial t^2}. \tag{2.13}$$

Hence, we use the Lorenz gauge to write

$$\frac{1}{c}\frac{\partial}{\partial t}\varphi_C = -\nabla \cdot \mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \gamma}{\partial t^2}. \tag{2.14}$$

Finally, with help of (11) we obtain

$$\frac{1}{c}\frac{\partial}{\partial t}\varphi_C = (\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\gamma. \tag{2.15}$$

The step involving the substitution of relation (11) in (14) is equivalent to the supposition of gauge invariance. This step cannot be done with the gauge transformations proposed by Jackson, defined by (1.1.5)-(1.1.6). Equations (12) and (15) must be simultaneously satisfied by the gauge function. In [4] Jackson wrote equation (15) as his equation (3.8) to prove that solutions obtained by his method do not involve arbitrary functions of space. However, there is not any mention of equation (12). Indeed, Jackson's methodology is completely different of ours because he starts from the assumption that (1.2.8) is no integrable. We expand on these matters in section 4 and 5. Our method suppose that a gauge function exists, so we deduce a pair of differential equations given by

$$\left(\Delta - \frac{\mathbf{1}}{\mathbf{c^2}}\frac{\partial^\mathbf{2}}{\partial \mathbf{t^2}}\right)\gamma = \frac{\mathbf{1}}{\mathbf{c}}\frac{\partial \varphi_\mathbf{C}}{\partial \mathbf{t}} \ ,$$

$$\Delta\gamma \ = \frac{1}{c}\frac{\partial \varphi_L}{\partial t}.$$

These equations are the integrability conditions of the 1-form (1.2.8). These equations are the crucial basis of any discussion about gauge invariance. Now, with the equations (12) and (15) we can deduce a field equation for the gauge function γ. The proposition that gauge invariance is fulfilled can be proven as not contradictory if this equation can be solved.

3 A field equation for the gauge function.

For solving equations (2.12) and (2.15) we shall consider a non-local point of view because we will suppose that the solutions are distributions. We shall derive a differential equation for the gauge function, but we propose that it is a linear functional of the form (2.6), whose kernel is what we want to obtain. To do so we must suppose that the gauge function is, at least, four times differentiable, *i.e.,* $\gamma \in C^4$. If this condition is fulfilled we can apply Laplace operator to (2.15) and D'Alembert operator to (2.12) and, with the use of the field equations for each potential in its respective gauge, we obtained the fourth order field equation for the gauge function

$$\Delta\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\gamma = -\frac{4\pi}{c}\frac{\partial}{\partial t}\rho, \tag{3.1}$$

The charge density is a BC^N function, but we require another hypothesis to proceed: all the integrals are understood in the Lebesgue sense in order to interchange limits and integrals and to employ Fubini's theorem on the order of integration. A useful reference on this subject is [7]. If we apply the ansatz (2.6) for the gauge function, we write

$$\Delta\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\gamma\left(\mathbf{x},t\right) = \int \Delta\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G\left(\mathbf{x},t;\mathbf{x}',t'\right)\rho\left(\mathbf{x}',t'\right)\mathrm{dV}'\mathrm{dt}', \tag{3.2}$$

So

$$\int \Delta\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G\left(\mathbf{x},t;\mathbf{x}',t'\right)\rho\left(\mathbf{x}',t'\right)\mathrm{dV}'dt = -\frac{4\pi}{c}\frac{\partial}{\partial t}\rho\left(\mathbf{x},t\right). \tag{3.3}$$

Then the following equation is satisfied by the Green's function

$$\Delta\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G\left(\mathbf{x},t;\mathbf{x}',t'\right) = \frac{4\pi}{c}\delta\left(\mathbf{x}-\mathbf{x}'\right)\frac{\partial}{\partial t}\delta\left(t-t'\right). \tag{3.4}$$

In the case treated the solution can be represented in the form

$$G\left(\mathbf{x},t;\mathbf{x}',t'\right) = \int G_0\left(\mathbf{x},t;\mathbf{x},t''\right)G_1\left(\mathbf{x}',t';\mathbf{x}'',t''\right)dV''dt'', \tag{3.5}$$

where the Green's functions involved satisfy

$$\Delta G_0\left(\mathbf{x},t;\mathbf{x}',t'\right) = -\frac{4\pi}{c}\delta\left(\mathbf{x}-\mathbf{x}'\right)\frac{\partial}{\partial t}\delta\left(t-t'\right), \tag{3.6}$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G_1\left(\mathbf{x},t;\mathbf{x}',t'\right) = \delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right). \tag{3.7}$$

If we apply the D'alembert and Laplace operators to equation (5) considering equations (6)-(7) we can see that, formally, we have obtained a solution. We can get the same result with the system

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G\left(\mathbf{x},t;\mathbf{x}',t'\right) = G_0\left(\mathbf{x},t;\mathbf{x}',t'\right), \tag{3.8}$$

$$\Delta G_0\left(\mathbf{x},t;\mathbf{x}',t'\right) = -\frac{4\pi}{c}\delta\left(\mathbf{x}-\mathbf{x}^{'}\right)\frac{\partial}{\partial t}\delta\left(t-t'\right). \tag{3.9}$$

It is important to note that the gauge function is considered as physically meaningless so, it is hard to see what sort of boundary conditions must be used to solve equations (6)-(7). We shall treat the unbounded space case for simplicity and because it allows us to derive Jackson's results. To solve (6) we consider as our domain of definition for the spatial variables the extended $\mathbf{R}^3$, hence as boundary conditions we suppose that our Green's function for unbounded space, together with all its derivatives satisfy

$$G_0\left(\mathbf{x},t;\mathbf{x}',t'\right) \to 0. \tag{3.10}$$

Therefore,

$$G_0\left(\mathbf{x},t;\mathbf{x}',t'\right) = -\frac{4\pi}{c\left|\mathbf{x}-\mathbf{x}'\right|}\frac{\partial}{\partial t}\delta\left(t-t'\ \right). \tag{3.11}$$

If Dirichlet boundary conditions are required, we change the previous Green's function by

$$G^0\left(\mathbf{x},t;\mathbf{x}',t'\right) = G_0\left(\mathbf{x},t;\mathbf{x}',t'\right) + f(\mathbf{x},\ t;\mathbf{x}',t), \tag{3.12}$$

where f is a solution to Laplace's equation with an adequate boundary condition. For more details see [5], we shall revisit these Green's functions in section 4. We will solve (6) for unbounded space, but in addition to (10) the Green's function requires the following condition

$$G_1\left(\mathbf{x},t;\mathbf{x}',t'\right) = 0\ for\ \left|\mathbf{x}-\mathbf{x}'\right|^2 - c^2(t-t')^2 > 0 \tag{3.13}$$

whose meaning is that the perturbation cannot be beyond the wave front or that it cannot propagate faster that the velocity of light. This condition is used by Rohrlich in [6] to derive the Jordan-Pauli invariant function. Thus, the usual retarded solution is

$$G_1\left(\mathbf{x},\ t;\mathbf{x}',t'\right) = \frac{H(\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}-(t-t'))}{4\pi}\frac{\delta(t-(t^{'}+\frac{\left|\mathbf{x}-\mathbf{x}'\right|}{c}))}{\left|\mathbf{x}-\mathbf{x}'\right|}, \tag{3.14}$$

here $H(t-t')$ is Heaviside's function, whose use guarantee that (13) is fulfilled. Of course, if we keep in mind that we restrict all the variables to time-like separations we may skip the Heaviside function. We will skip this function from now on. So, we can write using (5) and (11)

$$\begin{aligned} G(\mathbf{x},t;\mathbf{x}',t') &= -\frac{1}{4\pi}\int \frac{dV''}{|\mathbf{x}-\mathbf{x}''|}\frac{\partial}{\partial t}\delta(t-t'')\,G_{1}(\mathbf{x}',t';\mathbf{x}'',t'') \\ &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{dV''}{|\mathbf{x}-\mathbf{x}''|}\delta(t''-t)\,G_1(\mathbf{x}',t';\mathbf{x}'',t'') \\ &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{dV''}{|\mathbf{x}-\mathbf{x}''|}G_1(\mathbf{x}',t';\mathbf{x}'',t). \end{aligned}$$

Now we use (2.6) and (14) to get

$$\gamma(\mathbf{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int \frac{dV''}{|\mathbf{x}-\mathbf{x}''|}\frac{dV'dt'}{|\mathbf{x}'-\mathbf{x}''|}\delta(t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}))\rho(\mathbf{x}',t')\,. \tag{3.15}$$

We have used $\delta(t-t') = \delta(t'-t)$. For this solution we can adopt the following compact expression

$$\gamma(\mathbf{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\sigma(\mathbf{x},t)\,,$$

with

$$\sigma(\mathbf{x},t) = \int \frac{dV''}{|\mathbf{x}-\mathbf{x}''|}\frac{dV'dt'}{|\mathbf{x}'-\mathbf{x}''|}\delta(t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}))\rho(\mathbf{x}',t')\,. \tag{3.16}$$

Now, let's formally prove that this function satisfy equation (2.12). To do so we apply Laplace's operator to the left-hand side of (16) to get

$$\Delta\sigma(\mathbf{x},\mathbf{t}) = \int \Delta\left(\frac{\mathbf{1}}{|\mathbf{x}-\mathbf{x}''|}\right)\mathrm{dV}''\frac{\mathbf{dV'dt'}}{|\mathbf{x}'-\mathbf{x}''|}\delta(\mathbf{t}'-(\mathbf{t}\{\frac{|\mathbf{x}-\mathbf{x}'|}{\mathbf{c}}))\rho(\mathbf{x}',\mathbf{t}')\ .$$

Then

$$\Delta\sigma(\mathbf{x},\mathbf{t}) = -\mathbf{4}\pi\int \delta(\mathbf{x}-\mathbf{x}'')\,\mathrm{dV}''\frac{\mathbf{dV'dt'}}{|\mathbf{x}'-\mathbf{x}''|}\delta\left(\mathbf{t}'-\left(\mathbf{t}-\frac{|\mathbf{x}-\mathbf{x}'|}{\mathbf{c}}\right)\right)\rho(\mathbf{x}',\mathbf{t}')\ .$$

Therefore

$$\Delta\sigma(\mathbf{x},\mathbf{t}) = -\mathbf{4}\pi\int \frac{\mathbf{dV'dt'}}{|\mathbf{x}-\mathbf{x}'|}\delta\left(\mathbf{t}'-\left(\mathbf{t}-\frac{|\mathbf{x}-\mathbf{x}'|}{\mathbf{c}}\right)\right)\rho(\mathbf{x}',\mathbf{t}')\,.$$

In the right-hand side of this equation appears the usual retarded solution, as must be. It is also easy to prove that upon application of the

D'Alembert operator the equation (2.15) is also satisfied with the solution (15). The step of proving that this solution is equivalent to Jackson's gauge function is more intricate, fortunately Jackson himself did the job in section IV of [4], so we may skip this calculation. Then the methodology we have proposed to obtain the gauge function is more rigorous and general than Jackson's because, for instance, we have solved equation (4) for quite specific boundary conditions for the equations (6)-(7). Logically, our results would change for different boundary conditions. Conversely, in Jackson's method these conditions are not accounted for explicitly.

Besides, Jackson's procedure takes for granted that the many conditions involving the gauge function are consistent. Here we have derived a solution that allow us to test if such conditions are, indeed, consistent. By consistency we mean that for given conditions the integral (16) converges. But the main point is that there is no way to compare the obtained gauge function with other functions (obtained by any other method) because we have only one procedure: to solve equation (1). So, why so much ado about multiplicity of solutions? Because that is an independent hypothesis that we shall discuss in the next sections.

4 Violation of gauge invariance.

Regarding the question of the violation of gauge invariance, in this section we are going to reconstruct Chubykalo & Onoochin problem in order to obtain more generality and logical consistency. We repeat reasoning (O): if we suppose that gauge invariance is respected, then we can obtain a contradiction from the formalism. We have proved in the previous section that this is not correct: if we suppose gauge invariance, we obtain a gauge function defined by the integrability conditions of 1-form (1.2.8). In this respect Chubykalo & Onoochin are completely wrong. But their arguments may become consistent if they leave aside argument (O) to embrace the *ab initio* hypothesis that gauge invariance is violated. In order to follow this hypothesis, we require to deduce the differential conditions of existence of the 1-form ω^* defined by (1.2.10). It turns out that there are three equations, two for a_1 and one for a_2. Once this is done we can assert that the equations (19) and (24) of [1] are particular cases of these conditions, and the whole effort of Chubykalo & Onoochin consist in proving that $a_1 \neq a_2$. They are right, it is correct that the functions a_i are not equal, except for some particular cases, but our arguments will be more general. This is enough to prove that Jackson in [4] is also wrong. Let's now deduce the integrability conditions of (1.2.10). We start from the gauge transformation

$$\mathbf{A}_C = \mathbf{A}_L + \nabla a_1, \tag{4.1}$$

$$\varphi_C = \varphi_L - \frac{1}{c}\frac{\partial a_2}{\partial t}, \tag{4.2}$$

the field equations in the Lorenz gauge

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_L = -\frac{4\pi}{c}\mathbf{J}, \tag{4.3}$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi_L = -4\pi\rho, \tag{4.4}$$

and the Coulomb gauge

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_C = -\frac{4\pi}{c}\left(\mathbf{J} - \frac{1}{4\pi}\nabla\frac{\partial\varphi_C}{\partial t}\right), \tag{4.5}$$

$$\Delta\varphi_C = -4\pi\rho. \tag{4.6}$$

Now, we shall deduce the integrability conditions. We apply D'Alembert operator to equation (1) and we use field equations (3) and (5) to get

$$\nabla\left(\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)a_1 - \frac{1}{c}\frac{\partial\varphi_C}{\partial t}\right) = 0. \tag{4.7}$$

Therefore, with the use of the divergence operator in (1), and the Lorenz gauge equation, we straightforwardly get

$$\Delta a_1 = \frac{1}{c}\frac{\partial\varphi_L}{\partial t}. \tag{4.8}$$

To obtain conditions for a_2 we apply Laplace operator to (2) and we use (4) and (6), with the result

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\Delta a_2 - \frac{1}{c}\frac{\partial\varphi_L}{\partial t}\right) = 0. \tag{4.9}$$

If we use the operator $\frac{1}{c}\frac{\partial}{\partial t}$ on (2) and equation (8) we can write

$$\Delta a_1 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}a_2 = \frac{1}{c}\frac{\partial\varphi_C}{\partial t}. \tag{4.10}$$

We can write (7) and (9) as follows

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) a_1 - \frac{1}{c}\frac{\partial \varphi_C}{\partial t} = H(t), \tag{4.11}$$

$$\Delta a_2 - \frac{1}{c}\frac{\partial \varphi_L}{\partial t} = G(\mathbf{x}). \tag{4.12}$$

If we ask that for $a_2 = a_1$ the equations reduce to the equations (2.12)-(2.15) we can take $H = 0,\ G = 0$. In this way we have obtained the determining equations for the gauge functions $a_1,\ a_2$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) a_1 = \frac{1}{c}\frac{\partial \varphi_C}{\partial t}, \tag{4.13}$$

$$\Delta a_1 = \frac{1}{c}\frac{\partial \varphi_L}{\partial t}, \tag{4.14}$$

$$\Delta a_2 = \frac{1}{c}\frac{\partial \varphi_L}{\partial t}. \tag{4.15}$$

It is clear that if $a_1 = a_2$ these equations become (2.12)-(2.15). Hence, we can see that the supposition of violation of gauge invariance is much more complex than expected. Chubykalo & Onoochin, like Jackson, arrive at less general equations, even more, they did not settle the problem correctly because they did not account for the complete set of conditions. The consistency of the premise of violation of gauge invariance is clearly solved if we can find a solution to the deduced differential equations, which is certainly possible. But the question posed by Jackson in [4] is under what constraints $a_1 = a_2$. In order to solve this problem, we shall simplify the equations (13)-(14)-(15). We note that if we ask for the conditions of the equality of a_1 and a_2 we may skip equation (14). This is because under the conditions that allow the equality $a_1 = a_2$ it is clear that, because of the form of (15), a_1 will satisfy (14) if it satisfy (15). Other way to consider the problem is from two equations

$$\Delta\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) a_1 = -\frac{4\pi}{c}\frac{\partial}{\partial t}\rho, \tag{4.16}$$

$$\Delta a_2 = \frac{1}{c}\frac{\partial \varphi_L}{\partial t}. \tag{4.17}$$

But we shall not discuss this form of the problem.

5 Conditions for gauge invariance.

Consider the field equations

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) a_1 = \frac{1}{c}\frac{\partial \varphi_C}{\partial t} ,$$

$$\Delta a_2 = \frac{1}{c}\frac{\partial \varphi_L}{\partial t} .$$

The question is: under what conditions $a_1 = a_2$? Clearly one of the equations is of elliptic type, while the other is hyperbolic, so their solutions are not identically the same, but only equal in a region under certain conditions. Now, if we display the general solution of each equation, obtained with the help of Green's identity, we can see why the solutions are not, in all cases, equal

$$a_2(\mathbf{x},t) = \int \frac{\partial G^0}{\partial n}(\mathbf{x},t;\mathbf{x}',\ t')\, a_2(\mathbf{x}',\ t')\, dS^{'} dt' + \frac{1}{c}\int G^0(\mathbf{x},t;\mathbf{x}',\ t')\, \frac{\partial \varphi_L}{\partial t'}(\mathbf{x}',\ t') dV^{'} dt,$$

$$a_1\ (\mathbf{x},t) = \int \sum_{\alpha\mu} \eta_{\alpha\mu} n_\mu \left(\frac{\partial G^1}{\partial x_\alpha}(\mathbf{x},t;\mathbf{x}',\ t')\, a_1(\mathbf{x}',\ t')\right) dS^{'} dt' + \frac{1}{c}\int G^1(\mathbf{x},t;\mathbf{x}',\ t')\, \frac{\partial \varphi_C}{\partial t'}(\mathbf{x}',\ t') dV^{'} dt' ,$$

here $\eta_{\alpha\mu}$ is a diagonal matrix whose diagonal elements are $\left\langle -\frac{1}{c},\ 1,1,1\right\rangle$ and n_μ are the components of a normal vector field defined on the smooth surface S where the integrals are defined. We have inserted Green's functions for boundary value problems because we are not considering unbounded space, therefore in the solutions for $a_1,\ a_2$ the integrations are over a region $S \times I$ of $\mathbf{R}^3 \times \mathbf{R}^1$ and the normal derivatives along the boundary are not present. In order to obtain an explicit solution out of these formulae, boundary conditions are required. Hence, we must have for the Poisson equation, *e.g.* a Dirichlet problem

$$\forall \mathbf{x} \in S\ ,\ a_2(\mathbf{x},t) = f(\mathbf{x},t). \tag{5.1}$$

We can also take the same problem for the D'Alembert equation therefore,

$$\forall \mathbf{x} \in S\ ,\ a_1(\mathbf{x},t) = f(\mathbf{x},t) \tag{5.2}$$

Here is the nature of S that is different for each differential equation, in general. For an elliptic problem there is not a characteristic manifold in real space, therefore we can use a surface whose distance R to the source of the perturbation is $\frac{R}{t} > c$, *i.e.* a space-like separation. Conversely, for a hyperbolic problem there is an especially important characteristic surface: the light cone L_c, where all singularities of the initial value problem are located. So we must consider S as a subset of the lower half of the light cone L_c^-, otherwise the boundary value problem is not well defined for D'Alembert equation, or we must introduce an unphysical shock wave moving faster than light. Then, the boundary value problems (4.8) and (4.9) are consistent only for a well-defined problem for the D'Alembert equation. If we solve problem (4.8) for a sphere S_R^2 of radius such that $\frac{R}{t} > c$ we can see that, whatever the behavior of the volume integral in the solution for a_2, the surface terms indicate that a perturbation outside of the light cone appears because $L_c^- \subset S_R^2$. With the restrictions enunciated this unphysical behavior disappears. However, even if due to some property of the potentials in each gauge the volume integrals are equal, and the boundary value problem is enough to equalize the surface integrals, there is the question of the initial value problem, or Cauchy problem, for D'Alembert equation. Let's examine the term

$$\int \left(\frac{\partial G^1}{\partial t} \left(\mathbf{x}, t; \mathbf{x}', \ t'\right) a_1 \left(\mathbf{x}', \ t'\right) \right) n_0 dS^{'} dt', \tag{5.3}$$

which does not appear in the solution to a_2. Here initial value conditions are needed, but if we consider

$$a_1 \left(\mathbf{x}, \ 0\right) = a(\mathbf{x}), \tag{5.4}$$

we obtain, again, the result that $a_1 \neq a_2$. Then the whole discussion of Chubykalo & Onoochin is incomplete because when general arguments are used, it is easy to prove that gauge invariance of the electromagnetic field is violated. However, it is also easy to see how to remedy such a situation using boundary and initial value problems. For the volume integrals the situation seems to be more difficult.

Now let's revisit Chubykalo & Onoochin's key arguments from the standpoint of equations (4.13) and (4.15). We shall solve the equations for unbounded space, therefore all boundary terms are zero and we just have the volume integrals given by

$$\int \frac{\delta(t-(t^{'}-\frac{|\mathbf{x}-\mathbf{x}'|}{c}))}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\varphi_C}{\partial t'}(\mathbf{x}',\ t')\,dV'dt', \tag{5.5}$$

$$\int \frac{\delta\left(t-t^{'}\right)}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\varphi_L}{\partial t'}(\mathbf{x}',\ t')\,dV'dt'. \tag{5.6}$$

Now we solve the field equations for each potential, again for unbounded space and such that at infinity all boundary terms vanish, then

$$\varphi_L(\mathbf{x}',\ t') = \int \frac{\delta(t'-(t''-\frac{|\mathbf{x}'-\mathbf{x}''|}{c}))}{|\mathbf{x}'-\mathbf{x}''|}\rho(\mathbf{x}'',\ t'')\,dV''dt'', \tag{5.7}$$

$$\varphi_C(\mathbf{x}',\ t') = \int \frac{\delta(t'-t'')}{|\mathbf{x}'-\mathbf{x}''|}\rho(\mathbf{x}'',\ t'')\,dV''dt''. \tag{5.8}$$

The time derivative of (5)-(6) is no problem because it can be directly transferred, via the delta function, to the density function (7)-(8), so we can write for (5)-(6)

$$\int \frac{\delta(t-(t^{'}-\frac{|\mathbf{x}-\mathbf{x}'|}{c}))}{|\mathbf{x}-\mathbf{x}'|}\frac{\delta(t'-t'')}{|\mathbf{x}'-\mathbf{x}''|}\frac{\partial}{\partial t''}\rho(\mathbf{x}'',\ t'')\,dV''dt''dV'dt', \tag{5.9}$$

$$\int \frac{\delta\left(t-t^{'}\right)}{|\mathbf{x}-\mathbf{x}'|}\frac{\delta(t'-(t''-\frac{|\mathbf{x}'-\mathbf{x}''|}{c}))}{|\mathbf{x}'-\mathbf{x}''|}\frac{\partial}{\partial t''}\rho(\mathbf{x}'',\ t'')\,dV''dt''dV'dt'. \tag{5.10}$$

Integral (9) becomes

$$\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\delta(t-(t^{'}-\frac{|\mathbf{x}-\mathbf{x}'|}{c}))}{|\mathbf{x}'-\mathbf{x}''|}\frac{\partial}{\partial t'}\rho(\mathbf{x}'',\ t')\,dt'dV''dV' \tag{5.11}$$

and (10) is

$$\int \frac{1}{|\mathbf{x}'-\mathbf{x}''|}\frac{\delta(t-(t''-\frac{|\mathbf{x}'-\mathbf{x}''|}{c}))}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial}{\partial t''}\rho(\mathbf{x}'',\ t'')\,dt''\ dV''dV'. \tag{5.12}$$

It is clear that, because of the different arguments of integration in the delta function, the integrals (11) and (12) are not identically equal. Hence, we re-obtained, in our notation, the result that Chubykalo & Onoochin obtained with their equation (27) of [1].

6 The solutions of Maxwell's equations.

According to Chubykalo & Onoochin in [1] the potentials are the only way to solve electromagnetic problems. This is indeed correct but not for the reasons they claim. Their arguments are valid for point charges modeled with delta functions only, not for charges defined by a bounded smooth density. However, there are solid reasons to ground the belief that the potentials are the only way to solve Maxwell's equations, let's examine these reasons. The Maxwell's equations treated by Chubykalo & Onoochin are those where the Lorentz constitutive relations are valid, i.e., they have reduced the full set of Maxwell's equations for any medium with the help of

$$\mathbf{E} = \mathbf{D},\ \mathbf{B} = \mathbf{H}. \tag{6.1}$$

But this is only the special case of a non-magnetizable, non-polarizable, non-dispersive medium, for the general case Maxwell's equations involve two pairs of vector fields: $\langle \mathbf{E},\ \mathbf{B} \rangle$ and $\langle \mathbf{D},\ \mathbf{H} \rangle$. The general solution for the first pair of fields is $\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$, while for $\langle \mathbf{D},\ \mathbf{H} \rangle$ the equations are

$$\nabla \cdot \mathbf{D} = 4\pi\varrho\ ,$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}\ ,$$

whose solution is:

$$\mathbf{D} = \nabla \times \mathbf{A}' - \nabla\varphi',\ \mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}' - \nabla\varphi', \tag{6.2}$$

where $\langle\ \mathbf{A}'\ ,\ \varphi'\ \rangle$ are another set of potentials, not equal to $\langle\ \mathbf{A}\ ,\ \varphi\ \rangle$, and constrained by the equation: $\Delta\varphi' = -4\pi\rho$, which implies charge conservation; for more details see [13]. Then, in absence of constitutive relations the general solution of Maxwell's equations involves seven arbitrary functions plus a boundary value problem. Constitutive relations plus gauge conditions are used to reduce the number of arbitrary functions in the solution. With the help of Lorentz relations (1) in the general solution only four arbitrary functions remain; if gauge conditions and boundary conditions are introduced just one remains: the gauge function. Then Chubykalo & Onoochin are right, but not for the correct reasons, hence their belief is ungrounded.

7 Conclusions.

In order to highlight the logical structure of the hypothesis of gauge invariance we have suggested a mathematical formalism, which in outline is as follows. We introduced, in section (1.2) a 1-form ω that upon substitution of gauge transformations given by (2.4)-(2.5) becomes exact, so gauge invariance is automatic. For gauge transformations of the form (1.1.5)-(1.1.6) the 1-form is not closed, so gauge invariance is violated. Then we can express Chubykalo & Onoochin reasoning as follows:

1. ω is closed, then
2. gauge invariance is violated, in conclusion
3. ω is not closed, then the gauge function is inexistent and the solutions in different gauges are not identical [9].

To obtain the implication (1) $\rightarrow$ (2) Onoochin starts from $\mathbf{E}_C = \mathbf{E}_L$ which is equivalent to say (1), as we saw in (1.1.2), then he introduces solutions for the potentials in each gauge, taking care of not using a gauge function explicitly, to prove (2). This is fallacious, because what he, and Chubykalo, are using are the equations (4.13)-(4.15) which are the result of a different premise. If we outline their reasoning, we can see that the fallacy is the well-known *petitio principi*: they suppose what they want to prove. Jackson falls in the same fallacy:

4. ω^* is not closed, then
5. gauge invariance is respected, in conclusion
6. ω^* is closed, the gauge function exists and the solutions in any gauge are always identical.

The wrong steps in the implication (4) $\rightarrow$ (5) are clear in section 5: in general it is not possible to integrate ω^*, so, if we start from such a premise we shall not be able to obtain an integration except for one case: the motion of light, *i.e.* a phenomenon not involving sources of charge, with severe constraints. We achieve three major conclusions, which we believe worth stressing:

- If we suppose that Maxwell's equations are gauge invariant, then we do not get a contradiction with this assumption in the development of the mathematical formalism in general.

- If we suppose that Maxwell's equations are not gauge invariant, then we do not get a contradiction with this assumption in the development of the mathematical formalism in general.
- If we want to decide between these pair of mutually exclusive assumptions, we must review them under the light of physical reality, because *a priori* arguments neglect this reality.

Finally, we must add that in order to decide what is the correct hypothesis we must go to the world, to experience.

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Last revision Saturday 7 May 2020.