Reply to 'Comment on "On the ambiguity of solutions of the system of the Maxwell equations" by R. Alvarado-Flores and A. Espinoza

Vladimir Onoochin[1]

[1] "Sirius", Nikoloyamski lane 3A, Moscow, 109004, Russia.
E-mail: onoochin@gmail.com

In this short article one of the author replies to a Comment on "On the ambiguity of solutions of the system of the Maxwell equations" [1] by R. Alvarado-Flores and A. Espinoza that points out some 'weak points' of non-equivalence of the expressions for the electric field calculated in two gauges given in [2].

Objections of the Comment by R. Alvarado-Flores and A. Espinoza [1] on our recent paper [2] consist essentially of the following points:
1. One of two statements of [2], that the wave equation for the electric field cannot be solved, is wrong.
2. Our treatment of the gauge invariance is not correct.
3. The condition of the criterion of non-equivalence of the potentials (Eq. (19) of [2]) calculated in different gauges is not sufficient to have the complete proof of non-equivalence of the gauges.
4. The gauge function that provides transformation of one pair of the potentials into the other pair is not given in [2] meanwhile the explicit expression of this gauge function is given by Jackson [3].

I explain point-by-point why the objections of Alvarado-Flores and Espinoza are irrelevant to the results of our paper.

1. First, we do not state that the wave equation for the electric field cannot be solved. Jefimenko presented some examples of solutions of this equation for different sources and his solutions are in good agreement with the experimental data [4]. We state that some problems arise in attempt to derive the wave equation for the electric field directly from the Maxwell equations. For example, application of two Maxwell equations, that are responsible for the radiated component of the electric field, to well known Hertz's solution with the source the elementary dipole

gives partial differential equations for the radial and angular components of the E field. These differential equations are not compatible with Hertz's solution derived from the wave equations for the potentials. It means that two of the Maxwell equations perfectly describe the process of transfer of the bound components of the electric field into the radiated components but fail to describe the process of the energy transfer from the source to the fields, that is shown by the author[1].

2. The gauge invariance. On p. 4 of [1] Alvarado-Flores and Espinoza give own treatment of our statement regarding the 'gauge invariance': ' (O) *If we suppose that gauge invariance is fulfilled then all the solutions obtained through potentials in different gauges are equal. However, if these solutions are not equal, gauge invariance is violated.*'

In solving the Maxwell equations by means of the potentials, it is easy to find that the EM fields do not change if the potentials are transformed in accordance to

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda\,, \tag{1}$$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\,. \tag{2}$$

where Λ is an *arbitrary* scalar function. But to make the above transformations, one should determine the potentials at least. These potentials cannot be determined without choice of some relation between the potentials called the choice of the gauge. This relation reduces the system of the Maxwell equations to two wave equations for the potentials.

It was Lorentz who in 1094 noted that in order to have the potentials satisfy the ordinary wave equations they must be related by[2]

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0\,, \tag{3}$$

He discusses the arbitrariness in the potentials and then states that every other admissible pair $\mathbf{A}'$ and φ' can be related to the first pair $\mathbf{A}'$ and φ' via the transformations (1)-(2), where the function $\Lambda(\mathbf{r}, t)$ can be determined by subjecting $\mathbf{A}'$ and φ' to the condition (3) (Note 5 of [5])

$$\nabla\cdot\mathbf{A}' + \frac{1}{c}\frac{\partial\varphi'}{\partial t} = 0\,, \quad\to\quad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\,. \tag{4}$$

[1] https://www.researchgate.net/publication/347263752_on_application_of_hertz's_solution_to_the_maxwell_equations

[2] Lorenz, who first introduced this relation, used it to establish the correspondence between his wave equations for the potentials and the Maxwell equations.

Since these results of Lorentz, the function Λ is called the *gauge function.* It is important to note that this function transforms the potentials determined in one gauge (the Lorenz gauge (3)) to the potentials determined in the same gauge. It was reasonable to seek both pairs of the potentials in the Lorenz gauge because the equations for the potentials are fully separated and their solutions are known.

It follows from Eqs. (1) and (2) that one is always able to use *arbitrary* function Λ' to transform the potentials – but at the only point. The transformation in accordance to (1) and (2) in another point is acceptable by using an *arbitrary* function Λ''. But these operations of transformation are not productive because Λ' and Λ'' are not connected by any functional link. Therefore, they cannot be used in practical calculations. If one intends to find the function Λ that gives transformation of the pair of the potentials into corresponding pair of the potentials in all space, the choice of the gauge condition put certain restriction to *arbitrariness* of Λ. In the Lorenz gauge, Λ obeys the homogeneous wave equation. However, in the Coulomb gauge the transformations (1) and (2) do not work. In this gauge,

$$\varphi'_C = \varphi_C - \frac{1}{c}\frac{\partial \Lambda_C}{\partial t}\,. \tag{5}$$

where both φ_C and φ'_C should satisfy the Poisson equation $\nabla^2\Phi = -4\pi\rho$. So

$$\nabla^2\varphi'_C = -4\pi\rho \;\rightarrow\; \nabla^2\left(\varphi_C - \frac{1}{c}\frac{\partial \Lambda_C}{\partial t}\right) = -4\pi\rho \;\rightarrow\; \nabla^2\Lambda_C = 0\,, \tag{6}$$

since the charge density cannot change if one transforms the potentials. But if the boundary of the region, where φ is sought, is the closed surface[3], the Poisson equation has unique stable solution for φ that is completely determined by the source and the Dirichlet boundary conditions for the potential. Therefore, the conditions for the gauge function Λ_C should be equal to zero on this boundary because one cannot use these conditions twice. As a result, the only possible (unique) solution for the gauge function in the Coulomb gauge is $\Lambda_C = 0$. It means that in the Coulomb gauge the gauge invariance is absent.

But determination of the gauge function Λ in some gauge in order to provide transformation of one pair of the potentials into the other pair

[3] If the surface is open, one should put $\varphi_\infty = 0$

of the potentials is quite different problem and we do not consider it in [2]. We even do not use the term 'gauge invariance' in our work so the objections of Alvarado-Flores and Espinoza on our incorrect treatment of this term is not essential.

3. The third objection of Alvarado-Flores and Espinoza is that non-zero value of Eq. (19) in [2] is not sufficient to state on non-equivalence of the gauges.

Eq. (19) is obtained from the following equation

$$\nabla_R\,\varphi_{\rm C}(\mathbf{R};t)-\nabla_R\,\varphi_{\rm L}(\mathbf{R};t)+\frac{1}{4\pi c^2}\nabla_R\left(\int\frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial^2\varphi_C(\mathbf{r};t)}{\partial t^2}\right]\,\mathrm{d}\mathbf{r}\right)=0\,. \tag{7}$$

by omitting the operator ∇_R, that acts on all terms of (7). It leads to $\varphi_{\rm C}-\varphi_{\rm L}-\psi=0$

Alvarado-Flores and Espinoza state that this is a mistake because in opposite to our statement, (7) should give $(\varphi_{\rm C}-\varphi_{\rm L}-\psi)=d(t)$ where $d(t)$ is a function of time.

In the classical electrodynamics, electromagnetic potentials are treated as auxiliary mathematic quantities used to calculate the expressions for the physical quantities, the EM fields. So any functions which *do not change* the magnitudes of the fields can be added to the potentials. Therefore, any function which depends only on the time variable can be added to Eq (19) of [2].

Moreover, our arguments do not change if the operator ∇ is not eliminated, or the following expression

$$\nabla_R\left\{\varphi_L(\mathbf{R},t)-\varphi_C(\mathbf{R},t)-\frac{1}{4\pi c^2}\int\frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial^2\varphi_C(\mathbf{r},t)}{\partial t^2}\right]\,\mathrm{d}\mathbf{r}\right\}, \tag{8}$$

is used instead of Eq. (19) of [2]. Eqs. (20) – (24) of [2] represent transformation of the difference $\varphi_L-\varphi_C$. So Eq. (8) takes a form

$$\nabla_R\left\{\int\frac{1}{|\mathbf{R}-\mathbf{r}|}\frac{\partial^2\varphi_L(\mathbf{r},t)}{\partial t^2}\,\mathrm{d}\mathbf{r})-\int\frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial^2\varphi_C(\mathbf{r},t)}{\partial t^2}\right]\,\mathrm{d}\mathbf{r}\,.\right\} \tag{9}$$

Obviously, if the expression in the curly brackets is not equal to zero, the equality $\mathbf{E}_L=\mathbf{E}_C$ is not fulfilled which means that the gauges are not equivalent. It is the same as a statement on non-equivalence of the expressions for the E field calculated in two gauges.

4. On the objection of Alvarado-Flores and Espinoza that we omit the usage of the gauge function similar to that one Jackson gives in [3]

The non-usage of the gauge function follows from our result. If the expressions for electric field calculated in two gauges are different such a function cannot exist. Despite Jackson derives the gauge function which provides transformation of potentials determined in the Lorenz gauge to the potentials in the Coulomb gauge [3], the detailed analysis of the expressions for this gauge function shows that some errors are made in this derivation.

Jackson introduced two 'gauge functions', $\chi_C(\mathbf{x},t)$ and $\Psi(\mathbf{x},t)$, in Chs. III and IV of [3], respectively. The function Ψ is written as (Eq. (4.1) of [3])

$$\Psi(\mathbf{x},t) = \frac{1}{4\pi c}\int d^3x'' \frac{1}{|\mathbf{x}-\mathbf{x}''|}\left[\int d^3x' \frac{1}{|\mathbf{x}'-\mathbf{x}''|}\frac{\partial\rho(\mathbf{x}',t')}{\partial t'}\right], \quad (10)$$

where $t' = t - |\mathbf{x}-\mathbf{x}''|/c$.
Let us estimate this integral. For ρ describing the charge density of the classical electron,

$$\rho(\mathbf{x}',t') = q\delta^3\left[\mathbf{x}'-\mathbf{r}_0(t')\right] \;\to\; \frac{\partial\rho(\mathbf{x}',t')}{\partial t'} = -\left(\mathbf{v}_0(t')\cdot\nabla\delta^3(\mathbf{x}',t')\right).$$

where $\mathbf{r}_0$, $\mathbf{v}_0$ are the coordinate and the velocity of the electron (these quantities should be introduced 'by hands'), and the variable of integration x' does not enter into them. Because the derivative of the delta–function is ill–defined operation, one should make the integration by parts of the internal integral which gives

$$\int \frac{d^3x'}{|\mathbf{x}'-\mathbf{x}''|}\frac{\partial\rho(\mathbf{x}',t')}{\partial t'} = -q\int \mathbf{v}\cdot\nabla\frac{1}{|\mathbf{x}'-\mathbf{x}''|}\delta^3(\mathbf{x}',t')\,d^3x' = \\ = q\,\frac{\mathbf{v}_0(t')\cdot(\mathbf{x}''-\mathbf{r}_0(t'))}{|\mathbf{x}''-\mathbf{r}_0(t')|^3}.$$

So the expression (10) takes a form

$$\Psi(\mathbf{x},t) = \frac{q}{4\pi c}\int d^3x'' \frac{1}{|\mathbf{x}-\mathbf{x}''|}\frac{\mathbf{v}_0(t')\cdot(\mathbf{x}''-\mathbf{r}_0(t'))}{|\mathbf{x}''-\mathbf{r}_0(t')|^3}, \quad (11)$$

Obviously, this integral diverges at $x'' \to \pm\infty$ as

$$I \simeq \int_{x''\to\infty} \frac{d^3x''}{|\mathbf{x}-\mathbf{x}''|\,|\mathbf{x}''-\mathbf{r}_0(t')|^2} \simeq \lim_{x''\to\infty} \ln x'' \;\to\; \infty.$$

Let us consider Jackson's transformation of Eq. (3.3) of [3]

$$\frac{1}{c}\frac{\partial\chi_C}{\partial t} = \Phi_L(\mathbf{x},t) - \Phi_C(\mathbf{x},t) = \int \frac{d^3x'}{R}\left[\rho(\mathbf{x}',t-R/c) - \rho(\mathbf{x}',t)\right] . \quad (12)$$

and show that the author incorrectly processes the lower limit.

Jackson integrates both sides of this equation with respect to ct to obtain (Eq. (3.4))

$$\chi_C = c\int d^3x'\frac{1}{R}\left[\int\limits_{t_0}^{t-R/c} dt'\rho(\mathbf{x}',t') - \int\limits_{t_0}^{t} dt'\rho(\mathbf{x}',t)\right] ,$$

and 'This can be written more compactly as'

$$\chi_C = c\int d^3x'\frac{1}{R}\left[\int\limits_{t}^{t-R/c} dt'\rho(\mathbf{x}',t')\right] . \quad (13)$$

Let's consider the procedure of integration over t in Eq. (12). Since the first function ρ in the integrand depends on the shifted time $t-R/c$, one has

$$\int\limits_{T_0}^{T} dt\rho(\mathbf{x}',t-R/c) \sim \int\limits_{T_0}^{T} dtf(x,t-\alpha) = \int\limits_{T_0-\alpha}^{T-\alpha} dtf(x,t) .$$

So the lower limits are different and the correct expression for Jackson's gauge function should be

$$\chi_C = c\int d^3x'\frac{1}{R}\left[\int\limits_{t}^{t-R/c} dt'\rho(\mathbf{x}',t') - \int\limits_{t_0}^{t_0-R/c} dt'\rho(\mathbf{x}',t')\right] . \quad (14)$$

However, in [3] the second integral of (14) is omitted.

But this lower limits gives the function that depends on the current coordinates because there is no integration over the coordinates. Obviously, the second term in (14) cannot be ignored.

Let us add one more objection to derivation of the function χ_C: the functions $\rho(\mathbf{x}',t-R/c)$ and $\rho(\mathbf{x}',t)$ are different because they occupy different volume of the space at the same x', t. Therefore, the operation of integration over t cannot unify two different function ρ. Thus, both forms of the gauge function in [3] cannot be used to transform the potentials from one gauge to the other one.

The same error is made in work of Alvarado-Flores and Espinoza. It is easy to show that the integral (12), sec. 6 of [1], that represents their final result diverges. Let us assume $\rho(\mathbf{x}', t) = \delta^3(\mathbf{x}', t)$. The integral is defined in 7D space (three coordinates $\mathbf{x}'$, three and one time variable). After integration with four δ–functions, the integral takes a form

$$\int \frac{\mathrm{d}^3 x'}{|\mathbf{x}-\mathbf{x}'| \cdot |\mathbf{x}-\mathbf{x}'|} \simeq \lim_{x' \to \infty} x' \to \infty$$

It confirms the statement that the gauge function which provides transformation of the potentials determined in one gauge to the potentials in the other gauge cannot exist.

The Comment poses important questions regarding the gauge problem. However, the objections of Alvarado-Flores and Espinoza fails to identify mistakes in the commented paper and are thus superfluous.

References

[1] R. Alvarado-flores, A.A. Espinoza. Comment on "On the ambiguity of solutions of the system of the Maxwell equations". *Ann. Found. Lous de Broglie.* , , 2021.

[2] A. Chubykalo and V. Onoochin. On the ambiguity of solutions of the system of the Maxwell equations. *Ann. Fond. L. Broglie.* **44**, 147, (2019).

[3] J. D. Jackson. From Lorenz to Coulomb and other explicit gauge transformations. *Am. J. Phys.* **70**(9), 912, (2002).

[4] O. D. Jefimenko. *Electricity and Magnetism*, 2^{nd} ed., (Eectret Scientific Co., Star City, 1989), p. 335.

[5] Lorentz, H. A., *Theory of Electrons* (G. B. Teubner, Leipzig and G. E. Stechert, New York. 1909).[2^{nd} ed., 1916; reprinted by Dover Publications, New York. 1952]

(Manuscrit reçu le 28 décembre 2020)