On the direct solution of Maxwell’s equations for electromagnetic waves

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ABSTRACT. In this paper, a novel analytical method is proposed to directly obtain the solution of Maxwell's equations for the problem of electromagnetic radiation. This method does not make use of the Liénard-Wiecher theory of electrodynamic potentials. The solution obtained offers a better understanding of the phenomenon of electromagnetic radiation, and could lead to new methods of electromagnetic analysis. The method could impact other branches of physics and technology, which make use of potentials to explain phenomena of nature. Potentials, it seems, are no longer be necessary.

1 Introduction

Throughout history there have been several attempts to solve the equations of Maxwell, for electromagnetic radiation, but in none of them could it be done directly, so other procedures were used. Heinrich Rudolf Hertz obtained approximate solutions with the potentials that he defined and that today bear his name, [1]. Between 1898 and 1900, the Liénard-Wiechert electrodynamic potentials were accepted and are still used today, [2, 3]. Here we propose a simpler procedure for obtaining the solutions directly, without using them.

This paper is organized in six sections; the first is the present introduction. The second, entitled, the principle of linearity and wave equations, uses the linearity principle of Maxwell's equations to obtain the equations of radiated electromagnetic waves (EMW). The third corresponds to the, generalization of the Biot-Savart and Coulomb laws, and it extends these laws to the case of electromagnetic waves. The fourth section, called, solutions of the wave equations, develops the method of solving these equations. The fifth is a brief discussion about the solutions obtained in the previous section. Finally, in the conclusions section, a summary of the main results obtained. At the end of the paper the references used are listed.

2 The principle of linearity and wave equations

2.1 The principle of linearity of Maxwell's equations

Maxwell's equations are linear, therefore, they comply with the superposition principle, sum of two solutions is also a solution. Let us propose the principle of linearity as follows:

$$\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}_{Exc}}{\partial t} + \vec{J} \tag{1}$$

and

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} - \mu_0 \frac{\partial \vec{H}_{Exc}}{\partial t}, \tag{2}$$

where (1) is the Ampère-Maxwell equation, in which we make $\vec{J} = 0$, and (2) is the Faraday-Maxwell equation. In both, the curls operators of the left members can be equated to the superposition of the fields of the right members, where the second terms represent excitation fields, $\vec{E}_{Exc}$ and $\vec{H}_{Exc}$. The condition $\vec{J} = 0$ is valid in the small space between the arms of a dipole, feeding point in which there are only displacement currents.

2.2 Wave equation excited by magnetic field

Applying the curl operation to both members of (1):

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{E}\right) + \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{E}_{Exc}\right). \tag{3}$$

Expressing the left member of (3) according to the identity vector (4):

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}, \tag{4}$$

then, substituting (2) in (3), and knowing that $\nabla \cdot \vec{H} = 0$ and that this implies $\nabla(\nabla \cdot \vec{H}) = 0$, we get:

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} + \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}_{Exc}}{\partial t^2}, \tag{5}$$

however, for harmonic fields, $[\partial^n / \partial t^n] \to (i\omega)^n$, where $i = \sqrt{-1}$, the imaginary unit, ω the angular frequency, and n, an integer, in this case, $n = 2$, therefore (5), in complex form, we have:

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 (i\omega)^2 \vec{H} + \mu_0 \varepsilon_0 (i\omega)^2 \vec{H}_{Exc}, \tag{6}$$

on the other hand, the speed of electromagnetic waves in a vacuum is:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \tag{7}$$

where $\mu_0 = 4\pi \cdot 10^{-7}, [H/m]$ and $\varepsilon_0 = 8.85 \cdot 10^{-12}, [F/m]$ are the magnetic permeability and the dielectric permittivity of the vacuum respectively. The propagation constant k, of the EMW is:

$$k = \frac{\omega}{c}, \tag{8}$$

now, expressing (6) in its compact form:

$$\nabla^2 \vec{H} = -k^2 \vec{H} - k^2 \vec{H}_{Exc}, \tag{9}$$

organizing (9), we finally arrive at a Helmholtz wave equation:

$$\nabla^2 \vec{H} + k^2 \vec{H} = -k^2 \vec{H}_{Exc}. \tag{10}$$

That is, a nonhomogeneous differential equation for the intensity vector of the magnetic field $\vec{H}$ of a wave radiated by a magnetic source. On the right side of (10) we have the exciter function $\vec{H}_{Exc}$.

2.3 Wave equation excited by electric field

Following the same procedure as the previous case, in expression $(\mathbf{2})$:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right) - \mu_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}_{Exc}\right), \tag{11}$$

now, expressing the left member (11) as identity vector (12):

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}, \tag{12}$$

substituting (12) in (11) and knowing that for EMW the $\nabla \cdot \vec{E} = 0$ implies that $\nabla\left(\nabla \cdot \vec{E}\right) = 0$, so we get:

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}_{Exc}}{\partial t^2}, \tag{13}$$

we can express (13) in its phasorial form:

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 (i\omega)^2 \vec{E} + \mu_0 \varepsilon_0 (i\omega)^2 \vec{E}_{Exc}, \tag{14}$$

furthermore, putting (14) as a function of the constant k, seen in (8), it results:

$$\nabla^2 \vec{E} = -k^2 \vec{E} - k^2 \vec{E}_{Exc}, \tag{15}$$

rearranging it conveniently:

$$\nabla^2 \vec{E} + k^2 \vec{E} = -k^2 \vec{E}_{Exc}. \tag{16}$$

That is, a nonhomogeneous differential equation for the intensity vector of the electric field $\vec{E}$ of a wave radiated by an electric source. On the right side of (16) is the exciter function, $\vec{E}_{Exc}$.

3 Generalization of the Biot-Savart and Coulomb laws

3.1 Biot-Savart and Coulomb laws

For convenience we repeat equations (10) and (16) here:

$$\nabla^2\vec{H}+k^2\vec{H} = -k^2\vec{H}_{Exc}, \tag{10}$$

$$\nabla^2\vec{E}+k^2\vec{E} = -k^2\vec{E}_{Exc}, \tag{16}$$

now, as the excitation function $\vec{H}_{Exc}$ in (10) we take the field distribution of Biot-Savart, [4], whose expression is the following:

$$\vec{H}_{Exc} = \frac{I\cdot\delta}{4\pi}\left[\frac{1}{r^2}\right] sin(\theta)\ \vec{\imath}_{\varphi}, \tag{17}$$

and as excitation function $\vec{E}_{Exc}$, in (10), we take the field distribution of the Coulomb electric dipole, [5], its expression is:

$$\vec{E}_{Exc} = -i\frac{I\cdot\delta}{4\pi\omega\varepsilon_0 r^3}(2\,cos(\theta)\,\vec{\imath}_r - sin(\theta)\,\vec{\imath}_\theta). \tag{18}$$

In (17), δ is the length of the magnetic field support. The current flowing through support is I, and $\vec{\imath}_\varphi$ is the unit vector of the spherical coordinate system that indicates the direction of the $\vec{H}_{Exc}$. In(18), δ is the distance between charged ends of dipole, in this case electric field support. The relation $[-i\,I/\omega]$ is the electric charge q that accumulates at the ends of dipole. In this case $\vec{\imath}_r$ and $\vec{\imath}_\theta$, are the unit vectors that indicate the direction of the components of the $\vec{E}_{Exc}$.
Equations (17) and (18) are paired and either of them can be obtained from the other, using Maxwell's equations. When in equations (10) and (16) the excitatory functions (17) and (18) appear, they correspond to the field distributions of the Biot-Savart and Coulomb laws respectively, then these equations will be called, as, Helmholtz-Biot-Savart equation and Helmholtz-Coulomb equation, respectively. The first part of the name corresponds to the type of inhomogeneous differential equation, and the second, to the type of excitatory function used.

4 Solutions of the wave equations

4.1 Solution for magnetic field

Then, the solution of the wave equation (10) is as follows:

$$\vec{H} = C_0 e^{-ik\cdot r} + C_1 e^{ik\cdot r}, \tag{19}$$

where the first addend in (19) corresponds to a wave that is radiated from the source, and the second addend arrives from infinity. Now, applying the Sommerfeld radiation condition [6] to (19), only the first addend are taken into account. The parameter C_0 represents the function to be determined. On it, the initial conditions of the wave formation process in the area of fields linked to the radiator is described as:

$$\vec{H} = C_0 e^{-ik\cdot r}, \tag{20}$$

to determine C_0 the solution (20) is evaluated with the excitation function of the system $\vec{H}_{Exc}$:

$$\vec{H}_{Exc} = C_0 e^{-ik\cdot r}, \tag{21}$$

substituting (18) in (21) and solving for C_0 we obtain:

$$C_0 = \frac{I\cdot\delta}{4\pi}\left[\frac{e^{ik\cdot r}}{r^2}\right] sin(\theta)\ \vec{\imath}_{\varphi}, \tag{22}$$

expressing the complex exponential of (22) in its trigonometric form, we have:

$$C_0 = \frac{I\cdot\delta}{4\pi}\left[\frac{cos(k\cdot r)}{r^2} + i\frac{sin(k\cdot r)}{r^2}\right] sin(\theta)\ \vec{\imath}_{\varphi}, \tag{23}$$

multiplying in (23) the sine function, k up and down and conveniently separating the distance r in the denominator, you reach the expression (24):

$$C_0 = \frac{I\cdot\delta}{4\pi}\left[\frac{cos(k\cdot r)}{r^2} + i\frac{k}{r}\cdot\frac{sin(k\cdot r)}{(k\cdot r)}\right] sin(\theta)\ \vec{\imath}_{\varphi}. \tag{24}$$

In (24), when the electrical distance kr is extremely small we are the zone of fields linked to the radiator. In this zone $cos(kr) \rightarrow 1$ and $[sin(kr)/kr] \rightarrow 1$, which reduces (24) to:

$$C_0 = \frac{I \cdot \delta}{4\pi}\left[\frac{1}{r^2} + \frac{ik}{r}\right] sin(\theta)\ \vec{\imath}_{\varphi}. \qquad (25)$$

Finally, substituting (25) in (20) we arrive at the expression of the intensity vector of the magnetic field $\vec{H}$, solution of the wave equation (10). Inside the brackets, it has been multiplied up and down by k^2 and k respectively, and then k^2 has been taken as a common factor:

$$\vec{H} = \frac{I \cdot \delta}{4\pi} k^2 \left[\frac{1}{(kr)^2} + \frac{i}{kr}\right] e^{-ik \cdot r}\, sin(\theta)\ \vec{\imath}_{\varphi}. \qquad (26)$$

See that (26) coincides with the expression of the Hertz dipole magnetic field intensity vector, obtained by the traditional method, that is, making use of the well-established theory of the Liénard-Wiechert electrodynamic potentials. It is important to point out that the solution (26) has been obtained by applying the Biot-Savart law to the displacement current filament existing in the electric dipole, when it is fed by an alternating current signal. Said displacement current is nothing more than the time-varying electric field of the dipole given by the expression $\vec{J}_D = \varepsilon_0\, \partial\vec{E}/\partial t$. In this way, said current has all the characteristics of the electromagnetic field, and all the valid operations that can be carried out with Maxwell's system of equations are also applicable to it.

4.2 Another way to get the solution for the magnetic field

The solution (26), obtained previously, will now be obtained by a second method. Now, the wave equation for the magnetic field (11) will be solved by means of an expansion in series of powers of the complex exponential function (e^{ikr}). This method will allow to observe the terms that remain hidden in the accepted classical solution. This procedure will also be used to find the solution to the electric field wave equation in the next section.

Developing in power series the exponential in the bracket of (15):

$$\frac{e^{ikr}}{(kr)^2} = \frac{1}{(kr)^2}\sum_{n=0}^{\infty}\frac{(ikr)^n}{n!}\frac{1}{(kr)^2} + \frac{ikr}{(kr)^2} + \frac{(ikr)^2}{2!\,(kr)^2} + \frac{(ikr)^3}{3!\,(kr)^2} + \frac{(ikr)^4}{4!\,(kr)^2} + \cdots. \tag{27}$$

See that the denominator in (27) has been completed with the k constant. Performing:

$$\frac{e^{ikr}}{(kr)^2} = \frac{1}{(kr)^2}\sum_{n=0}^{\infty}\frac{(ikr)^n}{n!} = \frac{1}{(kr)^2} + \frac{i}{(kr)} - \frac{1}{2} - \frac{i(kr)}{6} + \frac{(kr)^2}{24} - \frac{(ikr)^3}{120} + \cdots, \tag{28}$$

and substituting (28) in (22):

$$C_0 = \frac{I \cdot \delta}{4\pi}\left[\frac{1}{(kr)^2} + \frac{i}{(kr)} - \frac{1}{2} - \frac{i(kr)}{6} + \frac{(kr)^2}{24} - \frac{(ikr)^3}{120} + \cdots\right] sin(\theta)\ \vec{\imath}_{\varphi}, \tag{29}$$

and taking the most influential terms when kr tends to zero, in the near fields zone:

$$C_0 = \frac{I \cdot \delta}{4\pi}\left[\frac{1}{(kr)^2} + \frac{i}{(kr)} - \frac{1}{2}\right] sin(\theta)\ \vec{\imath}_{\varphi}, \tag{30}$$

substituting (30) in (20) we has reached the expression of the $\vec{H}$, solution of the equation (10):

$$\vec{H} = \frac{I \cdot \delta}{4\pi}k^2\left[\frac{1}{(kr)^2} + \frac{i}{kr} - \frac{1}{2}\right] e^{-ik\cdot r}\, sin(\theta)\ \vec{\imath}_{\varphi}. \tag{31}$$

See in (31), that the difference with (26) consists in the appearance of a new term within the bracket, this term will be necessary for said field to match the electric field that will be obtained in the next section.

4.3 Solution for electric field

The solution of (16) is of the form:

$$\vec{E} = C_0 e^{-ik\cdot r} + C_1 e^{ik\cdot r}, \tag{32}$$

applying Sommerfeld's condition:

$$\vec{E} = C_0 e^{-ik\cdot r}, \tag{33}$$

and evaluating the solution $(\mathbf{33})$ with the excitation function $\vec{\boldsymbol{E}}_{\boldsymbol{Exc}}$:

$$\vec{E}_{Exc} = C_0 e^{-ik\cdot r}, \tag{34}$$

we get

$$C_0 = \vec{E}_{Exc} e^{ik\cdot r}, \tag{35}$$

substituting $(\mathbf{31})$ in $(\mathbf{35})$, it results:

$$C_0 = -i\frac{I\cdot\delta}{4\pi\omega\varepsilon_0}k^3\left[\frac{e^{ikr}}{(kr)^3}\right](2\,cos(\theta)\,\vec{\imath}_r - sin(\theta)\;\vec{\imath}_\theta). \tag{36}$$

Here, $(\mathbf{36})$ has been conveniently arranged and multiplied up and down by $\boldsymbol{k^3}$. Expressing the complex exponential as a series of powers divided by $(\boldsymbol{kr})^{\mathbf{3}}$:

$$\frac{e^{ikr}}{(kr)^3} = \frac{1}{(kr)^3}\sum_{n=0}^{\infty}\frac{(ikr)^n}{n!} = \frac{1}{(kr)^3} + \frac{ikr}{(kr)^3} + \frac{(ikr)^2}{2!\,(kr)^3} + \frac{(ikr)^3}{3!\,(kr)^3} + \frac{(ikr)^4}{4!\,(kr)^3} + \cdots, \tag{37}$$

solving:

$$\frac{e^{ikr}}{(kr)^3} = \frac{1}{(kr)^3}\sum_{n=0}^{\infty}\frac{(ikr)^n}{n!} = \underbrace{\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)}}_{TERMS\ OF\ NEAR\ FIELD} - \frac{i}{6} - \frac{(ikr)^2}{120} + \cdots, \tag{38}$$

substituting what is inside the bracket of $(\mathbf{36})$ by $(\mathbf{38})$, we have:

$$C_0 = -i\frac{I\cdot\delta}{4\pi\omega\varepsilon_0}k^3\cdot \cdot\left\{2\left[\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)} - \frac{i}{6} + \frac{(kr)}{24} - \frac{(ikr)^2}{120} + \cdots\right]cos(\theta)\,\vec{\imath}_r - \left[\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)} - \frac{i}{6} + \frac{(kr)}{24} - \frac{(ikr)^2}{120} + \cdots\right]sin(\theta)\ \vec{\imath}_\theta\right\}. \tag{39}$$

In$(\mathbf{39})$, when the electrical distance $\boldsymbol{kr}$ is extremely small, linked fields zone to the radiator, only the first three terms of the series contribute, reducing $(\mathbf{39})$ to:

$$C_0 = -i\frac{I\cdot\delta}{4\pi\omega\varepsilon_0}k^3\left\{2\left[\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)}\right]cos(\theta)\,\vec{\imath}_r - \left[\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)}\right]sin(\theta)\ \vec{\imath}_\theta\right\}, \tag{40}$$

finally, substituting $(\mathbf{40})$ in $(\mathbf{34})$, the expression of the electric field intensity vector of the radiated wave is obtained, see that the relation $\boldsymbol{k/\omega\varepsilon_0}$ has been replaced by the intrinsic impedance of the fields in vacuum, $\boldsymbol{\eta_0} = \mathbf{377}\ \boldsymbol{\Omega}$, then:

$$\vec{E} = -i\frac{I\cdot\delta\cdot\eta_0}{4\pi}k^2\left\{2\left[\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)}\right]cos(\theta)\,\vec{\imath}_r - \left[\frac{1}{(kr)^3} + \frac{i}{(kr)^2} - \frac{1}{2(kr)}\right]sin(\theta)\ \vec{\imath}_\theta\right\}e^{-ik\cdot r}. \tag{41}$$

The electric field solution given by $(\mathbf{41})$ is paired with the solution of magnetic field $(\mathbf{31})$. This does not happen between $(\mathbf{41})$ and $(\mathbf{26})$, by the fact hidden terms exist. These terms are only observed through the procedure of expanding the complex exponentials in series. Now expression $(\mathbf{41})$ can be obtained by Maxwell's equations from $(\mathbf{31})$ and vice versa.

5 Discussion

Expression (41) is similar to that of the Hertz dipole, obtained by the traditional method of electrodynamic potentials and applied to a conduction current filament, on a segment of a metallic conductor much shorter than the wavelength that it radiates. However, (41) differs from the previous one, in that it has longitudinal component in the field of the far zone, terms associated with $1/(kr)$. This component is necessary to explain the phenomenon of vortex closure of the electric field to form EMW. On the other hand, both equations (19) and (41) correspond to the radiation emitted by the displacement current filament that exists in the little space between the arms of a dipole. For this reason, it can be considered that the existence of this space is a mandatory condition for the phenomenon of electromagnetic radiation to take place. This space is also known as aperture and it has to meet the condition $\delta \ll \lambda$. Another important point is that throughout the analysis, we have not consider the conduction current densities, J, since equation (1). But in a real system, such as a half-wave dipole, the conduction currents on the conductors must be considered, since they would have an impact on system parameters, such as the spatial characteristics of radiation, the input impedance and others.

6 Conclusions

A novel analytical method of direct obtaining a solution of Maxwell's equations for the phenomenon of electromagnetic radiation from an elemental source has been presented. This method offers a solution that is better adapted to the nature of the radiation. The procedure is simple and analytical, which could be the basis for the emergence of electromagnetic analysis methods, analytical and numerical, more precise and efficient than the current ones. It is interesting to note that the field associated with the Biot-Savart law, used as an excitatory function, dates from the year 1820, when the unified nature of the electromagnetic field had not yet been accepted. This came much later, with the theoretical contributions of James Clerk Maxwell, around the year 1848, and the birth as such, of the electromagnetic theory. On the other hand, the method presented here and the solutions obtained could have an impact on other fields of physics and technology. The Liénard-Wiechert theory of electrodynamic potentials, in the light of the evidences, will no longer be necessary, since from now on Maxwell's equations can be solved directly. The theory presented here could also be considered as the generalization of the Biot-Savart and Coulomb laws to the electrodynamic case, something that professor Oleg Jeffimenko tried to demonstrate in 1966 [7], but that its foundation contains various inconsistencies and conceptual errors, as already analyzed, by professor David Griffight et al, in [8]. In future research, the densities of the conduction currents should be taken into account in the Maxwell equations associated with any real radiation system. This will allow to study with greater theoretical rigor the mechanisms and physical principles that intervene in the phenomenon of the formation and radiation of electromagnetic waves.

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