Finding one's way through de Broglie's double solution program.

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RÉSUMÉ. *Dans les années '50,stimulé par les travaux de David Bohm, Louis de Broglie remet en doute l'interprétation orthodoxe de la théorie quantique. Il reformule alors son "programme de double solution" en supposant que la particule obéit à une équation d'onde non-linéaire. Dans cette approche, la particule quantique est semblable à un soliton, stabilisé par une non-linéarité de type "self-focusing". Loin du soliton, là où l'amplitude de l'onde tend vers zéro, la non-linéarité serait désactivée, et l'on retrouverait en bonne approximation l' équation d'onde linéaire de Schrödinger. de Broglie impose aussi la condition d'accord des phases entre le soliton et l'onde de Schrödinger, censée garantir le guidage de la particule par l'onde linéaire, en accord avec l'équation de guidance qu'il avait déjà postulée en 1926. Nous nous interrogeons ici sur le bien-fondé de ce programme, et sur la possibilité de satisfaire aux nombreuses contraintes nécessaires à sa réalisation.*

ABSTRACT. *In the 1950's, stimulated by the work of David Bohm, Louis de Broglie puts into question the orthodox interpretation of the quantum theory. He reformulates his "double solution program", assuming that the particle obeys a non-linear wave equation. In this approach the quantum particle behaves as a soliton, stabilized by a non-linearity of the self-focusing type. Far away from the soliton, in regions where the amplitude of the wave goes to zero, the non-linearity would be desactivated so that in good approximation the linear Schrödinger equation would be satisfied. de Broglie also imposes the condition of phase haromony between the soliton and the linear wave in order to guarantee the guidance of the particle by Schrödinger's wave, in agreement with the guidance equation already postulated by him in 1926. In the present paper we question the realizability and the self-consistency* *of this program, and also the possibility to satisfy the numerous constraints required by its fulfillment.*

Key words double solution, soliton, non-linear Schrödinger equation, wave monism, trajectories in configuration space.

Preamble: thirty years of quest, finding my way through de Broglie's double solution program.

At the beginning of the completion of my Ph.D, in 1992, I was invited to participate to the colloquium *La physique quantique -"Pour raison garder" (Centenaire de Louis de Broglie)* in Les Treilles[1], a magnificent domain in the Haut-Var, in the heart of Provence. There I met distinguished quantum physicists, among which Leslie Ballentine, Asim Barut, Anthony Leggett, Philip Pearle and Anton Zeilinger. This event definitively anchored the direction of my research towards the foundations of the quantum theory. It also constituted my first contact with the Fondation de Broglie, and the beginning of a long collaboration. In particular I met in Les Treilles Pierre Lochak, Simon Diner and Jacques Robert, who belonged to the Fondation. Our discussions incited me to read a book of Louis De Broglie entitled *Non-linear Wave Mechanics: A Causal Interpretation*, where he developed his double solution program [1]. To be fully frank, although I highly appreciated the clarity and originality of this book, I was not convinced by de Broglie's arguments concerning phase harmony and the derivation of the guidance equation, which pushed me in a first time to question specialists of these questions at the Fondation de Broglie. I received a large variety of advice and suggestions; for instance I was invited to consult numerous references among which books of Lichnerowicz, Hadamard and others. Unfortunately, I did not find a satisfying answer to my questions in this literature.

At that time, the group of theoretical physics at the Vrije Universiteit Brussels had not yet been "restructured" and it was divided in two subgroups, the first one (to which I belonged) dealing with foundations, the second one with non-linear dynamics and solitons. I also consulted by then my distinguished colleagues about the analogy between solitons and particles but the answer was rather negative, partly due to Derrick's theorem [2] (that will be analysed in depth in appendix), and partly due to the fact that many interesting properties of non-linear equations do

[1] https://lestreilles.hypotheses.org/501

not survive in presence of noise. The general advice that I received was that dealing with non-linear partial differential equations was like “jumping from a plane without parachute”.

Some years later I entered in contact with the work of Penrose and Diosi concerning the Schrödinger-Newton equation [3, 4, 5], in which a non-linear and self-focusing potential is present, due to an hypothetical self-gravitational interaction. Their work suggested that the collapse of the wave function had something to do with this non-linear potential, which is in direct correspondence with some ideas developed by de Broglie in the 60’s around his double solution program. In collaboration with Samuel Colin and Ralph Willox, in a first time [6, 7], and with Mohamed Hatifi more recently [8], we spent several years investigating these questions [9, 10], with the support of two Templeton projects and one FQXI project. We understood however that the Schrödinger-Newton suffers from a widespread property, shared by the overwhelming variety of non-linear potentials considered so far in the framework of the double solution program: it is not self-accelerating [8] and therefore, in virtue of Ehrenfest’s theorem, the dynamics of the soliton associated to solitonic solutions of the non-linear Schrödinger equation is classical (Newtonian). The guidance equation is thus fulfilled in this (classical) situation only when Bohm’s quantum potential is absent. Even in the case of free particles, when no external potential acts on the system, the quantum potential vanishes only when the solution is a plane wave, which considerably reduces the field of application of the double solution program.

It is only two years ago [11] that I found a formal solution of the double solution program that does not suffer from the previously mentioned drewbacks, solution that will be sketched in the present paper. It is not fully satisfying however, because it establishes from the beginning a difference of principle between the (non-linear) soliton and the (linear) pilot wave. This is in a sense a reappearance of the wave particle duality, which contradicts wave monism, a fundamental motivation for the double solution program.

After more than thirty years, here ends my quest about de Broglie’s double solution program, with a somewhat mitigated conclusion. Now, one could imagine that other ways remain open, as I will discuss in the last section of the paper, but the problem is not simple and it is even, maybe, ill-posed.

1 Introduction

Louis de Broglie proposed in 1926 [12] a realistic interpretation of the quantum theory in which particles are guided by the solution of the linear Schrödinger equation (Ψ_L). The theory was generalised by David Bohm in 1952 [13, 14]. Certain ingredients of de Broglie's original idea disappeared in Bohm's formulation, in particular the double solution program, according to which the particle is associated to a wave u distinct from the pilot-wave Ψ_L, u being sometimes treated as a moving singularity [15], and sometimes as a solution ϕ_{NL} of a non-linear equation of very high amplitude, a "hump" (see also refs. [1, 16]). In the present paper we will focus on the second alternative ("hump") that would be associated to a non-linear wave equation about which de Broglie wrote [1]

" ... a set of two coupled solutions of the wave equation: one, the Ψ wave, definite in phase, but, because of the continuous character of its amplitude, having only a statistical and subjective meaning; the other, the u wave of the same phase as the Ψ wave but with an amplitude having very large values around a point in space and which ($\cdots$) can be used to describe the particle objectively."...

We are thus looking for a solitonic solution of a non-linear self-focusing equation, represented here by ϕ_{NL}, which supposedly has a very small size, which is reminiscent of Bohm's description of particles as material points.

We consider that de Broglie's double solution program [1, 15] is very appealing for several reasons.

The first reason is that this program is wave monist and that wave monism potentially solves some paradoxes inherent to the wave particle duality [7].

It is also reminiscent of Poincaré's ideas about the electron [17] which he considered to consist of a field of forces, which pioneers the modern approach of particles physics which is based on fundamental interactions.

Another reason is that, if de Broglie's ideas are right, the quantum theory would be an emergent theory, as we will explain now. The pilot wave interpretation (also commonly called de Broglie-Bohm (dBB) interpretation or simply Bohm interpretation [13, 14]) which is the backbone of the double solution program postulates[2] that

[2]Mathematical details and precise definitions can be found in section 2.

-(i) particles follow trajectories which obey the guidance equation (or the quantum potential in Bohm's approach);

-(ii) the distribution of positions at a certain time t_0 obeys the Born rule

-(iii) each measurement is in the last resort a measurement of position.

As the guidance equation is derived from the equation of conservation associated to Schrödinger's equation, combining postulates (i) and (ii) ensures that the Born rule is satisfied at any time, at least when the observable that we consider is the position of the particle.

Taken together, (i), (ii) and (iii) ensure that the dBB interpretation leads to exactly the same predictions as the orthodox quantum theory. Some years ago, important results were obtained by Antony Valentini and coworkers [18, 19, 20, 21, 22, 23, 24, 25], who established that the Born rule is the consequence of (i): after a sufficiently long time, for any initial distribution of position, the chaotic nature of the dBB dynamics ensures that the distribution will converge to the distribution in $|\Psi|^2$, in accordance with the Born rule. This process is called the onset of quantum equilibrium [26].

The postulate (iii) is in a sense always true and unfalsifiable: whenever we print the result of a measurement on a piece of paper, or whenever it is displayed on the screen of a computer, and that we look at this result, our eyes will perform a measurement of position.

Taking account of the aforementioned results about the onset of quantum equilibrium [26], the dBB interpretation is thus a mere consequence of the guidance equation (postulate (i)). From this point of view, if the guidance equation could be derived from a well-chosen non-linear self-focusing equation in agreement with de Broglie's double solution program, one would be in right to consider that the quantum theory emerges from a non-linear wave equation. It is obviously worth establishing firmly such a result which explains why your servitor devoted, as explained in preamble, some of his best years to this quest. This program is severely constrained however and leaves not much room for credible candidates as we shall discuss throughout the present paper.

The paper is structured as follows:

-In section 2 we introduce preliminary concepts and basic ingredients of the double solution program (de Broglie-Bohm dynamics; equivariance, guidance equation and quantum potential).

-In section 3 we consider the velocities in the light of Ehrenfest's theorem, and the constraints to be fulfilled by every non-linear potential aimed at realizing the double solution program. We also introduce the factorization ansatz which plays a fundamental role in our analysis.

-In section 4 we pursue this analysis for what concerns accelerations.

-In section 5 we study in depth the implications of the factorization ansatz regarding the double solution program.

-In section 6, we present a non-linear potential which fulfills the constraints imposed by Ehrenfest's theorem.

-The last section (7) is devoted to open questions and conclusions.

Some technicalities related to the factorization ansatz, are treated in appendix (section 8) as well as Derrick's no-go theorem (section 9), and reference is made to Barut's program [29] (section 10) which also incorporates the factorisation ansatz.

2 Prerequisites: the (deterministic) de Broglie-Bohm dynamics; equivariance, guidance equation and quantum potential.

The dBB interpretation [27] is a dynamical and deterministic formulation of quantum mechanics in which it is assumed that the positions of the particle exist at all times, i.e. independently of the observer. We will consider, in what follows, a single spinless and non-relativistic particle for which a quantum wave function (also called the pilot wave) $\Psi_L(\mathbf{x},t)$ solves the (linear) Schrödinger equation:

$$i\hbar\frac{\partial\Psi_L(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_L(\mathbf{x},t) + V^L(\mathbf{x},t)\Psi_L(\mathbf{x},t), \tag{1}$$

where $V^L(\mathbf{x},t)$ is an external potential[3]. In the standard formulation of quantum mechanics, the probability distribution of all particle positions $P_{dB}(\mathbf{x},t)$ obeys the Born rule $P_{dB}(\mathbf{x},t) = |\Psi_L(\mathbf{x},t)|^2$. For convenience, let us express the wave function in polar form:

$$\Psi_L(\mathbf{x},t) = R_L(\mathbf{x},t)e^{i\,\varphi_L(\mathbf{x},t)}, \tag{2}$$

[3]We attributed the label L to this potential because it is assumed that it does not depend on Ψ. It acts thus linearly on the wave function, due to the fact that complex multiplication is distributive relative to addition and commutative.

where $R_L(\mathbf{x},t)$ and $\varphi_L(\mathbf{x},t)$ are two real functions. The probability distribution is then given by $P_{dB}(\mathbf{x},t) = R_L(\mathbf{x},t)^2$ and is conserved through the continuity equation:

$$\frac{\partial R_L(\mathbf{x},t)^2}{\partial t} + \boldsymbol{\nabla}\cdot\left(R_L(\mathbf{x},t)^2\frac{\hbar\boldsymbol{\nabla}\varphi_L}{m}\right) = 0, \tag{3}$$

By analogy with classical hydrodynamics, the phase-function $\varphi(\mathbf{x},t)$ is associated to a velocity field $\mathbf{v}(\mathbf{x},t)$ given by:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_L(\mathbf{x},t)\Big|_{\mathbf{x}=\mathbf{x}(t)}, \tag{4}$$

which is also called the **guidance equation** of de Broglie. This equation expresses how the pilot wave guides the trajectories of the particles. After integration of (4), the deterministic dB trajectory $\mathbf{x}(t)$ is obtained.

In order to mimick the distribution of positions in $(R_L)^2$ predicted in the standard interpretation, that is to say in order to mimick the Born rule, it suffices to impose that at a certain time t_0, $P_{dB} = (R_L)^2$ everywhere. Then, in virtue of equations (3) and (4), $P_{dB} = (R_L)^2$ everywhere at any time, which is also called the **equivariance** property.

As we already mentioned in the introduction, it can be shown that for a very large class of hamiltonians the distributions of positions will converge in time to the Born distribution $P_{dB} = (R_L)^2$, even when initially they depart from it (a process also called "quantum equilibrium" [26]). Ultimately the onset of the quantum equilibrium is due to the chaotic nature of the dBB dynamics in the vicinity of zeros of the pilot wave [7, 25].

Note that the guidance equation (4) is of the first order in time because it deals with velocities. David Bohm considered the accelerations associated to these velocities and showed that they derive from a non-classical potential, the so called quantum potential Q [13, 14], from now on denoted Q^L in order the emphasize the fact that is related to the pilot wave Ψ_L. As can indeed be shown by a lengthy but straightforward computation, combining equations (1), (2) and (4), implies that

$$m\frac{d^2\mathbf{x}(t)}{dt^2} = -\nabla(V^L(\mathbf{x},t) + Q^L(\mathbf{x},t)), \tag{5}$$

where

$$Q^L(\mathbf{x},t)) = \frac{-\hbar^2}{2m}\frac{\Delta R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)} \tag{6}$$

Remark: dB versus dBB dynamics.

As already noted, de Broglie's guidance equation is of the first order in time while Bohm's equation is of the second order. From now on we shall refer to this fine structure in the dynamics by labelling by the label dB "de Broglie" velocities and dynamics as encapsulated in the guidance equation (4), while we shall associate the label dBB to the acceleration as encapsulated in the generalised Newton equation (5). The label dBB will also refer to the "pilot wave interpretation" outlined in the previous paragraphs, in which no particular assumption is made about the structure of the particles, that we can treat FAPP as material points.

In the rest of the paper, we shall take for granted the result according to which equilibrium is reached after a sufficiently long time ONLY when initial velocities obey equation (4). On the contrary, when they are distributed arbitrarily, while accelerations obey equation (5), quantum equilibrium does not occur and is even unstable [28].

3 Double solution program, Ehrenfest's theorem and velocities

Let us consider a non-linear Schrödinger equation of the type

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = -\hbar^2\frac{\Delta\Psi(\mathbf{x},t)}{2m} + V^L(\mathbf{x},t)\Psi(\mathbf{x},t) + V^{NL}(\Psi)\Psi(\mathbf{x},t), \tag{7}$$

where V^L represents an arbitrary linear potential, of the type commonly considered when solving the linear Schrödinger equation (for instance an electro-magnetic potential) while V^{NL} represents a non-linear self-focusing potential which supposedly concentrates the wave function of the particle over a tiny region of space, in accordance with de Broglie's double solution program.

3.1 Ehrenfest's theorem.

Ehrenfest's theorem establishes that the time derivative of the average value of an observable $\hat{O}$ obeys

$\frac{d<\hat{O}>}{dt} =< \frac{\partial\hat{O}}{\partial t} > + \frac{1}{i\hbar} < [H,\hat{O}] >,$

where
$< .... >= \int \mathbf{d^3x}\Psi(\mathbf{x},t)^*....\Psi(\mathbf{x},t)$,
and
$H\Psi(\mathbf{x},t) = -\hbar^2\frac{\Delta\Psi(\mathbf{x},t)}{2m} + V^L(\mathbf{x},t)\Psi(\mathbf{x},t) + V^{NL}(\Psi)\Psi(\mathbf{x},t)$.

In particular, the time derivative of the position of the barycentre of a solitonic solution $\Psi(\mathbf{x},t)$ obeys $\frac{d<\hat{\mathbf{x}}(t)>}{dt} =< \hat{\mathbf{p}} > /m$ where $\hat{\mathbf{x}}$ represents the position operator, multiplicative in position representation and $\hat{\mathbf{p}}$ the momentum operator which is equal, in position representation to $\frac{\hbar}{i}\nabla$. Note that in order to apply Eherenfest's theorem we made use of the fact that the non-linear potential is assumed here to be a multiplicative, real valued, potential.

In order to fulfill de Broglie's guidance equation (4), we must impose that

$$\frac{d < \hat{\mathbf{x}}(t) >}{dt} =< \hat{\mathbf{p}} > /m = \mathbf{v}_{dB}(< \mathbf{x} >, t) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_L(\mathbf{x},t)\Big|_{\mathbf{x}=<\mathbf{x}>(t)}, \quad (8)$$

where $\Psi_L(\mathbf{x},t) = R_L(\mathbf{x},t)e^{i\,\varphi_L(\mathbf{x},t)}$ is solution of the linear Schrödinger equation (1) and $< \hat{\mathbf{x}}(t) >$ is the barycentre of the particle[4] ($< \hat{\mathbf{x}}(t) >= \int \mathbf{d^3x}|\Psi(\mathbf{x},t)|^2 \cdot \hat{\mathbf{x}}(t)$).

3.2 Factorisation ansatz and generalized guidance equation.

Now, the soliton is never located in places where the amplitude of the pilot wave Ψ_L (Ψ_L is solution of the linear Schrödinger equation (1)) vanishes; therefore, without loss of generality, we can impose that Ψ is the product of Ψ_L with a peaked function Φ_{NL} associated to the soliton (this is the so called **factorization ansatz** studied by us in several papers in the past and originally proposed by Asim Barut [29] in a slightly different context, as explained in appendix, section 10).

For convenience, let us express these wave functions in polar form:

$$\Psi_L(\mathbf{x},t) = R_L(\mathbf{x},t)e^{i\,\varphi_L(\mathbf{x},t)}, \quad (9)$$

[4]Remark that the bra-ket notation introduced here should not necessarily be interpreted as a quantum statistical average in the usual sense; it rather indicates an average quantity in regard of the weight (density of stuff) $|\,\phi_{NL}(t,\mathbf{x})\,|^2$. In the same order of ideas, the $\mathcal{L}_2$ norm of Ψ, $\int \mathbf{d^3x}|\Psi(\mathbf{x},t)|^2$ is assumed here to be normalized to unity, by convenience, in order not to overload the equations, but this choice is not imperative. In any case, this $\mathcal{L}_2$ norm is constant throughout time because the dynamics is unitary.

$$\Phi_{NL}(\mathbf{x},t) = R_{NL}(\mathbf{x},t)e^{i\,\varphi_{NL}(\mathbf{x},t)}, \tag{10}$$

$$\Psi(\mathbf{x},t) = R(\mathbf{x},t)e^{i\,\varphi(\mathbf{x},t)}, \tag{11}$$

then, according to the factorisation ansatz

$$\Psi(\mathbf{x},t) = \Psi_L(\mathbf{x},t)\cdot\Phi_{NL}(\mathbf{x},t), \tag{12}$$

so that $R = R_L \cdot R_{NL}$, and $\varphi = \varphi_L + \varphi_{NL}$.

Let us estimate the velocity of the barycentre of this solitonic solution:

$$< \hat{\mathbf{p}} > /m = \int \mathbf{d^3x}\Psi^*(\mathbf{x},t)\frac{\hbar}{im}\nabla\Psi(\mathbf{x},t) = \frac{\hbar}{m}\int \mathbf{d^3x}R^2(\mathbf{x},t)\nabla\varphi(\mathbf{x},t) \tag{13}$$

We shall assume from now on that R and R_{NL}, the amplitudes of the wave functions Ψ and Φ_{NL} are peaked around their barycentre, and also that the amplitude as well as the phase of the pilot wave Ψ_L remain constant in good approximation in the region where the weight of Ψ (Φ_{NL}) is concentrated.

Henceforth, in good approximation,

$$< \hat{\mathbf{p}} > /m = \frac{\hbar}{m}(\nabla\varphi^L(< \hat{\mathbf{x}}(t) >) + \int \mathbf{d^3x}R^2(\mathbf{x},t)\nabla\varphi_{NL}(\mathbf{x},t)), \tag{14}$$

which constitutes a generalisation of the guidance equation (4), due to the presence of a new contribution taking account of the phase of the soliton.

Denoting $\mathbf{v}_{dB}$ the de Broglie-bohm velocity $\frac{\hbar}{m}(\nabla\varphi^L(< \hat{\mathbf{x}}(t) >)$, $\mathbf{v}_{int.}$ the solitonic contribution to the velocity of the barycentre of the wave function, $\frac{\hbar}{m}(\mathbf{d^3x}R^2(\mathbf{x},t)\nabla\varphi_{NL}(\mathbf{x},t)(\mathbf{x},t))$, and their sum $\mathbf{v}_{drift}$, we get

$$\mathbf{v}_{drift} = \mathbf{v}_{dB}(\mathbf{x}_0(t)) + \mathbf{v}_{int.}, \tag{15}$$

where $\mathbf{x}_0(t) = \mathbf{x}_0(t=0) + \int_0^t dt\mathbf{v}_{drift}$.

This is a generalization of de Broglie's guidance equation with an extra-contribution due to the internal degrees of freedom of the soliton.

As we have discussed elsewhere [8], the solitonic contribution to the velocity of the barycentre of the wave function, $\mathbf{v}_{int.}$ is most often NOT negligible. Henceforth, for the overwhelming majority of non-linearities considered so far in order to tackle de Broglie's double solution program, the solitons DO NOT obey the guidance equation (4). Their dynamics is actually classical due to the absence of an ad hoc self-acceleration as we shall show in the next section.

4 Double solution program, Ehrenfest's theorem and the problem of (self)acceleration.

4.1 Logarithmic self-interaction potential.

Other authors in the past considered seriously the possibility of a non-linear generalisation of Schrödinger's equation, among which Bialynicki-Birula and Mycielski who studied in depth the case of a logarithmic non-linearity [30]. They showed that the gaussian solitons associated to this non-linear self-focusing potential ("gaussons") do not obey the guidance equation; instead their dynamics is classical. Here is a sketch of the proof of this result in a special case[5]. For convenience, we shall assume that the problem is formulated in one dimension of space (passing to three dimensions does not bring any fundamental novelty here). We shall also assume that the potential linear in Ψ is harmonic ($V^L = kx^2/2$) and that the nonlinear potential is logarithmic in $|\Psi|$ ($V^{NL} = -\kappa \cdot ln(|\Psi|)$ with κ a real number taken to be positive in order to ensure self-focusing). Then, equation (7) admits gaussian solutions (gaussons) of the type

$\Psi(x,t) = exp^{-(Ax^2+Bx+C)}$,

with A, B and C complex functions of time; to get localized gaussons requires $Re.A > 0$.

The reason therefore is that for such solutions the logarithmic potential $-\kappa ln(|\Psi|)$ is quadratic in x: $V^{NL}(x,t) = \kappa(Re.(A(t))\cdot x^2 + Re.(B(t))\cdot x + Re.(C(t)))$. This self-interaction is also "comoving" in the sense that it can be written in the form $\kappa(Re.(A(t)) \cdot (x - <x>)^2 + \kappa(Re.(\tilde{C}(t))$ with $<x>$ representing the barycentre of Ψ at time t:

$<x> = -Re.(B(t))/(2Re.(A(t)))$,

[5] Bialynicki-Birula and Mycielski derived this result in a very general case in their paper [30], making use of Ehrenfest's theorem as we shall do later. Accordingly, they wrote the following: *...The most elaborate program to create an intrinsically nonlinear wave mechanics has been developed by de Broglie and his collaborators(...). This was a very ambitious program aimed at creating a causal underlying nonlinear structure basically different from the linear theory. The linear theory was to describe only the statistical behavior of the new structure. No specific nonlinear equation, however, has emerged from those investigations...*The last sentence was underlined by us.

and $Re.(\tilde{C}(t)) = Re.(C(t)) - <x>^2$.

Let us now pursue with Ehrenfest's theorem in order to estimate the acceleration of the soliton.

$$\frac{d<\hat{\mathbf{p}}(t)>}{dt} = <-\nabla(V^L(\mathbf{x},t)+V^{NL}(\mathbf{x},t)> = \mathbf{a}_{classical}(\mathbf{x},t)+\mathbf{a}_{self}(\mathbf{x},t), \tag{16}$$

where $\mathbf{a}_{classical}(\mathbf{x},t)$ represents the classical acceleration due to the presence of the linear potential (here $\mathbf{a}_{classical}(\mathbf{x},t) = -k < \mathbf{x} >$), and a self-acceleration which is due to the presence of the non-linear, self-focusing, potential.

This self-acceleration is equal to 0 at all times however because it is equal to $< -\kappa(\mathbf{x}- < \mathbf{x} >) >$.

If the guidance equation was satisfied, we should impose, in accordance with equations (5,6) that

$$\frac{d<\hat{\mathbf{p}}(t)>}{dt} = <-\nabla(V^L(\mathbf{x},t)+Q^L(\mathbf{x},t))> = \mathbf{a}_{classical}(\mathbf{x},t)+\mathbf{a}_{dB}(\mathbf{x},t), \tag{17}$$

which imposes, taking account of (16), that the self-acceleration and the dB acceleration are equal, in good approximation[6]:

$< -\nabla Q^L(<\mathbf{x}>,t) >= \mathbf{a}_{dB}(<\mathbf{x}>,t)$

$= \mathbf{a}_{self}(\mathbf{x},t) =< -\nabla V^{NL}(\mathbf{x},t) >$

It is easy to check that the gradient of the quantum potential (6) in the case of a gaussian wave function is equal to 0 only at the center of the gaussian packet ($x =< x >$). In all other locations, the guidance equation is thus not satisfied because if it was the case this would imply that $\mathbf{a}_{self}(\mathbf{x},t) = \mathbf{a}_{dB}(<\mathbf{x}>,t) \neq 0$ which contradicts the fact that everywhere $\mathbf{a}_{self}(\mathbf{x},t) = - < \nabla V^{NL}(\mathbf{x},t) >= 0$.

In ref. [8], we have shown through accurate, semi-analytic simulations that the solitonic contribution to the velocity of the barycentre of the wave function "conspires" to erase the influence of the quantum potential at the level of the dynamics. Expressed in terms of the drift velocity, the dB velocity and the solitonic contribution to the velocity (15), this means that $\mathbf{v}_{drift} = \mathbf{v}_{dB}(\mathbf{x}_0(t)) + \mathbf{v}_{int.} = \mathbf{v}_{classical}$ as illustrated by the figure 1, reproduced from ref. [10].

[6]When the soliton is peaked enough, we are free to neglect quantum fluctuations: assuming that a function $f(\mathbf{x})$ varies slowly over the extent of the soliton, we get $< f(x) >= f(< x >)$, in good approximation.

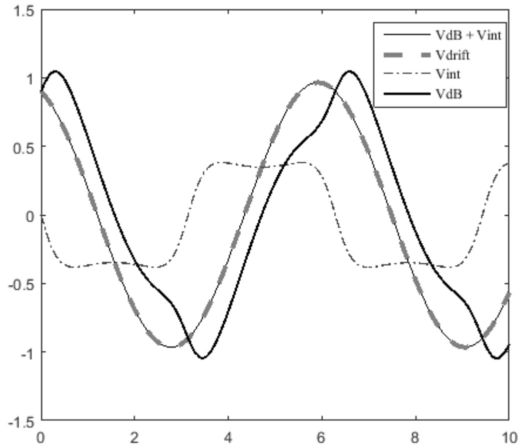

Figure 1: Plot of $v_{dB}, v_{int.}, v_{dB} + v_{int.}$ and $v_{drift} = \frac{d}{dt}(x_0)$ in function of time, obtained from semi-analytic gaussian solutions (gaussons) of the 1D equation (7) in presence of a harmonic linear potential and self-gravitation of an homogeneous spherical mass distribution (equivalent to a logarithmic non-linearity in the case of coherent gaussons and when the size of the gausson is quite smaller than the radius of the sphere); space and time were rescaled and are of the order of unity. The drift velocity is obvioulsy classical here (it is an harmonic function of time).

In other words, in order to realize the double solution program we must find a non-linear potential such that everywhere $< V^{NL}(\mathbf{x}, t) >= Q^L(\mathbf{x}, t)$ up to a constant. This condition is not fulfilled by gaussons self-focused by the logarithmic potential because this potential is not self-accelerating.

Actually, it is easy to estimate, by direct computation, even when the solitonic solution is not a gausson, the self-acceleration of the logarithmic potential:

$$< -\nabla V^{NL}(\mathbf{x}, t) >= - \int \mathbf{d^3x} R^2(\mathbf{x}, t) \nabla V^{NL}(\mathbf{x}, t)$$
$$= -\kappa \int \mathbf{d^3x} R^2(\mathbf{x}, t) \frac{\nabla R(\mathbf{x}, t)}{R(\mathbf{x}, t)} = -\kappa \int \mathbf{d^3x} R(\mathbf{x}, t) \nabla R(\mathbf{x}, t)$$
$$= -\frac{\kappa}{2} \int \mathbf{d^3x} \nabla R^2(\mathbf{x}, t).$$

Now, $R^2(\mathbf{x},t)$ obviously vanishes at infinity, because we deal here with localized, self-focused solutions of equation (7), which implies that the self-acceleration of the logarithmic potential is equal to zero always and everywhere, as originally noted in ref.[30].

4.2 Generalized Schrödinger-Newton self-interaction potential.

In a previous paper [6], we studied the gravitational self-interaction, in the case of a homogeneous rigid sphere of mass M and radius R. We showed that when the extent of the wave function is quite smaller than the size of the sphere, the gravitational self-interaction can be expressed through the non-linear potential $V^{NL}(\mathbf{x},t) = \int \mathbf{d^3x'}|\Psi(\mathbf{x'},t)|^2 \cdot \frac{G.M^2}{2R^3} \cdot ||\mathbf{x}-\mathbf{x'}||^2$ where $k = G.M^2/R^3$, with G Newton's gravitational constant, and $||...||$ the 3D-euclidean norm[7].

Applying Ehrenfest's theorem to this interaction we find that

$< \nabla V^{NL}(\mathbf{x},t) >= \frac{G.M^2}{R^3} \cdot < \mathbf{x}- < \mathbf{x} >>= 0$ so that the self-gravitational interaction is also non-accelerating in the regime where the extent of the wave function is quite smaller than the size of the sphere. This result is in agreement with the plot of figure 1 which shows that the acceleration of the barycentre of the gausson is classical (harmonic). We generalized this result: every self-interaction of the type $V^{NL}(\mathbf{x},t) = \int \mathbf{d^3x'}|\Psi(\mathbf{x'},t)|^2 \cdot f(||\mathbf{x}-\mathbf{x'}||^2)$ is non-accelerating whichever choice we could make for the kernel function f. We can indeed prove by direct computation that, in virtue of Ehrenfest's theorem, the self-acceleration is then equal to

$$\frac{-1}{m}\int \mathbf{d^3x}\nabla V^{NL}(\mathbf{x},t)|\Psi(\mathbf{x},t)|^2$$
$$=\frac{-1}{m}\int \mathbf{d^3x}\int \mathbf{d^3x'}|\Psi(\mathbf{x},t)|^2|\Psi(\mathbf{x'},t)|^2 \cdot \nabla f(||\mathbf{x}-\mathbf{x'}||^2)$$
$$=\frac{-1}{m}\int \mathbf{d^3x}\int \mathbf{d^3x'}|\Psi(\mathbf{x},t)|^2|\Psi(\mathbf{x'},t)|^2 \cdot 2(\mathbf{x}-\mathbf{x'})\frac{df}{du}|_{u=||\mathbf{x}-\mathbf{x'}||^2},$$

where $||...||$ represents the euclidean norm in R^3.

Now, $|\Psi(\mathbf{x},t)|^2|\Psi(\mathbf{x'},t)|^2\frac{df}{du}|_{u=||\mathbf{x}-\mathbf{x'}||^2}$ is an even function of $(x-x')^2$, $(y-y')^2$ and $(z-z')^2$, while the cartesian components of the vector $\mathbf{x}-\mathbf{x'}$ are odd functions of $(x-x')$, $(y-y')$ and $(z-z')$. As we integrate over a symmetric domain $\mathbf{d^3xd^3x'}$, the self-acceleration is always equal to 0.

[7] Retrospectively, we realized that, when the external, linear potential is the harmonic potential quadratic in the position, this interaction is the same as the logarithmic self-interaction ($V^{NL} = -\kappa ln(|\Psi|)$), in the case of coherent gaussons for which $Re.A$ does not vary with time, provided we impose that $\kappa Re.A = \frac{GM^2}{2R^3}$.

In particular, when the kernel is equal to $f(u) = -GM^2/u$ with, as here above, G Newton's gravitational constant and M the mass of an elementary, structureless, particle, we deal with the single particle Schrödinger-Newton dynamics. This result concerning the absence of self-acceleration is also valid if we deal with the gravitational self-interaction of any rigid body provided its mass ditribution is isotropic. If we consider the limit where $f(u)$ is proportional to a Dirac delta function of u, our analysis also applies to the so called NLS interaction ($V_{NL}(\mathbf{x}, t)$ proportional to $|\Psi(\mathbf{x}, t)|^2$ that was studied in the past by D.Fargue in the 1D case [31, 32].

4.3 Self-interaction potential analytic in $|\Psi(\mathbf{x}, t)|$.

Let us now consider that the self-interaction is an analytic function in the variable $|\Psi(\mathbf{x}, t)|$: $V^{NL}(\mathbf{x}, t) = \sum_{m=0,1,2...}^{\infty} v_m|\Psi(\mathbf{x}, t)|^m$, where $\sum_{m=0,1,2...}^{\infty} v_m u^m$ is supposedly the Taylor development of a real, analytic function of $u \in C$.

Then, if we apply Ehrenfest's theorem in the same way that we did in the previous sections, we find that the self-acceleration of the barycentre of the wave function obeys

$$\frac{d^2<\mathbf{x}>}{dt^2} = \frac{-1}{m}\int \mathbf{d^3x}(\nabla V^{NL}(\mathbf{x}, t))|\Psi(\mathbf{x}, t)|^2 =$$
$$\frac{-1}{m}\int \mathbf{d^3x}(\nabla(\sum_{n=0,1,2...}^{\infty} v_n|\Psi(\mathbf{x}, t)|^n))|\Psi(\mathbf{x}, t)|^2 =$$
$$\frac{-1}{m}\int \mathbf{d^3x}(\sum_{n=0,1,2...}^{\infty} n \cdot v_n|\Psi(\mathbf{x}, t)|^{n+1})\nabla\Psi(\mathbf{x}, t) =$$
$$\frac{-1}{m}\int \mathbf{d^3x}\nabla(\sum_{n=0,1,2...}^{\infty} \frac{(n)}{(n+2)} v_n|\Psi(\mathbf{x}, t)|^{n+2}),$$

which is equal to zero because the wave function is supposedly localized and is thus equal to zero at infinity.

4.3.1 Combining Noether's theorem and Ehrenfest's theorem into a no-go theorem.

As is well-known, fundamental symmetries are related to conservation laws, in virtue of Noether's theorem. In particular it is generally so that if the action is invariant under translations in space, the total momentum of the system is a concerved quantity. This property explains why the potentials considered so far are not self-accelerating. To show this, let us reconsider the non-linear equation (7); it can be derived from a variational principle for the action

$$A_{NL}(\Psi) = \int_{-\infty}^{+\infty} dt \int \mathbf{d^3x} \frac{i}{2}(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}) - \frac{\hbar^2}{2m}|\nabla \Psi|^2 + \Psi^*(V^L + V^{NL})\Psi.$$

The contribution of the non-linear potential to the Lagrangian is thus equal to $\int \mathbf{d^3x}\Psi^*(\mathbf{x},t)V^{NL}(\mathbf{x},t)\Psi(\mathbf{x},t)$. Now, one can check that for potentials considered so far, this contribution is invariant if we replace $\Psi(\mathbf{x},t)$ by $\Psi(\mathbf{x}+\delta\mathbf{x},t)$:

$\int \mathbf{d^3x} \int \mathbf{d^3x'} |\Psi(\mathbf{x},t)|^2 |\Psi(\mathbf{x'},t)|^2 \cdot f(||\mathbf{x}-\mathbf{x'}||^2)$

$= \int \mathbf{d^3x} \int \mathbf{d^3x'} |\Psi(\mathbf{x}+\delta\mathbf{x},t)|^2 |\Psi(\mathbf{x'}+\delta\mathbf{x},t)|^2 \cdot f(||\mathbf{x}+\delta\mathbf{x} - (\mathbf{x'}+\delta\mathbf{x})||^2)$

and

$\int \mathbf{d^3x} \sum_{n=0,1,2...}^{\infty} v_n |\Psi(\mathbf{x},t)|^n = \int \mathbf{d^3x} \sum_{n=0,1,2...}^{\infty} v_n |\Psi(\mathbf{x}+\delta\mathbf{x},t)|^n$

The absence of self-acceleration is thus a consequence of the invariance of the self-interaction under a homogeneous translation in the 3D space. From this point of view, our analysis constitutes a no-go theorem: if the self-interaction is invariant under global translations in space, no self-acceleration is there to mimick the influence of the quantum potential and we will fail to realize de Broglie's double solution program.

Another no-go theorem was derived in the past by Derrick [2], in which space dilations play a central role. This theorem is however flawed to a large extent as we will show in appendix (section 9), and is in particular not relevant in the context of the double solution program, unless we consider non-unitary evolutions, which is clearly out of context here. Most of all, even if, as shown in appendix, this theorem, conveniently reformulated, tells us something about the stability of the solitonic solutions of the non-linear Schrödinger equation, it remains mute concerning the question of self-acceleration which is a key issue regarding the double solution program. Its relevance remains therefore very limited in the present context.

5 Factorisation ansatz and guidance equation.

5.1 Factorisation ansatz.

In addition to the non-linear evolution equation (7), let us also impose the factorization ansatz (12):

$$\Psi(\mathbf{x},t) = \Psi_L(\mathbf{x},t) \cdot \phi_{NL}(\mathbf{x},t),$$

where Ψ_L, the pilot wave, is a solution of the linear Schrödinger equation (1) while $\phi_{NL}(\mathbf{x},t)$ is supposed to be localized over a very small region of space. Our aim is to represent the particle by $\phi_{NL}(\mathbf{x},t)$, a soliton guided by the pilot wave according to de Broglie's guidance equation (4):

$$\mathbf{v} = \mathbf{v}_{dB} \equiv \frac{\hbar}{m}\frac{Im.(\Psi_L(t,\mathbf{x})^*\nabla\Psi_L(t,\mathbf{x}))}{\mid \Psi_L(t,\mathbf{x})\mid^2}, \tag{18}$$

where $\mathbf{v}$ represents the velocity of the (barycentre of) the soliton. Combining equations (1,7,8), expressing $\Psi_L(\mathbf{x},t)$ in function of its modulus and its phase through $R_L(\mathbf{x},t)e^{i\varphi_L(\mathbf{x},t)}$, and also making use of the identity $\bigtriangledown\Psi_L(\mathbf{x},t) = (\bigtriangledown R_L(\mathbf{x},t))e^{i\varphi_L(\mathbf{x},t)} + \Psi_L(\mathbf{x},t)i\bigtriangledown\varphi_L(\mathbf{x},t)$, it is straightforward to show that ϕ_{NL} obeys the non-linear equation

$$\begin{aligned} &i\hbar\cdot\frac{\partial\phi_{NL}(\mathbf{x},t)}{\partial t} = \\ &-\frac{\hbar^2}{2m}\cdot\Delta\phi_{NL}(\mathbf{x},t) - \frac{\hbar^2}{m}\cdot(i\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown\phi_{NL}(\mathbf{x},t) \\ &+\frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\bigtriangledown\phi_{NL}(\mathbf{x},t)) + V^{NL}(\Psi)\phi_{NL}(\mathbf{x},t). \end{aligned} \tag{19}$$

By doing so we replace thus equation (7) by a system of three equations (1,8,19). This replacement is one to one and can be done without loss of generality whenever $\mathbf{x}$ is not a node of the pilot-wave $\Psi_L(\mathbf{x},t)$ which happens "nearly everywhere".

5.2 Guidance equation.

It is worth noting that, to the difference with the equations (1,7) which are unitary, the $\mathcal{L}_2$ norm of the non-linear wave ϕ_{NL} is not preserved with time, because the terms mixing Ψ_L and ϕ_{NL} in (19) are not hermitian. The change of norm of ϕ_{NL} can be shown [10], to obey

$$\begin{aligned} &\frac{d<\phi_{NL}\mid\phi_{NL}>}{dt} \approx \frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot<\phi_{NL}\mid\phi_{NL}> \\ &-2\frac{\bigtriangledown R_L(t,\mathbf{x_0})}{R_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(\mathbf{x},t), \end{aligned} \tag{20}$$

where we define the (position of the) barycentre $\mathbf{x_0}$ of the soliton as follows:

$\mathbf{x_0} \equiv \frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$.

Let us define the velocity $\mathbf{v}_{drift}$ of the barycentre $\mathbf{x_0}$ as follows:

$$\mathbf{v}_{drift} \equiv \frac{d(\frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>})}{dt} \tag{21}$$

Then, as shown in reference ..., in the limit where the soliton is peaked enough around its barycentre, $\mathbf{v}_{drift}$ obeys

$$\begin{aligned}\mathbf{v}_{drift} &= \frac{\hbar}{m}\nabla\varphi_L(t,\mathbf{x_0}(t)) + \frac{<\phi_{NL} \mid \frac{\hbar}{im}\nabla \mid \phi_{NL}>}{<\phi_{NL} \mid \phi_{NL}>} \\ &= \mathbf{v}_{dB} + \mathbf{v}_{int.},\end{aligned} \tag{22}$$

where $\mathbf{x}_0(t) = \mathbf{x}_0(t=0) + \int_0^t dt\mathbf{v}_{drift}$.

The drift velocity contains the well-known Madelung-de Broglie-Bohm contribution ($\mathbf{v}_{dB} = \frac{\hbar}{m}\nabla\varphi_L(t,\mathbf{x_0}(t))$) plus a new contribution due to the internal structure of the soliton ($\mathbf{v}_{int.} = \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$).

As the weight of Ψ is essentially concentrated at the level of the soliton, we also find that $\mathbf{v}_{int.}$ is nothing else than the solitonic contribution to the velocity of the barycentre of the wave function that we discussed previously (15):

$\frac{1}{m}(\mathbf{d^3x}R^2(\mathbf{x},t)\nabla S^{NL}(\mathbf{x},t)(\mathbf{x},t))=\mathbf{v}_{int.} = \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$

In reference [10], we also established the following result:

-in the limit where the soliton is peaked enough around its barycentre, we get:

$$\frac{<\phi_{NL} \mid \phi_{NL}>(t)}{<\phi_{NL} \mid \phi_{NL}>(t=0)} = \frac{R_L^2(\mathbf{x_0},t=0)}{R_L^2(t,\mathbf{x_0})}. \tag{23}$$

Keeping in mind that all aforementioned results were derived in the limit where the width of the peaked soliton is quite smaller than the typical scales of variation of $R_L(\mathbf{x},t))$ and $\varphi_L(\mathbf{x},t)$ over space, the previous results imply that the full wave function Ψ solution of (19) has the form

$$\Psi(x,y,z,t) \approx \phi'_{NL}(t,\mathbf{x})e^{i\varphi_L(t,\mathbf{x})}, \tag{24}$$

where $\phi'_{NL}(\mathbf{x},t)$ is centered in $\mathbf{x}_0(t=0) + \int_0^t dt\mathbf{v}_{drift}$ and is of constant $\mathcal{L}_2$ norm (throughout the present paper we chose to normalize Ψ and ϕ' to unity).

Details of all these computations are reproduced in appendix.

6 A (formal) realization of the double solution program.

6.1 Realizing the double solution program with the factorisation ansatz and a purely real solitonic wave function.

These results strongly suggests the possible existence of a purely real solitonic solution of equation (19) such that $\mathbf{v}_{int.} = 0$ in which case the guidance equation of de Broglie (18) is satisfied: $\mathbf{v}_{drift} = \frac{\hbar}{m}\bigtriangledown\varphi_L(t, \mathbf{x_0}(t))$. In the present section, we shall identify a well-chosen non-linearity aimed at guaranteeing that the there exist purely real solitonic solution of equation (19). This result was already derived in ref. [11] where we showed that, in presence of this well-chosen non-linearity and when the pilot wave is a gaussian coherent state of a 3D harmonic oscillator, (the barycentres of) exact analytic, solitonic solutions of the equation (7) obey dB dynamics. Roughly summarized, the reasoning which brought us to identify the ad hoc non-linearity goes as follows: let us impose that at all times and everywhere $\varphi_{NL} = 0 = Im.(\phi_{NL})$. Making use of equation (19), we find

$$-\frac{\partial Im.(\phi_{NL}(\mathbf{x},t))}{\partial t} = \frac{1}{\hbar}Re.(i\hbar\cdot\frac{\partial\phi_{NL}(\mathbf{x},t)}{\partial t}) =$$
$$\frac{1}{\hbar}Re.(-\frac{\hbar^2}{2m}\cdot\Delta\phi_{NL}(\mathbf{x},t) - \frac{\hbar^2}{m}\cdot(i\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown\phi_{NL}(\mathbf{x},t)$$
$$+\frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\bigtriangledown\phi_{NL}(\mathbf{x},t)) + V^{NL}(\Psi)\phi_{NL}(\mathbf{x},t)) = 0. \tag{25}$$

Making use of the fact that ϕ_{NL} and V^{NL} are supposedly purely real function, $\phi_{NL}(\mathbf{x},t) = R_{NL}(\mathbf{x},t)$ and the non-linear potential is fixed unambisguously by the constraint (25):

$$V^{NL}(\Psi)\phi_{NL}(\mathbf{x},t) = \frac{\hbar^2}{2m}\cdot(\frac{\Delta R_{NL}(\mathbf{x},t)}{R_{NL}(\mathbf{x},t)} + 2\frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\frac{\bigtriangledown R_{NL}(\mathbf{x},t)}{R_{NL}(\mathbf{x},t)}) \tag{26}$$

Then, equation (19) reduces to a system of two equations

$$Im.(\phi^{NL}) = 0 \tag{27}$$

$$\frac{\partial Re.(\phi^{NL})}{\partial t} = \frac{\partial R_{NL}(\mathbf{x},t)}{\partial t} = \frac{\hbar}{m}\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown R_{NL}(\mathbf{x},t) \tag{28}$$

Assuming again that the width of the peaked soliton is quite smaller than the typical scales of variation of $\varphi_L(\mathbf{x},t))$, equation (28) is nothing else than the guidance equation. Indeed, imposing that $\bigtriangledown\varphi_L(\mathbf{x},t) = \bigtriangledown\varphi_L(\mathbf{x}_0(t))$, with $\mathbf{x}_0(t) = \mathbf{x}_0(t=0) + \int_0^t dt\mathbf{v}_{dB}(t)$ and $\mathbf{v}_{dB}(t) = \frac{\hbar}{m}\bigtriangledown\varphi_L(\mathbf{x}_0(t))$, it is straightforward to check that the solution of (28) is a solitary wave (we call in the present context a solitary wave a wave that keeps the same shape at all times, what is sometimes called a soliton in other contexts):

$R_{NL}(\mathbf{x},t) = R_{NL}(\mathbf{x} - (\mathbf{x}_0(t) - \mathbf{x}_0(t=0))), t=0)$.

Remarkably, there is no constraint at all concerning the shape of the soliton $R_{NL}(\mathbf{x},t=0)$.

Generalising equation (6), let us define the quantum potential of the Ψ wave as follows:

$$Q(\mathbf{x},t)) = \frac{-\hbar^2}{2m}\frac{\Delta R(\mathbf{x},t)}{R(\mathbf{x},t)} \tag{29}$$

The non-linearity $V^{NL}(\Psi)$ (26) can as well be expressed in terms of the quantum potential of the pilot wave (6) and of the quantum potential of the Ψ wave:

$$V^{NL}(\Psi) = Q^L - Q. \tag{30}$$

Intuitively, the quantum potential associated to the pilot wave, Q^L, provides the self-acceleration required in order to obey the dB dynamics. It is not self-focusing however. Beside this, we expect the contribution proportional to the quantum potential associated to the full wave, $-Q$, to be self-focusing but not self-accelerating[8] because when the non-linear

[8]Under the constraints considered here, the non-linear potential is unambiguously expressed through equation (30). Possibly, any potential of the form $Q^L + U^{NL}$ where U^{NL} is self-focusing but not self-accelerating would deliver the correct acceleration although it does not deliver the correct velocity. For instance U^{NL} could be a logarithmic non-linearity [30], or of the Schrödinger-Newton type [6, 7]. Now, as the onset of quantum equilibrium is guaranteed only [28] when initial velocitites do obey (4), the solution of de Broglie's double solution found here (30) is seemingly the only one. This remark should be put in relation with ref. [33] where J.R. Croca proposed some years ago to add to the linear Schrödinger equation a non-linear potential of the form $V^{NL} = Q^L$. We actually consider that self-focusing is required in order to realize de Broglie's double solution program. According to us there is thus a problem with Croca's proposal: in general Q^L is not self-focusing. It can even be self-defocusing as shown by the example treated in section 6.2.

potential is an analytic function of $R(\mathbf{x},t)$ the self-acceleration is equal to 0, as shown in appendix. This is illustrated by the following example (see ref. [11] for a more detailed treatment) where the pilot wave is a coherent state of a 1D harmonic oscillator, and $R(\mathbf{x},t)$ is gaussian, of constant size.

6.2 Special case: coherent states of a 1D harmonic potential.

It is easy to figure out what are the respective physical contributions of Q^L and Q, when the linear potential is harmonic $V^L(x) = k\cdot x^2/2$, in the case where the soliton is a coherent gausson, and when the pilot wave is a gaussian coherent state of a 1D harmonic oscillator. The quantum potential in the case of a gaussian wave function $exp^{-(Ax^2+Bx+C)}$ is equal to $(4(Re.A)^2(x- < x >)^2 - 2Re.A)$. Therefore, up to an additive constant, $Q^L = -\frac{k}{2}(x - x_0^L(t))^2$, and $Q = +\frac{\tilde{k}}{2}(x - x_0(t))^2$ where $x_0^L(t)$ represents the peak of the coherent state while $x_0(t)$ represents the peak/barycentre of the gausson, and $\tilde{k}$ is an effective spring constant associated to the gausson. $\tilde{k}$ is quite larger than k because the gausson is supposedly quite narrower than the pilot wave (here a coherent state). Q^L plays the same role here as an external time-dependent potential because it does not depend on the wave function of the gausson. It is accelerating but not self-focusing. On the contrary $-Q$ is not self-accelerating but self-focusing; its presence guarantees that the gausson keeps the same shape throughout time.

In virtue of Eherenfest's theorem, the peak of the gausson undergoes an acceleration equal to the sum of

-the classical contribution ($\frac{1}{m} < -\nabla V^L(\mathbf{x},t) >$) with

-the self-acceleration ($\frac{1}{m} < -\nabla V^{NL}(\mathbf{x},t) >= \frac{1}{m} < -\nabla Q^L(\mathbf{x},t) >$).

This is equal to $-\frac{k}{2}x + \frac{k}{2}(x - \mathbf{x}_0^L(t)) = -k\mathbf{x}_0(t)$. In every point, the velocity and acceleration are the same as the velocity and acceleration of the peak of the coherent state (pilot wave). All gaussons move thus as a block, following the peak of the coherent state. This explains why in the dBB picture the shape of the coherent state does not change with time. At the level of the faraway tails of the pilot wave for instance the barycentres of the gaussons oscillate around a virtual harmonic potential of spring constant k but centered around a position which is the position occupied by the barycentre of the gausson at the time where the peak of the coherent state passes through $x = 0$. The dynamics is obviously not classical here, excepted for what concerns the peak of the pilot wave/coherent state.

7 Conclusions and open questions.

7.1 Phase harmony.

In his book [1], de Broglie introduced the condition of phase harmony according to which the phase of the soliton is the same as the phase of the pilot wave. This constraint can be understood in terms of Ehrenfest's theorem: the velocity of the barycentre of the soliton is equal to the average value of the gradient of its phase multiplied by $\hbar/m$. In virtue of the guidance equation (4), it must also be equal to the dB velocity, that is to say, the gradient of the phase of the pilot wave multiplied by $\hbar/m$. If the condition of phase harmony is satisfied, and that the phase of the pilot wave varies very slowly at the scale of the soliton, then, indeed, the velocity of the barycentre of the soliton is well equal to the dB velocity. However, as we have shown by many examples, it is very difficult to achieve phase harmony, and the majority of the non-linear potentials considered so far do NEITHER respect phase harmony, NOR the guidance equation.

The non-linear potential (30), on the contrary, makes it possible to satisfy phase harmony (as can be seen after combining (24) with (27)), and its solitonic solutions respect, in good approximation, the guidance equation.

7.2 Galilean invariance.

As has been noted in [10], when there is no external potential (V^L), which means that the particle is only submitted to its self-interaction, and when equation (7) is invariant under Galilean transformations (boosts)[9], which is the case whenever the non-linear potential V^{NL} only depends on the modulus of Ψ, we find, after performing a Galilean boost on a static soliton $\phi^0_{NL}(\mathbf{x})$ the exact solution

$\phi^0_{NL}(\mathbf{x}-\mathbf{v}\cdot t)e^{-i((E_0+\hbar\omega)\cdot t-\hbar\mathbf{k}\cdot\mathbf{x})/\hbar}$. This solution satisfies the factorization ansatz with a pilot wave which is a plane wave. It also respects phase harmony because static solitonic solutions $\phi^0_{NL}(\mathbf{x})e^{-iE_0t/\hbar}$ are de facto real functions (up to an irrelevant, position-independent,

[9]See for instance ref. [7] where we studied in depth the Galilean invariance of the 1 D NLS potential.

global phase). This trick makes it possible to generate solitons moving at velocity v_{dB}. As already noted by Fargue [31] many years ago, such solutions exist for a large class of different non-linearities (e.g. a non-linearity proportional to $|\Psi|^2$), and they all agree with the principle of phase harmony already invoked by de Broglie in 1927. Obviously, this property is seen to be a direct consequence of the Galilean invariance of the non-linear potential V^{NL}. Difficulties arise however when an external potential V^L is present, and/or when the pilot wave is not a plane wave. The abovementioned trick is then useless because the quantum potential does no longer vanish and accelerations are present. This approach is thus a dead-end, but other approaches maybe provide satisfactory realizations of the double solution program. One of them was proposed by Barut [29]. It has some relativistic flavour because it incorporates the principle of mass-energy equivalence [34]. However, as discussed in appendix (section 10), it is only valid when the pilot wave is a plane wave. Other, fully relativistic approaches remain possible in principle, but even there serious obstacles limit the realization of the double solution program as we discuss in the next paragraph.

7.3 Fully Lorentz covariant approaches.

Special relativity played an essential role in the genesis of de Broglie's wave mechanics. In the 30's, de Broglie was fascinated by Dirac's equation and throughout his life he repeatedly refered to Klein-Gordon equation [16] (which, by the way, could as well have been attributed to Schrödinger and/or de Broglie as explained in ref. [35]). As is well-known however, the dBB interpretation meets several obstacles in the case of bosons. For instance, the conserved quantity associated to the Klein-Gordon equation is not positive-definite. Other approaches are possible, one of them being for instance to treat Glauber's first order correlation function as a photon wave function [36]. As this wave function obeys Maxwell's equations, one can associate to it a conservation equation which is a complexified version of Poynting's conservation equation. In this approach, however, the conserved density does not transform under Lorentz boosts as the time component of a relativistic 4-vector [27]. If we try to associate trajectories to photons through the dBB usual trick consisting of interpreting the velocity of the particle as the ratio between the current vector and the conserved density, we must accept that trajectories do not transform as quadridimensional, minkoskian, lines of universe. Dirac's equation does not suffer from this drewback, but recently it has been shown that the same problem occurs when we consider two non-interacting Dirac fermions:

-either their dynamics is described by the Bohm-Hiley dynamics to which it is possible to associate a conservation equation with a positive definite density; however, as has been shown by Pierre Pelcé [37], the Bohm-Hiley dynamics is not invariant under Lorentz boosts;

-or their dynamics is described the Durt-Pelcé dynamics [38] which is well Lorentz invariant; however the conserved density is, in this case, not positive definite.

More generally, as was recognized by Bohm and later by Bell [13, 14, 39], if we wish to associate dBB trajectories to entangled systems, trajectories must be defined in the configuration space which is a source of intrinsic non-locality. This problem had already been identified by de Broglie himself in 1926 [12]. Many years later, de Broglie attempted, together with Andrade e Silva, to get rid of the configuration space and to reformulate his interpretation in real, 3D physical space [40, 41, 42, 43, 44]. The violation of Bell's inequalities [39] shows however, according to us, that this attempt is condemned to fail from the beginning.

7.4 Other ways to realize the double solution program.

In a previous paper [7], we wrote...*Since a couple of decades, de Broglie's point of view has been revived, be it indirectly, by experimental observations in hydrodynamics, which show that certain macroscopic objects, so called walkers (bouncing oil droplets), exhibit many of the features of the de Broglie-Bohm (dB) dynamics [45, 46, 47, 48].* ... It is not clear however to figure out how such systems could exhibit quantum non-locality. In particular, having in mind that the majority of the theoretical models that are used to describe bouncing droplets is based on hydrodynamics and fluid dynamics, which is always formulated in 3D, it is hard to go beyond a formulation of trajectories in 3D space. It seems thus impossible mission to formulate the dynamics of bouncing droplets in configuration space. Retrospectively, wave monism does not appear to have played a role in the theory of bouncing oil droplets, even in "masselotte" models *à la* Borghesi [49] where the particle is represented by a massive material point, and the bath on which it bounces by a wave, similar to the pilot wave interpretation. We are thus in right to consider that, also here, the price to pay in order to realize the double solution program is to sacrifice wave monism.

7.5 Conclusions

Our paper illustrates to which extent the double solution program is severely constrained. We showed that the most obvious way to realize it is to look for a wave function which is the product of the pilot wave with a purely real solitonic wave. This, in turn, fixes the form of the non-linear potential[10]. To conclude our analysis it seems well that in order to realize de Broglie's double solution program it is required to sacrifice wave monism, because the pilot wave and the particle are not treated on the same footing: the non-linearity "makes a difference" between them from the beginning and they do not naturally "emerge" from the non-linearity.

Now, the dBB interpretation is rejected by the majority of quantum physicists because of its ad hoc character, of its artificial degree of complexity and of its non-relativistic flavour. Unfortunately, the same could be said, at this level, about the double solution program, in contradiction with the fact that this program was aimed at restoring simplicity and coherence at a fundamental level. In a sense such conclusions are not surprising: we know today, with a degree of certainty that was absent 60 years ago, that quantum mechanics is a non-local theory and that it is very difficult to formulate it in terms of trajectories without passing to the configuration space. Some of the obstacles described in the present paper reflect all these difficulties. Our solution can be extended to the configuration space, at least in the classical limit [36], which shows that passing to the configuration space is not the main problem here[11]. Despite of this, two main question remains open at this level:

[10] In the context of the first order in time dB dynamics the potential is unambiguously specified by equation (30). If we generalize this result in the case of the second order in time dBB dynamics, we found that the non-linear potential is the sum of Q^L with a self-focusing potential U^{NL} still to be specified. Now, quantum equilibrium is seemingly unstable in this case [28] so that those solutions ought to be rejected.

[11] As explained elsewhere with more details [36, 50], de Broglie remained reluctant throughout all his life to the possibility to express quantum trajectories in the configuration space [51]. For instance, in the 50's, he wrote the following [1]: *...Or, la méthode de Schrödinger implique nécessairement l'emploi de l'espace de configuration et ne permet pas de se représenter le phénomène physique constitué par le mouvement des corpuscules dans le cadre de l'espace physique. Sans doute la Mécanique classique se servait-elle souvent, elle aussi, de l'espace de configuration, mais ce n'était pas pour elle une nécessité: elle pouvait raisonner en considérant le mouvement des points matériels du système dans l'espace à trois dimensions et elle n'employait l'espace de configuration que comme un artifice mathématique permettant de présenter plus élégamment ou d'effectuer plus aisément certains calculs. Dès l'apparition des Mémoires de Schrödinger, tout en reconnaissant l'exactitude*

-Q1: is it possible to find a Lorentz invariant realization of the double solution program ?

-Q2: is it possible then to respect wave monism?

It is not clear whether a relativistic formulation of the double solution program is possible, because even in the case of the pilot wave/dBB interpretation such a formulation does not exist yet. One is in right, however, to expect that a solution of the double solution program ought to be robust in the non-relativistic limit, in which case the answer to the question Q2 is, definitively, negative.

8 Appendix 1: Factorisation ansatz and generalized guidance equation: additional computations.

8.1 Non-unitarity/change of norm.

Let us denote H_L the linear part of the the full Hamiltonian in (19). It is not hermitian, so that $\sqrt{<\phi_{NL}|\phi_{NL}>}$, the $\mathcal{L}_2$ norm of its solution $\phi_{NL}(\mathbf{x},t))$ is not constant throughout time. The non-linear potentials considered by us preserve the $\mathcal{L}_2$ norm however. We can thus evaluate the time derivative of $<\phi_{NL}|\phi_{NL}>$ by direct computation, either integrating by parts, or making use of the formula

$$\begin{aligned}
&\frac{d<\phi|O|\phi>}{dt}= \\
&<\phi|\frac{\partial O}{\partial t}|\phi>+\frac{1}{i\hbar}<\phi|OH_L-H_L^{\dagger}O|\phi> \\
&=<\phi|\frac{\partial O}{\partial t}|\phi>+\frac{1}{i\hbar}(<\phi|[O,Re.H_L]_-|\phi> \\
&+\frac{1}{\hbar}(<\phi|[O,Im.H_L]_+|\phi>),
\end{aligned} \tag{31}$$

where O is an arbitrary observable, described by a self-adjoint operator, while $Re.H_L$ and $Im.H_L$, the real and imaginary parts of H_L are self-adjoint operators defined through $2\cdot Re.H_L=H_L+H_L^{\dagger}$ and $2i\cdot Im.H_L=H_L-H_L^{\dagger}$. Here, the symbol $[,]_-$ ($[,]_+$) represents the (anti)commutator.

We find by direct computation that

des résultats obtenus par sa méthode, j'avais trouvé paradoxal le principe même de cette méthode. Following Bohm and others [52], we do not share de Broglie's opinion, even if this opens the door to non-locality, that we consider to constitute an essential feature of quantum physics.

$$Re.(-\frac{\hbar^2}{m}i\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown)$$
$$=(-\frac{\hbar^2}{m}i\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown)-(\frac{\hbar^2}{2m}i\Delta\varphi_L(\mathbf{x},t)) \quad (32)$$

and $Im.(-\frac{\hbar^2}{m}i\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown)=(\frac{\hbar^2}{2m}\Delta\varphi_L(\mathbf{x},t))$.

Therefore the guidance potential contributes to $\frac{d<\phi_{NL}|\phi_{NL}>}{dt}=\frac{d<\phi_{NL}|1|\phi_{NL}>}{dt}$ by a quantity

$$<\phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(\mathbf{x},t))|\phi_{NL}>\approx(\frac{\hbar}{m}\Delta\varphi_L(<\mathbf{x}>,t))<\phi_{NL}|\phi_{NL}>),$$

due to the fact that, over the size of the soliton, $\varphi_L(\mathbf{x},t)$ and its derivatives are supposed to vary so slowly that we can consistently neglect their variation and put them in front of the $\mathcal{L}_2$ integral.

The contribution of the $R_L-\phi_{NL}$ coupling to $\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$ is $\frac{\hbar^2}{m}\frac{1}{i\hbar}\int d^3\mathbf{x}(\frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\bigtriangledown(\phi_{NL}(\mathbf{x},t))^*\phi_{NL}(\mathbf{x},t)-(\phi_{NL}(\mathbf{x},t))^*\frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\bigtriangledown\phi_{NL}(\mathbf{x},t))$.

We now suppose that we are in right to neglect the variation of $\frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}$ in the integral above and to replace it by $\frac{\bigtriangledown R_L(\mathbf{x_0},t)}{R_L(\mathbf{x_0},t)}$. Then we find, after integrating by parts, a contribution $-2\frac{\bigtriangledown R_L(t,\mathbf{x_0})}{R_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(\mathbf{x},t)$

Putting all these results together, we find [10] that

$$\frac{d<\phi_{NL}|\phi_{NL}>}{dt}\approx\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot<\phi_{NL}|\phi_{NL}>$$
$$-2\frac{\bigtriangledown R_L(t,\mathbf{x_0})}{R_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(\mathbf{x},t). \quad (33)$$

8.2 Property 1: drift velocity.

Let us now consider the barycentre $\mathbf{x_0}$ of the soliton: $\mathbf{x_0}\equiv\frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$ in order to estimate its velocity $\mathbf{v}_{drift}$:

$$\mathbf{v}_{drift} \equiv \frac{d(\frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>})}{dt} \tag{34}$$

For instance, if we consider its z component:
$z_0 = \frac{<\phi_{NL}|z|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$
and
$\frac{dz_0}{dt} = \frac{1}{<\phi_{NL}|\phi_{NL}>}\frac{d<\phi_{NL}|z|\phi_{NL}>}{dt} - \frac{z_0}{<\phi_{NL}|\phi_{NL}>}\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$,
so that we find (making use of (20) as well as of results in the previous section 8.1)

$$\begin{aligned}
\frac{dz_0}{dt} &= \frac{1}{<\phi_{NL}|\phi_{NL}>}\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\nabla_{\mathbf{z}}}{mi}\cdot\phi_{NL}(\mathbf{x},t)\\
&+\frac{1}{<\phi_{NL}|\phi_{NL}>}\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*(\frac{\hbar\nabla_{\mathbf{z}}}{m}\cdot\varphi_L(\mathbf{x},t))\phi_{NL}(\mathbf{x},t)\\
&+\frac{1}{<\phi_{NL}|\phi_{NL}>}<\phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(\mathbf{x},t))\cdot z|\phi_{NL}>\\
&+\frac{1}{<\phi_{NL}|\phi_{NL}>}\frac{\hbar}{im}\int d^3\mathbf{x}\frac{\nabla R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\nabla(\phi_{NL}(\mathbf{x},t))^*\cdot z\cdot\phi_{NL}(\mathbf{x},t)\\
&-\frac{1}{<\phi_{NL}|\phi_{NL}>}\frac{\hbar}{im}\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\cdot z\cdot\frac{\nabla R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\nabla\phi_{NL}(\mathbf{x},t))\\
&-\frac{z_0}{<\phi_{NL}|\phi_{NL}>}\cdot(\frac{\hbar}{m})\Delta\varphi_L(t,\mathbf{x_0})\cdot<\phi_{NL}|\phi_{NL}> \qquad (35)\\
&+2\frac{z_0}{<\phi_{NL}|\phi_{NL}>}\frac{\nabla R_L(t,\mathbf{x_0})}{R_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(\mathbf{x},t))
\end{aligned}$$

Now, $\frac{\hbar}{im}\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\cdot z\cdot\frac{\nabla R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\nabla\phi_{NL}(\mathbf{x},t)$
$\approx z_0\frac{\nabla R_L(t,\mathbf{x_0})}{R_L(t,\mathbf{x_0})}\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(\mathbf{x},t))$,
while
$<\phi_{NL}|(\frac{\hbar}{m})\Delta\varphi_L(\mathbf{x},t)\cdot z|\phi_{NL}> \approx z_0\cdot(\frac{\hbar}{m})\Delta\varphi_L(t,\mathbf{x_0})\cdot<\phi_{NL}|\phi_{NL}>$
and so on so that finally only the two first lines of (35) survive.

We get thus [10] the generalized dB guidance equation (22), which constitutes the

Property 1:

$$\mathbf{v}_{drift} = \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t) + \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$$
$$= \mathbf{v}_{dB} + \mathbf{v}_{int.}.$$

$\mathbf{v}_{drift}$ contains the de Broglie-Bohm velocity

$$\mathbf{v}_{dB} \equiv \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t),, \tag{36}$$

and the internal velocity

$$\mathbf{v}_{int.} \equiv \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}. \tag{37}$$

(36) is nothing else [27] than de Broglie velocity as imposed through the guidance equation (4), while $\mathbf{v}_{int.}$ can be considered as a contribution to the average velocity originating from the internal structure of the soliton. Both contributions to the drift are evaluated at the barycentre of the soliton, $\mathbf{x_0}$.

8.3 Property 2: rescaling.

Let us now reconsider the change of norm of ϕ_{NL}.

To do so, we introduce the total time derivative of R_L ($\frac{dR_L}{dt} = \frac{\partial R_L}{\partial t} + \mathbf{v}_{drift}\cdot\nabla R_L$) where $\mathbf{v}_{drift} = \frac{d\frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}}{dt}$ obeys the generalized dB guidance equation (22).

By a direct computation, we find

$$\frac{\frac{dR_L}{dt}}{R_L} = \frac{1}{R_L}(\frac{\partial R_L}{\partial t} + \nabla R_L\cdot\frac{\hbar\nabla}{m}\cdot\varphi_L(t,\mathbf{x_0})) +$$
$$\frac{1}{R_L}\nabla R_L\cdot\frac{1}{<\phi_{NL}|\phi_{NL}>}\int d^3\mathbf{x}(\phi_{NL}(\mathbf{x},t))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(\mathbf{x},t) \tag{38}$$

Making use of the conservation equation of the linear Schrödinger equation $\frac{\partial(R_L)^2}{\partial t} = -div((R_L)^2\frac{\hbar\nabla}{m}\cdot\varphi_L(t,\mathbf{x_0}))$ we find

$\frac{1}{R_L}(\frac{\partial R_L}{\partial t} + \bigtriangledown R_L \cdot \frac{\hbar\bigtriangledown}{m} \cdot \varphi_L(t, \mathbf{x_0})) = \frac{-1}{2} div(\frac{\hbar\bigtriangledown}{m} \cdot \varphi_L(t, \mathbf{x_0}))$ and we can rewrite (38) as follows:

$$\frac{\frac{dR_L}{dt}}{R_L} = \frac{-1}{2}\frac{\hbar}{m}\Delta\varphi_L(t, \mathbf{x_0})$$
$$+\frac{\bigtriangledown R_L}{R_L} \cdot \frac{1}{< \phi_{NL}|\phi_{NL} >} \int d^3\mathbf{x}(\phi_{NL}(\mathbf{x}, t))^* \frac{\hbar\bigtriangledown}{mi} \cdot \phi_{NL}(\mathbf{x}, t) \quad (39)$$

Making use of (20) (derived in section 8.1), we obtain at the end $\frac{\frac{dR_L}{dt}}{R_L} = \frac{-1}{2}\frac{1}{<\phi_{NL}|\phi_{NL}>}\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$ so that, finally,

$$\frac{\frac{d<\phi_{NL}|\phi_{NL}>}{dt}}{< \phi_{NL}|\phi_{NL} >} = -2\frac{\frac{dR_L}{dt}}{R_L}. \quad (40)$$

From the constraint (40) we infer [10] the

Property 2

$$\frac{< \phi_{NL}|\phi_{NL} > (t)}{< \phi_{NL}|\phi_{NL} > (t = 0)} = \frac{(R_L)^2(t = 0)}{(R_L)^2(t)}, \quad (41)$$

where we evaluate $(R_L)^2(t)$ at the barycentre of ϕ_{NL}, which moves according to the generalized dB guidance equation (22). Let us rescale $\phi_{NL}(\mathbf{x}, t)$ by defining ϕ'_{NL} through $\phi_{NL}(\mathbf{x}, t) \equiv \phi'_{NL}(\mathbf{x}, t)/R_L$; we can thus predict in general that, if it exists and remains peaked during its evolution, the solution of (19) has the form

$$\Psi(x, y, z, t) \approx \phi'_{NL}(\mathbf{x}, t)e^{i\varphi_L(\mathbf{x},t)}, \quad (42)$$

where $\phi'_{NL}(\mathbf{x}, t)$ is centred in $\mathbf{x}_0(t = 0) + \int_0^t dt\mathbf{v}_{drift}$ and is of constant $\mathcal{L}_2$ norm.

In general, the non-linearity does not depend on the phase of Ψ so that $V^{NL}(\Psi) = V^{NL}(\phi'_{NL}(\mathbf{x},t))$, and (19) can then be cast in the form

$$i\hbar \cdot \frac{\partial(\phi'_{NL}(\mathbf{x},t)/R_L(t,\mathbf{x_0}))}{\partial t} =$$
$$-\frac{\hbar^2}{2m} \cdot \Delta(\phi'_{NL}(\mathbf{x},t)/R_L(t,\mathbf{x_0})) + V^{NL}(\phi'_{NL})(\phi'_{NL}(\mathbf{x},t)/R_L(t,\mathbf{x_0}))$$
$$-\frac{\hbar^2}{m} \cdot i\bigtriangledown\varphi_L(\mathbf{x},t) \cdot \bigtriangledown(\phi'_{NL}(\mathbf{x},t)/R_L(t,\mathbf{x_0}))$$
$$-\frac{\hbar^2}{m} \cdot \frac{\bigtriangledown R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)} \cdot \bigtriangledown(\phi'_{NL}(\mathbf{x},t)/R_L(t,\mathbf{x_0})). \quad (43)$$

In order to say more about $\phi'_{NL}(\mathbf{x},t)$ we must solve equation (43) which is a complicated problem, beyond the scope of our paper.

9 Appendix 2: Derrick's theorem and linear power-law potentials.

9.1 Derrick's no go theorem.

Derrick [2] considered static self-interaction and localized static solitonic solutions of the time-independent non-linear Schrödinger equation:

$$E \cdot \Psi(\mathbf{x}) = \frac{-\hbar^2}{2m}\Delta\Psi(\mathbf{x}) + V^{NL}(\mathbf{x})\Psi(\mathbf{x}). \quad (44)$$

Multiplying this equation at the left by Ψ^* and integrating over the whole space we get

$$E \cdot \int \mathbf{d^3x}\Psi^*(\mathbf{x})\Psi(\mathbf{x}) = \frac{-\hbar^2}{2m} \int \mathbf{d^3x}\Psi^*(\mathbf{x})\Delta\Psi(\mathbf{x}) + \int \mathbf{d^3x}\Psi^*(\mathbf{x})V^{NL}(\mathbf{x})\Psi(\mathbf{x}). \quad (45)$$

Following Derrick let us interpret the functional $\frac{-\hbar^2}{2m} \int \mathbf{d^3x}\Psi^*(\mathbf{x})\Delta\Psi(\mathbf{x}) + \int \mathbf{d^3x}\Psi^*(\mathbf{x})V^{NL}(\mathbf{x})\Psi(\mathbf{x})$, as the energy of the soliton:

$$E = \frac{-\hbar^2}{2m} \int \mathbf{d^3x}\Psi^*(\mathbf{x})\Delta\Psi(\mathbf{x}) + \int \mathbf{d^3x}\Psi^*(\mathbf{x})V^{NL}(\mathbf{x})\Psi(\mathbf{x}). \quad (46)$$

Assuming here that $V^{NL}(\mathbf{x}) = V^{NL}(\Psi(\mathbf{x}))$, let us consider dilated functions $\Psi_\lambda(\mathbf{x}) = \Psi(\lambda\mathbf{x})$.

Their energy obeys

$$E_\lambda = \frac{-\hbar^2}{2m}\int \mathbf{d^3x}\Psi^*_\lambda(\mathbf{x})\Delta\Psi_\lambda(\mathbf{x}) + \int \mathbf{d^3x}\Psi^*_\lambda(\mathbf{x})V^{NL}(\mathbf{x})\Psi_\lambda(\mathbf{x}). \quad (47)$$

It is straightfoward to show that $E_\lambda = I_1/\lambda + I_2/\lambda^3$ where $I_1 = \frac{-\hbar^2}{2m}\int \mathbf{d^3x}\Psi^*(\mathbf{x})\Delta\Psi(\mathbf{x})$ and $I_2 = \int \mathbf{d^3x}\Psi^*(\mathbf{x})V^{NL}(\mathbf{x})\Psi(\mathbf{x})$.

Let us assume that $\Psi(\mathbf{x})$ is a minimal energy state; then $\frac{dE_\lambda}{d\lambda}\Big|_{\lambda=1}$ =0 and $\frac{d^2E_\lambda}{d\lambda^2}\Big|_{\lambda=1} \leq 0$. These constraints impose that

$-I_1 - 3I_2 = 0$ and $2I_1 + 12I_2 \leq 0$; combining both constraints implies that $-2I_1 \leq 0$. However, the soliton is supposed to be localized in a finite region of space so that, integrating by parts, we find that $I_1 = \frac{+\hbar^2}{2m}\int \mathbf{d^3x}|\nabla\Psi(\mathbf{x})|^2 > 0$.

According to Derrick, the soliton is thus unstable; in other words it is impossible to find a minimal energy state that would be a localized static solitonic solution of the time-independent non-linear Schrödinger equation; this is the so called Derrick no-go theorem. Note that this theorem can be generalized to the case where the dimension of space is N; then we get $E_\lambda = I_1/\lambda^{(N-2)} + I_2/\lambda^N$, and it is easy to show that the no-go theorem is still valid for all dimensions strictly larger than 2.

This theorem played the role of a scarecrow during many years. In the version presented above, which is its commonly accepted version, it rules out the possibility to apply the double solution program in the physical, 3D, space. Nevertheless, we showed in the past by two explicit examples, the 3D Schrödinger-Newton potential [6] and the 1 D NLS potential in $|\Psi|^2$ [7] , that the premices of Derrick's no-go theorem are fundamentally flawed.

Before considering more general situations, let us in a first time present an oversimplified version of our argument in the case where the potential is linear and is a power-law.

9.2 Derrick's theorem, linear power-law potentials, and the Virial quantum theorem.

Let us assume that $V^{NL} = 0$ and $V^L(\mathbf{x})$=$a \cdot |\mathbf{x}|^b$, with a and b constants. Repeating Derrick's procedure in this case, that is to say, varying the energy of (46) with dilated functions $\Psi_\lambda(\mathbf{x}) = \Psi(\lambda\mathbf{x})$, we get, in dimension N, $E_\lambda = I_1/\lambda^{(N-2)} + I_2/\lambda^{N+b}$, where $I_1 = \frac{-\hbar^2}{2m}\int \mathbf{d^Nx}\Psi^*_E(\mathbf{x})\Delta\Psi_E(\mathbf{x})$

and $I_2 = \int \mathbf{d^N x} \Psi_L^*(\mathbf{x}) V^L(\mathbf{x}) \Psi_L(\mathbf{x})$. Imposing that $\frac{dE_\lambda}{d\lambda}\Big|_{\lambda=1}$ =0 we get $(2-N)I_1 = (N+b)\cdot I_2$

Now, we are free to consider in the present context the quantum version of the Virial theorem. To do so, let us consider a solution of the time-dependent equation linear Schrödinger equation (1) $i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = -\hbar^2\frac{\Delta\Psi(\mathbf{x},t)}{2m} + V^L(\mathbf{x})\Psi(\mathbf{x},t)$. In virtue of Ehrenfest's theorem, we find that

$\frac{d}{dt} < \Psi|\mathbf{x}\cdot\mathbf{p}|\Psi > = \frac{1}{i\hbar}\int \mathbf{d^N x}\Psi^*(\mathbf{x},t))[-\hbar^2\frac{\Delta\Psi(\mathbf{x},t)}{2m} + V^L(\mathbf{x}), \mathbf{x}\cdot\mathbf{p}]\Psi(\mathbf{x},t))$

$= \int \mathbf{d^N x}\Psi^*(\mathbf{x},t))(2\hbar^2\frac{\Delta\Psi(\mathbf{x},t)}{m} + \mathbf{x}\cdot\nabla V^L(\mathbf{x})\Psi(\mathbf{x},t))$

If $V^L(\Psi)|_{\mathbf{x},t} = a\cdot|\mathbf{x}|^b$, and if we consider an eigenstate of the Hamiltonian, that is to say a wave function $\Psi_E(\mathbf{x})$ such that

$E\Psi_E(\mathbf{x}) = -\frac{\hbar^2}{2m}\Delta\Psi_E(\mathbf{x}) + V^L(\mathbf{x})\Psi_E(\mathbf{x}),$

we find

$0 = \int \mathbf{d^N x}\Psi_E^*(\mathbf{x})(2\hbar^2\frac{\Delta\Psi(\mathbf{x},t)}{m} + b\cdot V^L(\mathbf{x}))\Psi_E(\mathbf{x}).$

If we reformulate this constraint in terms of I_1 and I_2, we find that $2I_1 = b\cdot I_2$, which contradicts Derrick's constraint $(2-N)I_1 = (N+b)\cdot I_2$ excepted for the particular value $(2-N)/(N+b) = 2/b \leftrightarrow N = 0$. In particular, Derrick's "theorem" leads to the prediction that the ground state of the 1D harmonic potential as well as the ground state of the 3D Coulomb potential are unstable, a puzzling and obviously misleading prediction. This apparent paradox can be solved if we observe that the contradiction is present from the beginning at the level of equation (46), which differs from (45) by a factor $< \Psi|\Psi >$. Repeating Derrick's rescaling based on equation (45), instead of (46), we get, in dimension N,

$\tilde{E}_\lambda = I_1\lambda^2 + I_2\lambda^{-b}$ where

$\tilde{E}_\lambda = \frac{\frac{-\hbar^2}{2m}\int \mathbf{d^3x}\Psi^*(\lambda\mathbf{x})\Delta\Psi(\lambda\mathbf{x}) + \int \mathbf{d^3x}\Psi^*(\lambda\mathbf{x})V^L(\lambda\mathbf{x})\Psi(\lambda\mathbf{x})}{\int \mathbf{d^3x}\Psi^*(\lambda\mathbf{x})\Psi(\lambda\mathbf{x})}.$

Imposing $\frac{d\tilde{E}_\lambda}{d\lambda}|_{\lambda=1} = 0$ delivers, when $V^L(\mathbf{x}) = a\cdot|\mathbf{x}|^b$, $2I_1 = bI_2$ which is nothing else than the quantum Virial theorem (valid in arbitrary dimension N). The second derivative of $\tilde{E}_\lambda$ relatively to λ, estimated at $\lambda = 1$, is now equal to $2I_1 + b(b+1)I_2 = 2I_1\cdot(1+b+1)$.

Now, $I_1 > 0$, so that if we impose the second derivative to be positive we finally find $b+2 > 0$, which is for instance satisfied by the harmonic potential and the Coulomb potential, in any dimension N. In particular, this analysis confirms the stability of the ground states of the 1D harmonic potential and of the 3D Coulomb potential.

To conclude, Derrick's theorem is to a large extent ill-founded. As we discussed elsewhere [6, 7] , in the case of the 3D Schrödinger-Newton potential and the 1 D NLS potential in $|\Psi|^2$, the Derrick's theorem does not take account of the fact that the energy of the ground state of a set of equinormalized functions depends on their norm. For the two aforementioned potentials for instance, the minimal energy decreases when the norm increases. It is however impossible for a state initially prepared with a certain $\mathcal{L}_2$ norm $< \Psi|\Psi >$ to reach states with a higher norm, because the quantum evolution is unitary[12]. What actually occurs in the case of the 1 D NLS potential is that, as is well-known, the system will radiate some weight and decrease its norm until he reaches the ground state associated to its final norm and stabilizes. In ref. [7], we studied in depth the symmetries of the 1 D NLS potential regarding rescaling. Here, we shall extend this analysis, focusing on the rescaling properties of the non-linear power-law potentials.

9.3 Derrick's theorem and non-linear power-law potentials.

Let us assume that $V^L = 0$ and $V^{NL}(\mathbf{x}) = V^{NL}(\Psi(\mathbf{x}))=a \cdot |\Psi(\mathbf{x})|^b$, with a and b constants. Repeating Derrick's procedure, but this time with the correct functional, that is to say, varying the energy of (45) rather than of (46) with dilated functions $\Psi_\lambda(\mathbf{x}) = \Psi(\lambda\mathbf{x})$, we would get $E_\lambda = I_1/\lambda^{(N-2)} + I_2/\lambda^{(N-b)}$, where $I_1 = \frac{-\hbar^2}{2m}\int \mathbf{d^N x}\Psi_E^*(\mathbf{x})\Delta\Psi_E(\mathbf{x})$ and $I_2 = \int \mathbf{d^N x}\Psi_L^*(\mathbf{x})V^{NL}(\Psi(\mathbf{x}))\Psi_L(\mathbf{x})$. However, the dilated functions $\Psi(\lambda\mathbf{x})$ do not share the same $\mathcal{L}_2$ norm as $\Psi(\mathbf{x})$, and there is no reason, as we already remarked before, to impose that $\Psi(\lambda\mathbf{x})$ minimizes the energy, disregarding the norm of the trial function $\Psi_\lambda(\mathbf{x})$.

A correct stability analysis requires instead to minimize the energy under variations of $\Psi(\mathbf{x})$ that preserve the $\mathcal{L}_2$ norm.

We should thus rather impose that

$\tilde{E}_\lambda = \frac{\frac{-\hbar^2}{2m}\int \mathbf{d^N x}\tilde{\Psi}^*(\lambda\mathbf{x})\Delta\tilde{\Psi}(\lambda\mathbf{x})+\int \mathbf{d^N x}\tilde{\Psi}^*(\lambda\mathbf{x})V^{NL}(\tilde{\Psi}(\lambda\mathbf{x}))\tilde{\Psi}(\lambda\mathbf{x})}{\int \mathbf{d^N x}\tilde{\Psi}^*(\lambda\mathbf{x})\tilde{\Psi}(\lambda\mathbf{x})}$ is minimal,

with $\tilde{\Psi}(\lambda\mathbf{x}) = \lambda^{N/2}\Psi(\lambda\mathbf{x})$.

Now, if $V^{NL}(\mathbf{x})=a \cdot |\Psi(\mathbf{x})|^b$, we get $\tilde{E}_\lambda = \tilde{I}_1\lambda^2 + \tilde{I}_2\lambda^{N \cdot b/2}$ where

[12]This is also true in the case of the logarithmic potential as originally noted by Bialynicki-Birula and Mycielski [30] : if we rescale the wave function by multiplying it by a complex number z, the size and shape of the self-focused gaussons does not change but the potential is shifted by a positive quantity when $|z| < 1$, and by a negative quantity when $|z| > 1$. Lower energy gaussons are thus associated to larger $\mathcal{L}_2$ norms, which, once again, nullifies Derrick's theorem in this case.

$\tilde{I}_1 = \frac{\frac{-\hbar^2}{2m}\int \mathbf{d^N x}\tilde{\Psi}^*(\mathbf{x})\Delta\tilde{\Psi}(\mathbf{x})}{\int \mathbf{d^N x}\tilde{\Psi}^*(\mathbf{x})\tilde{\Psi}(\mathbf{x})}$

and

$\tilde{I}_2 = \frac{\int \mathbf{d^N x}\tilde{\Psi}^*(\mathbf{x})V^{NL}(\Psi(\mathbf{x}))\tilde{\Psi}(\mathbf{x})}{\int \mathbf{d^N x}\tilde{\Psi}^*(\mathbf{x})\tilde{\Psi}(\mathbf{x})}$.

Imposing $\frac{d\tilde{E}_\lambda}{d\lambda}|_{\lambda=1} = 0$ delivers[13] the Viral-like relation

$2\tilde{I}_1 = -(N \cdot b/2)\tilde{I}_2$.

Stability requires that $\frac{d^2\tilde{E}_\lambda}{d\lambda^2}|_{\lambda=1} > 0$, which implies that $2\tilde{I}_1 + (N \cdot b/2)(N \cdot b/2 - 1)\tilde{I}_2 > 0$. Combining this constraint with $2\tilde{I}_1 = -(N \cdot b/2)\tilde{I}_2$ delivers $2\tilde{I}_1(1 - Nb/2 + 1) > 0$ and, because $I_1 > 0$ we finally find $4 - Nb > 0$.

In order to complete our stability analysis, it is still necessary to show that states with lower energy have a $\mathcal{L}_2$ norm higher than the $\mathcal{L}_2$ norm of $\Psi(\mathbf{x})$. To do so, let us consider rescaled states $\tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2}(\mathbf{x}) = \lambda_2\Psi(\lambda_1\mathbf{x})$, with $\lambda_1, \lambda_2 \in R$ and $\lambda_1^2 = \lambda_2^b$. One can check by direct computation that if $\Psi(\mathbf{x})$ satisfies (44), and $V^{NL}(\mathbf{x})$=$a \cdot |\Psi(\mathbf{x})|^b$, then

$$\tilde{\tilde{E}}\tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2}(\mathbf{x}) = \frac{-\hbar^2}{2m}\Delta\tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2}(\mathbf{x}) + V^{NL}(\mathbf{x})\tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2}(\mathbf{x}).,\tag{48}$$

where the rescaled energy $\tilde{\tilde{E}}$ obeys $\tilde{\tilde{E}} = \lambda_2 \cdot \lambda_1^2 \cdot E = \lambda_2^{b+1} \cdot E$. Now, if we assume that Ψ is a stable localized soliton, then E is negative. Thus, $\frac{d\tilde{\tilde{E}}}{d\lambda_2}|_{\lambda_2=1} = (b+1) \cdot E$, with $E < 0$.

Moreover, $< \tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2}|\tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2} >$, the $\mathcal{L}_2$ norm of $\tilde{\tilde{\Psi}}_{\lambda_1,\lambda_2}(\mathbf{x})$ is equal to $\lambda_2^2\lambda_1^{-N} \cdot < \Psi|\Psi > = \lambda_2^{(2-bN/2)} \cdot < \Psi|\Psi >$.

Also imposing that in the vicinity of $\lambda_1 = \lambda_2 = 1$ the wave functions with energy smaller than E have a $\mathcal{L}_2$ norm larger than the non-varied norm $< \Psi|\Psi >$ requires

$(2 - bN/2) > 0 \leftrightarrow bN < 4$, when $b + 1 > 0$

and

$(2 - bN/2) < 0 \leftrightarrow bN > 4$, when $b + 1 < 0$.

[13]Note that, if we consider non-linear potentials of the form $V^{NL}(\mathbf{x}, t) = \int \mathbf{d^3x'}|\Psi(\mathbf{x'}, t)|^2 \cdot f(||\mathbf{x} - \mathbf{x'}||^2)$ with $=f(||\mathbf{x} - \mathbf{x'}||^2) = a \cdot ||\mathbf{x} - \mathbf{x'}||^b$, we obtain the Viral-like relation $2I_1 = bI_2$ and all the stability analysis is exactly identical to the one performed in the case of linear power-law potentials. In particular when $b = -1$ the single particle Schrödinger-Newton potential possess a stable gorund state, in accordance with Lieb's analysis [53] .

Now, we must also fulfill the constraint $4 - Nb > 0$ so that there is no stability when $b < -1$. Even when $b > -1$, stability is guaranteed according to our criteria only if $bN < 4$. For instance when $b = 2$ (NLS potential), it is only in a 1D space that stability is guaranteed.

The 3D NLS potential is for instance ruled out by this analysis of stability but there still exist 3D potentials that pass the test, contrary to Derrick's claim.

To conclude this section, although the original version of Derrick's theorem is flawed for several reasons, it appears to provide useful tools for performing a stability analysis of the solitonic solutions of (44), when this equation possesses specific invariance properties under rescaling, which is for example the case when $V^{NL}(\mathbf{x}){=}a \cdot |\Psi(\mathbf{x})|^b$, or when $V^{NL}(\mathbf{x},t) = \int \mathbf{d^3x'}|\Psi(\mathbf{x'},t)|^2 \cdot f(||\mathbf{x}-\mathbf{x'}||^2)$ with $=f(||\mathbf{x}-\mathbf{x'}||^2) = a \cdot ||\mathbf{x}-\mathbf{x'}||^b$.

Our analysis, in which we take account of the unitarity of the evolution, confirms Derrick's analysis in some cases (e.g. there is no stable 3D soliton for the NLS equation with $V^{NL}(\mathbf{x}){=}a \cdot |\Psi(\mathbf{x})|^2$, but for instance, in the case of the 3D single particle Schrödinger-Newton potential, it ensures the stability, of the solitonic ground state, in accordance with Lieb's results.

10 Appendix 3: Factorisation ansatz, and Barut's program.

The factorization ansatz was considered in the past by A.Barut, in the case of a free particle [29]. Here, we reproduce the derivation of L.Bindel [34], which differs from the original one by a factor 2, but is essentially the same. To do so, let us consider the non-relativistic Schrödinger equation for a free particle ($V^L = 0$) in absence of self-interaction ($V^{NL} = 0$):

$$i\hbar\frac{\partial\Psi_L(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_L(\mathbf{x},t) \tag{49}$$

Let us search for a solution of the form $\Psi_L(\mathbf{x},t) = F(\mathbf{x},t) \cdot e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-E\cdot t)}$. Mass-energy imposes, in the non-relativistic limit, that $E = mc^2 + \frac{p^2}{2m}$ so that we obtain two equations, considering the real and imaginary parts of equation (49):

$$\Delta F(\mathbf{x},t) + 2(\frac{m\cdot c}{\hbar})^2 F(\mathbf{x},t) = 0, \tag{50}$$

and

$$\frac{\partial F(\mathbf{x},t)}{\partial t} + (\frac{\mathbf{p}}{m}) \cdot \nabla \cdot F(\mathbf{x},t) = 0. \tag{51}$$

The second one is nothing else than the guidance equation, with a constant velocity $\mathbf{v} = (\frac{\mathbf{p}}{m})$; the first one is the Helmholtz equation and makes appear the Compton length of the particle. In order to ensure the consistency of this approach we must require that F is a purely real function. Then the wave $\Psi_L(\mathbf{x},t)$ is the product of a plane wave with a non-dispersive solution, that is to say a solitary wave which propagates without changing it shape: $F(\mathbf{x},t) = F(\mathbf{x} - \mathbf{v} \cdot t)$.

Surprisingly, this example shows that linear equations possess non-dispersive solutions. The solution considered here should be put in relation with the solutions considered in the section 7.2, that are built by "boosting" a static soliton (here $F(\mathbf{x}, t = 0)$) thanks to a Galilean transformation. It faces the same problem, however: it is difficult to figure out how to extend this result to the case where an external potential is present ($V^L \neq 0$), in which case Galilean invariance is broken. Bindel proposed to incorporate Barut's ideas to a perturbative approach in which the free propagator is expressed in terms of non-dispersive solutions, while the influence of the potential V^L is taken account through a development in series of the propagator, as is often done in the case of a scattering process. This approach is certainly valid in the case of a scattering process where the in and out states are asymptotically free states. However it is difficult to extend it to the case of bound states, like e.g. the case of a coherent state of a harmonic oscillator that we decribed in section 6.2.

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