

## Double-scale theory: Louis de Broglie's statistical wave and matter wave

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RÉSUMÉ. Nous présentons une nouvelle interprétation de la mécanique quantique, la théorie de la double échelle, qui complète l'interprétation standard et la théorie de Broglie-Bohm (dBB). Cette théorie peut apporter une réponse au programme *double solution* de Louis de Broglie. Elle est basée, pour tout système quantique, sur l'existence simultanée de deux fonctions d'onde : une fonction d'onde externe (l'onde statistique) dans le référentiel du laboratoire et une onde interne (l'onde de matière) dans le référentiel du centre de masse.

La fonction d'onde externe correspond à un champ non local qui semble piloter le centre de masse du système quantique. Cette onde s'étale dans l'espace au fil du temps. Mathématiquement, l'équation de Schrödinger converge vers les équations statistiques d'Hamilton-Jacobi lorsque la constante de Planck tend vers zéro et que le centre de masse correspond aux trajectoires dBB.

La fonction d'onde interne correspond à l'onde de matière dont la densité représente la masse volumétrique d'un système quantique étendu. Cette onde reste confinée dans l'espace. Elle converge, lorsque  $\hbar \rightarrow 0$ , vers une distribution de Dirac.

Nous montrons qu'il existe des solutions non stationnaires telles que l'espace de configuration à  $3N$  dimensions de la fonction d'onde interne peut être réécrit comme le produit de  $N$  fonctions d'onde individuelles de matière à 3 dimensions.

*ABSTRACT. We present a new interpretation of quantum mechanics, the double-scale theory, which complements the standard interpretation and the de Broglie-Bohm theory (dBB). This theory may provide an answer to Louis de Broglie's double solution program. It is based, for any quantum system, on the simultaneous existence of two wave functions : an external wave function (the statistical wave) in the laboratory reference frame and an internal wave (the matter wave) in the center-of-mass reference frame.*

*The external wave function corresponds to a non-local field that seems to pilot the center-of-mass of the quantum system. This wave spreads out in space over time. Mathematically, Schrödinger's equation converges to the Hamilton-Jacobi statistical equations when the Planck constant tends towards zero and the center-of-mass corresponds to the DBB trajectories.*

*The internal wave function corresponds to the matter wave whose the density represents the volumetric mass of an extended quantum system. This wave remains confined in space. It converges, when  $\hbar \rightarrow 0$ , to a Dirac distribution.*

*We show that non-stationary solutions can exist such that the  $3N$ -dimensional configuration space of the internal wave function can be rewritten as the product of  $N$  3-dimensional matter individual wave functions.*

## 1 Introduction

"*He has lifted a corner of the great veil*". This was Einstein's reply [1] to Paul Langevin, who had just sent him a manuscript of Louis de Broglie's thesis [2], in which the latter associates two waves with any mobile of mass  $m$  moving at speed  $v$  : "*a phase wave*", which for him is "fictitious" and "non-material", and "*a material wave*" corresponding to the mobile, these two waves being linked by a phase matching. Subsequently, in 1926, de Broglie conjectured that there was an interpretation of quantum mechanics based on a *double solution*, derived from the two preceding waves, but this double solution was a working theory that remained to be defined and refined. At the Solvay Congress in 1927, he presented only a truncated version, the pilot wave theory, which was poorly received and which he abandoned from 1928 until 1952. This pilot-wave theory was rediscovered in 1952 by David Bohm. And since 1952, there have been two research programs, one centred around Bohm and his collaborators [3, 4, 5, 6, 7] on the pilot wave theory, the other around Louis de Broglie and his collaborators [8, 9, 10, 11, 12] on the double solution theory.

In the case of the de Broglie-Bohm's *pilot wave*, the representation of the corpuscle's state corresponds to the association of a well-defined position (of a point particle) and an accompanying pilot wave [ $\psi(\mathbf{x}, t)$ ,  $\mathbf{X}(t)$ ]. In the case of Louis de Broglie's *double solution*, the representation of the corpuscle's state comprises two waves [ $\psi(\mathbf{x}, t)$ ,  $u(\mathbf{x}, t)$ ] as de Broglie pointed out in 1971 [13] (cf. Figure 1) :

*“I introduced, under the name of "theory of the double solution", the idea that it was necessary to differentiate between two solutions, distinct but intimately connected to the wave equation, one of which I called the  $u$ -wave, being a real and non-normalizable physical wave with a local accident defining the particle and represented by a singularity, the other one, Schrödinger’s  $\psi$  wave, normable and without singularity, which would be only a representation of probabilities.” [13]*

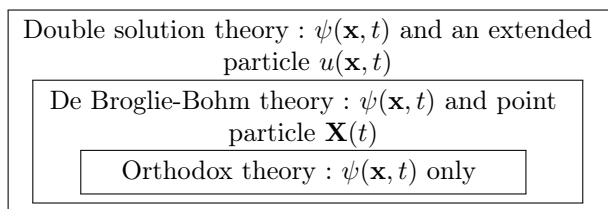


FIGURE 1 – The de Broglie-Bohm theory adds the position  $\mathbf{X}(t)$  of the center-of-mass to the orthodox description of a particle by its wave function  $\psi(\mathbf{x}, t)$  [14]. The double solution theory adds to the dBB description a matter function,  $u(\mathbf{x}, t)$ , for which the center of mass is the dBB position  $\mathbf{X}(t)$ .

Einstein’s reply in a letter of May 1953 to a note by Louis de Broglie [1] on these two programs is very enlightening:

*“The point of view you take in your note seems very clear. You don’t believe, if I understand correctly, in the possibility of adopting the program put forward again by Mr. Bohm:*

- a- *Solution of the Schrödinger equation for a  $\Psi$  field*
- b- *Addition of a trajectory compatible with the  $\Psi$  function.*

*Instead, you propose a representation of physical reality (complete description) of the form*

$$\Psi = \psi \cdot u \tag{1}$$

*This constitutes a form of product in which one of the factors leads to the particle structure and the other to the wave structure. This would in fact be a satisfactory representation of the dual structure imposed on us by experience. It would be a truly new theory, not a complement to the old ones. As*

*far as I can see, do you think that the product must satisfy Schrödinger's initial equation, or must only the "wave" factor possess this property, or both factors, or both factors and their product ?*

*Your goal would also be achieved if the function you're looking for could be represented by a sum of such products. In the end, it doesn't seem necessary for the whole to be represented by a single function (a component), but perhaps by a set of several components. [...] we have in common the idea that the possibility of an entirely objective representation of physical reality must be maintained."*

The aim of this article is to show that a double solution theory can be developed following Louis de Broglie's program as revised by Einstein. As Einstein thought, the solution will be more complex than the simple product of equation (1), and it will be necessary to use "a set of several components".

This article follows the different stages of this completion of quantum mechanics. In section 2, we recall the well-known existence of the two wave functions : the center-of-mass wave function in the laboratory reference frame and the relative wave function in the center-of-mass reference frame. We show that the center-of-mass wave function (external wave) corresponds to  $\psi$ , the probability wave of the corpuscle, and that the relative wave function (internal wave) corresponds to  $u$ , the matter wave of the corpuscle.

In section 3, we show that there are alternative solutions to stationary solutions for the relative wave function. These are time-dependent dynamic solutions associated with each of the  $N$  individual particles of a molecule. The matter wave function  $u$  of a molecule is then equal to the product of these  $N$  individual matter wave functions. This is the answer to one of Louis de Broglie's key questions : unlike Bohm, de Broglie disapproved of configuration spaces. Finally, for the hydrogen atom, the Ehrenfest theorem applied to individual wave functions shows that the centers of mass of these individual matter waves correspond to the electron and proton trajectories of the Bohr atom.

In section 4, we demonstrate that the molecule's center-of-mass wave function corresponds to a field that appears to drive the molecule's center-of-mass. This is Louis de Broglie's  $\psi$  statistical wave function, which has the same properties as the dBB pilot wave.

These two functions have very different physical behaviors : the statistical wave function spreads out over time contrary to the matter wave function which remains confined. To justify the distinction between these two wave functions, we demonstrate that the mathematical convergences are fundamentally different when  $\hbar \rightarrow 0$ . For the statistical wave function, its phase and the square of its modulus converge towards an action and a classical density satisfying the Hamilton-Jacobi statistical equations. For the matter wave function, the square of its modulus converges to a Dirac distribution, and we lose the internal structure of the molecule.

In section 5, we generalize previous results, showing that any elementary particle, such as an electron, two waves (in the laboratory reference frame) can also be associated : a statistical wave and a matter wave.

In section 6, we show that the matter wave function carries all the energy and momentum of the individual particles in the molecule. We also show how the simultaneous existence of these two functions allows us to understand and explain simply many phenomena of quantum mechanics, such as wave-corpuscle duality, quantum measurements and the nonlocality of the EPR-B experiment.

## 2 Center-of-mass wave and relative wave of molecules

It is well known, in all quantum mechanic textbooks, that the total wave function can be decomposed into a *center-of-mass wave function* in the laboratory reference frame and a *relative wave function* in the center-of-mass reference frame.

Let us consider a quantum system (atom or molecule) of  $N$  spinless particles of masses  $m_j$  and charges  $q_j$  (with  $j = 1..N$ ), of coordinates  $\mathbf{x}_j$ , subjected to an external gravitational field  $V_g(\mathbf{x}_j)$  and a zero external electric field, and to mutual interactions described by the potentials  $U_{ji}(|\mathbf{x}_j - \mathbf{x}_i|), \forall i = 1..N, j \neq i$ . This quantum system is then described by the wave function  $\Psi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t)$  which verifies the Schrödinger equation:

$$i\hbar \frac{\partial \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)}{\partial t} = H \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t) \quad (2)$$

with the Hamiltonian:

$$H = \sum_{j=1}^N \left( \frac{\mathbf{P}_j^2}{2m_j} + m_j V_g(\mathbf{x}_j) \right) + \sum_{j=1}^N \sum_{\substack{i=1 \\ i \neq j}}^N U_{ji}(|\mathbf{x}_j - \mathbf{x}_i|) \quad (3)$$

and the initial condition :

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, 0) = \Psi_0(\mathbf{x}_1, \dots, \mathbf{x}_N). \quad (4)$$

We separate the motion of these  $N$  particles from the motion of their center-of-mass: let  $\mathbf{x}_G = (\sum_{j=1}^N m_j \mathbf{x}_j) / (\sum_{j=1}^N m_j)$  be the position of the center-of-mass,  $\mathbf{x}'_j = \mathbf{x}_j - \mathbf{x}_G$  be the relative position of the  $j$  particle and  $M = \sum_{j=1}^N m_j$  be the total mass.

Then the Hamiltonian  $H$  is written as a function of the total impulse ( $\mathbf{p}_G = \sum_{j=1}^N \mathbf{p}_j$ ) and relative impulses ( $\mathbf{p}'_j = \mathbf{p}_j - m_j/M \mathbf{p}_G$ ) taking into account small variations of the gravitational field  $V_g(\mathbf{x}_j) \simeq V_g(\mathbf{x}_G) + \mathbf{x}'_j \nabla V_g(\mathbf{x}_G)$ :

$$\begin{aligned} H &= \left( \frac{\mathbf{p}_G^2}{2M} + M V_g(\mathbf{x}_G) \right) + \sum_{i=1}^N \left( \frac{\mathbf{p}'_i{}^2}{2m_i} + \sum_{\substack{j=1 \\ j \neq i}}^N U_{ij}(|\mathbf{x}'_i - \mathbf{x}'_j|) \right) \quad (5) \\ &:= H^{ext} + \sum_{i=1}^N H_i^{int}. \end{aligned}$$

**Remark 1** - *By introducing a link between the centre of mass and the relative positions, the previous decomposition does not preserve the  $3N$  degrees of freedom of the problem, as the Jacobi coordinates do, but they are very cumbersome to use [15]. In the following, we assume that this link is weak and that inertial terms such as Coriolis and centrifugal can be neglected [16]. This is particularly the case for atoms or molecules that allow the Born-Oppenheimer approximation. This is the framework for the discussion of adiabatic representations with dynamic couplings in molecular physics [16].*

*The interest of this hypothesis is methodological rather than computational. It will make it possible to show that there is a double solution in certain cases, even if the total wave function is not the product of these two solutions, as Einstein thought in the end of his letter to de Broglie. This hypothesis is an approximation that will simplify the analysis of wave functions.*

**Proposition 1** - *If the initial wave function  $\Psi_0(\mathbf{x}_1, \dots, \mathbf{x}_N)$  factorizes to the form:*

$$\Psi_0(\mathbf{x}_1, \dots, \mathbf{x}_N) = \psi_0(\mathbf{x}_G) \varphi_0(\mathbf{x}'_1, \dots, \mathbf{x}'_N). \quad (6)$$

then  $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)$ , a solution to (2),(3) and (4), is written as the product of the center-of-mass wave function  $\psi(\mathbf{x}_G, t)$  and the relative function  $\varphi(\mathbf{x}'_1, \dots, \mathbf{x}'_N, t)$  :

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t) = \psi(\mathbf{x}_G, t)\varphi(\mathbf{x}'_1, \dots, \mathbf{x}'_N, t) \quad (7)$$

where  $\psi(\mathbf{x}_G, t)$  is the solution to the Schrödinger equations of the center-of-mass:

$$i\hbar \frac{\partial \psi(\mathbf{x}_G, t)}{\partial t} = -\frac{\hbar^2}{2M} \Delta_{\mathbf{x}_G} \psi(\mathbf{x}_G, t) + MV_g(\mathbf{x}_G)\psi(\mathbf{x}_G, t) \quad (8)$$

with the initial condition:

$$\psi(\mathbf{x}_G, 0) = \psi_0(\mathbf{x}_G) \quad (9)$$

and where  $\varphi(\mathbf{x}'_1, \dots, \mathbf{x}'_N, t)$  is the Schrödinger relative equations:

$$i\hbar \frac{\partial \varphi(\mathbf{x}'_1, \dots, \mathbf{x}'_N, t)}{\partial t} = -\sum_{i=1}^N \left( \frac{\hbar^2}{2m_i} \Delta_{\mathbf{x}'_i} + \sum_{\substack{j=1 \\ j \neq i}}^N U_{ij}(|\mathbf{x}'_i - \mathbf{x}'_j|) \right) \varphi(\mathbf{x}'_1, \dots, \mathbf{x}'_N, t) \quad (10)$$

with the initial condition:

$$\varphi(\mathbf{x}'_1, \dots, \mathbf{x}'_N, 0) = \varphi_0(\mathbf{x}'_1, \dots, \mathbf{x}'_N). \quad (11)$$

This proposition shows the relevance of Einstein's hypothesis for molecules for which there is simultaneous existence of both wave functions and their product.

**Remark 2** - In classical mechanics, we find the simultaneous existence of two densities, one statistical density and one mass density. Consider, for example, the simple case of a solid in  $R^3$ . We assume that the initial position of its center of mass  $X(0)$  is not known precisely, but that we only know  $\rho_0^p(\mathbf{x})$  the **probability density** that the center of mass is in  $\mathbf{x}$  at the initial instant:  $\rho_0^p(\mathbf{x}) = P(X(0) = \mathbf{x})$ . The solid is assumed to have a velocity  $\mathbf{v}_0$  at the initial instant, and to be subject to a constant force (gravity)  $\mathbf{g}$ . So  $X(t) = X(0) + \mathbf{v}_0 t + 1/2gt^2$ . From this we deduce that at time  $t$  the probability density of the center of mass being present in  $\mathbf{x}$  is equal to  $\rho^p(\mathbf{x}, t) = P(X(t) = \mathbf{x}) = \rho_0^p(\mathbf{x} - X(t))$ . If the solid is real and not a point, we must associate with it at the initial instant

a mass density  $\rho_0^m(\mathbf{x})$ . At time  $t$ , the mass density in  $\mathbf{x}$  will be equal to  $\rho^m(\mathbf{x}, t) = \rho_0^m(\mathbf{x} - \mathbf{v}_0 t - 1/2\mathbf{g}t^2)$ . In classical mechanics, there are already two types of density: statistical and volumic mass. In classical mechanics, we forget the mass density by considering the particle as a point, i.e. by implicitly taking the mass density concentrated at the center of mass with a Dirac function. It is these two different types of density that we are extending to quantum mechanics.

### 3 Louis de Broglie matter-wave functions

To simplify notation in this section, we will note  $\mathbf{x}_j$  (and not  $\mathbf{x}'_j$ ) the variable linked to particle  $j$ .

By studying the equations (10-11) of the relative wave function, we will show how Louis de Broglie's matter waves can be introduced mathematically. The usual interpretation is that the relative wave is a stationary solution to the form:

$$\varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t) = \varphi_0(\mathbf{x}_1, \dots, \mathbf{x}_N) e^{-i \frac{E_p t}{\hbar}}. \quad (12)$$

And all quantum mechanics textbooks solve the eigenvalue and eigenvector equation to determine  $\varphi_0(\mathbf{x})$ :

$$-\sum_{i=1}^N \left( \frac{\hbar^2}{2m_i} \Delta_{\mathbf{x}'_i} + \sum_{\substack{j=1 \\ j \neq i}}^N U_{ij}(|\mathbf{x}_i - \mathbf{x}_j|) \right) \varphi_0(\mathbf{x}_1, \dots, \mathbf{x}_N) = E \varphi_0(\mathbf{x}_1, \dots, \mathbf{x}_N). \quad (13)$$

Indeed, it is not necessary to know the initial condition (11): it is obtained by solving the eigenvalue equation (13).

However, there is another solution to Schrödinger's relative equations (10-11) that seems not to have been taken into account in the literature. This is a time-dependent dynamic solution. It is based on the assumption that the initial wave function (11) can be written as a product and that there are  $N$  functions  $\varphi_0^j(\mathbf{x})$  such that:

$$\varphi_0(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{j=1}^N \varphi_0^j(\mathbf{x}_j) \quad (14)$$

This is a natural assumption for individual matter waves (particles) which, unlike molecular orbitals, do not overlap.

The existence of such a dynamical solution is consistent with Einstein's method of "logical simplicity". We will see in Proposition 2 that



this assumption for initial condition (14) allows us to write the internal wave function, for any  $t$ , in the form of the product:

$$\varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t) = \prod_{j=1}^N \varphi^j(\mathbf{x}_j, t) \quad (15)$$

where each of the  $N$  individual particles is associated with an individual time-dependent wave function  $\varphi^j(\mathbf{x}_j, t)$  in the center-of-mass reference frame of the molecule such that  $|\varphi^j(\mathbf{x}_j, t)|^2$  represents the voluminal mass (density) of particle  $j$  at time  $t$  in the center-of-mass reference frame.

The existence of such a solution is made possible by the orders of magnitude between the size of the different wave functions: Bohr radius  $a_0$  for the molecular orbitals of the hydrogen atom compared to the Compton length  $\alpha a_0$  or the classical electron radius  $\alpha^2 a_0$  for the physical size of an extended electron.

This assumption considers extended particles, with individual wave functions  $\varphi^i(\mathbf{x}, t)$  that do not overlap and that therefore have disjoint supports:  $\varphi^j(\mathbf{x}, t)\varphi^i(\mathbf{x}, t) = 0$  for all  $i \neq j$ .

This assumption is not respected for the stationary wave functions of the electrons of an atom because the orbitals (molecule size) can overlap.

**Remark 3** - *The hypothesis of an extended particle requires the introduction of forces to maintain the cohesion of the particle. A model of the extended and deformable electron is proposed by Poincaré in his famous Palermo memoir [17]. Dirac makes the same argument in his article of 1962 “An extensible model of the electron” [18]. It is also the basis of Delmelt’s work on the size of the electron [19].*

The spatial extension of the electron is also consistent with the introduction of cut-offs in quantum electrodynamics (QED) to eliminate the infinities due to the hypothesis of point particles.

**Proposition 2** - *If the  $N$  individual time-dependent wave functions  $\varphi^j(\mathbf{x}_j, t)$  are solutions to the  $N$  individual non linear Schrödinger equations:*

$$i\hbar \frac{\partial \varphi^j(\mathbf{x}_j, t)}{\partial t} = -\frac{\hbar^2}{2m_j} \Delta_{\mathbf{x}_j} \varphi^j(\mathbf{x}_j, t) + \left( \sum_{\substack{i=1 \\ i \neq j}}^N U_{ji}(|\mathbf{x}_j - \mathbf{x}^i(t)|) \right) \varphi^j(\mathbf{x}_j, t) \quad (16)$$

$$\mathbf{x}^i(t) = \int \mathbf{x} |\varphi^i(\mathbf{x}, t)|^2 d\mathbf{x} \quad (17)$$

with the initial conditions :

$$\varphi^j(\mathbf{x}_j, 0) = \varphi_0^j(\mathbf{x}_j), \quad (18)$$

then the function :

$$\varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t) = \prod_{j=1}^N \varphi^j(\mathbf{x}_j, t) \quad (19)$$

is an approximative solution to the relative Schrödinger equation (10) with the initial condition (14).

**Demonstration:** We multiply equation (16) by  $\prod_{i \neq j} \varphi^i(\mathbf{x}_i, t)$  and sum these equations for all  $j = 1..N$ . First, we have  $\frac{\partial}{\partial t} \prod_j \varphi^j(\mathbf{x}_j, t) = \sum_j \frac{\partial}{\partial t} \varphi^j(\mathbf{x}_j, t) \prod_{i \neq j} \varphi^i(\mathbf{x}_i, t)$ . Secondly, taking into account that the functions  $\varphi^i(\mathbf{x}, t)$  have disjoint supports, we obtain for all  $i = 1..N$  different from  $j$ ,  $\nabla_{\mathbf{x}_i} \varphi^j(\mathbf{x}_j, t) = 0$ . Then :

$$\begin{aligned} \nabla_{\mathbf{x}_i} \prod_j \varphi^j(\mathbf{x}_j, t) &= \nabla_{\mathbf{x}_i} \varphi^i(\mathbf{x}_i, t) \prod_{j \neq i} \varphi^j(\mathbf{x}_j, t) + \nabla_{\mathbf{x}_i} \varphi^j(\mathbf{x}_j, t) \prod_{k \neq i, j} \varphi^k(\mathbf{x}_k, t) \\ &= \nabla_{\mathbf{x}_i} \varphi^i(\mathbf{x}_i, t) \prod_{j \neq i} \varphi^j(\mathbf{x}_j, t). \end{aligned}$$

We deduce :

$$\begin{aligned} \Delta_{\mathbf{x}_i} \prod_j \varphi^j(\mathbf{x}_j, t) &= \Delta_{\mathbf{x}_i} \varphi^i(\mathbf{x}_i, t) \prod_{j \neq i} \varphi^j(\mathbf{x}_j, t) + \nabla_{\mathbf{x}_i} \varphi^j(\mathbf{x}_j, t) \prod_{k \neq i, j} \varphi^k(\mathbf{x}_k, t) \\ &= \Delta_{\mathbf{x}_i} \varphi^i(\mathbf{x}_i, t) \prod_{j \neq i} \varphi^j(\mathbf{x}_j, t). \end{aligned}$$

Finally the function  $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)$  satisfies the equation :

$$i\hbar \frac{\partial \varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)}{\partial t} = - \sum_{i=1}^N \left( \frac{\hbar^2}{2m_i} \Delta_{\mathbf{x}_i} + \sum_{\substack{i=1 \\ j \neq i}}^N U_{ji}(|\mathbf{x}_j - \mathbf{x}^i(t)|) \right) \varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t). \quad (20)$$

Because the size of each particle  $i = 1..N$  (i.e. the support of  $\varphi^i$ ) is very small compared to the size of a atom or molecule, we can assume that we have for all  $t$ ,  $|\varphi^i(\mathbf{x}, t)|^2 \simeq \delta(\mathbf{x} - \mathbf{x}^i(t))$  where  $\mathbf{x}^i(t)$  of equation (17) is the position of the center-of-mass of particle  $i$  in the center-of-mass reference frame de la molecule. In this case, we can replace in equation (10)  $U_{ji}(|\mathbf{x}_j - \mathbf{x}_i|)$  by  $U_{ji}(|\mathbf{x}_j - \mathbf{x}^i(t)|)$  and so the relative wave function satisfies equation (10).  $\square$

**Remark 4** - *With this result, the matter wave function  $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)$  in the  $3N$ -configuration space is replaced by the product  $\prod_j \varphi^j(\mathbf{x}_j, t)$  of the  $N$  individual matter wave functions in the real 3-space.*

**Remark 5** - *The initial conditions (18) are not known in practice. As  $|\varphi_0^j(\mathbf{x})|^2$  corresponds to the initial mass density of the particle  $j$  in  $\mathbf{x}$ , it satisfied the equation*

$$m_j = \int |\varphi_0^j(\mathbf{x})|^2 d\mathbf{x}. \quad (21)$$

The  $N$  individual matter wave functions can be written  $\varphi^j(\mathbf{x}, t) = \sqrt{\rho^j(\mathbf{x}, t)} \exp(\frac{iS^j(\mathbf{x}, t)}{\hbar})$ . With the change of variables, we deduce from Schrödinger equation (16):

$$\forall i = 1 \dots N, \quad \frac{\partial \rho^j}{\partial t}(\mathbf{x}, t) + \nabla \cdot \left( \rho^j(\mathbf{x}, t) \frac{\nabla S^j(\mathbf{x}, t)}{m_j} \right) = 0. \quad (22)$$

**Proposition 3** - *If we give at time  $t$  at the point  $\mathbf{x}$  of the particle  $j$ , the velocity :*

$$\mathbf{v}_j(\mathbf{x}, t) = \frac{\nabla S^j(\mathbf{x}, t)}{m_j}, \quad (23)$$

*then we obtain (from equation (22)) the equation of conservation of the mass :*

$$\frac{\partial \rho^j}{\partial t}(\mathbf{x}, t) + \nabla \cdot (\rho^j(\mathbf{x}, t) \mathbf{v}_j(\mathbf{x}, t)) = 0. \quad (24)$$

*Moreover, we can deduce :*

$$\int |\varphi^j(\mathbf{x}, t)|^2 d\mathbf{x} = \int |\varphi_0^j(\mathbf{x})|^2 d\mathbf{x} = m_j. \quad (25)$$

The conservation of the mass is obtained with the velocity of equation (23).

With the femto-seconde then atto-seconde light pulse technique developed by l'Huillier and Agostini, such a solution has been observed experimentally for the trajectories of atoms [20], and is announced for electrons [21].

**Remark 6** *The existence of such a solution can also be assumed theoretically from an extension of Floquet's theorem. This shows that if the*

*Hamiltonian is periodic with period  $T$ , there are solutions to the Schrödinger equation of the form  $\phi(\mathbf{x}, t) = \varphi(\mathbf{x}, t)e^{-i\frac{Et}{\hbar}}$  where  $\varphi(\mathbf{x}, t)$  is periodic with period  $T$ . We assume the validity of extending this theorem to the case of a stationary (i.e. periodic for all values of  $T$ ) and multivariate Hamiltonian.*

We now show that the individual relative Schrödinger equations (16-19) can be approximated, at each time  $t$ , using the positions of the centers of mass  $\mathbf{x}^i(t)$  of the individual particle and using the  $N$  potentials  $V_j(\mathbf{x}, t) = \sum_{\substack{i=1 \\ i \neq j}}^N U_{ji}(|\mathbf{x} - \mathbf{x}^i(t)|)$

The approximate calculation of equations (16-20) can be obtained from Ehrenfest's theorem applied simultaneously to  $N$  individual matter wave functions.

Let us recall that it relates the time derivative of the expectation values of the position and momentum operators  $\mathbf{x}$  and  $\mathbf{p}$  to the expectation value of the force  $F_j = -\nabla_{\mathbf{x}}V_j(\mathbf{x}, t)$  on a mass particle moving in scalar potentials  $V_j(\mathbf{x}, t)$ :

$$m \frac{d}{dt} \langle \mathbf{x} \rangle_j(t) = \langle \mathbf{p} \rangle_j(t), \quad \frac{d}{dt} \langle \mathbf{p} \rangle_j(t) = -\langle \nabla_{\mathbf{x}} V_j(\mathbf{x}, t) \rangle_j \quad (26)$$

where the expectation values are calculated for the particle  $j$  with the matter wave function  $\varphi^j$ ,  $\langle \mathbf{x} \rangle_j = \int \mathbf{x} |\varphi^j(\mathbf{x}, t)|^2 d\mathbf{x} = \mathbf{x}^j(t)$  and not with the total wave function  $\Psi$ . We then obtain:

$$\begin{aligned} \int \mathbf{x}_j |\varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)|^2 dx^1 \dots dx^N &= \prod_{i \neq j} \int |\varphi^i(\mathbf{x}_i, t)|^2 d\mathbf{x}_i \cdot \int \mathbf{x}_j |\varphi^j(\mathbf{x}_j, t)|^2 d\mathbf{x}_j \\ &= \prod_{i \neq j} m_i \cdot \langle \mathbf{x} \rangle_j \end{aligned} \quad (27)$$

By applying Ehrenfest's theorem to the  $N$  individual equations (16-18), we deduce Newton's equations for the mass centers  $\langle \mathbf{x} \rangle_j = \mathbf{x}^j(t)$  of the  $N$  particles:

$$m_j \frac{d^2}{dt^2} \mathbf{x}^j(t) = -\nabla V_j(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{x}^j(t)} \quad (28)$$

with initial conditions that depend on the initial conditions of the matter

individual wave functions  $\varphi_0^j(\mathbf{x})$  :

$$\mathbf{x}^j(0) = \int \mathbf{x} |\varphi_0^j(\mathbf{x})|^2 d\mathbf{x} \quad \text{and} \quad \frac{d}{dt} \mathbf{x}^j(0) = \frac{1}{m_j} \int \varphi_0^{*j}(\mathbf{x}) (-i\hbar \nabla \varphi_0^j(\mathbf{x})) d\mathbf{x}. \quad (29)$$

The initial conditions (29) are not known. Nonetheless, if we do not take the spin into account, we can conclude with the following proposition.

**Proposition 4** - *For a set of  $N$  particles (atomic nuclei and electrons), the evolution of the center-of-mass of the  $N$  particles is strongly approximated by Newton's equations (28), similarly to a  $N$ -body classical problem.*

Newton's equations (28) do not use Planck's constant  $h$ , only the initial conditions (29). The behaviour of the  $N$  particles is therefore classic.

In the case of the hydrogen atom, in the Born-Oppenheimer approximation (with  $m_p$  infinite), equations (28) are sufficient to find Bohr trajectories of the electron [22].

**Remark 7** - *The Bell's beables [23] are also defined by Norsen [7, 24] using the conditional wave functions introduced by Dürr, Goldstein and Zanghi [25, 26]. From the time-dependent function of  $N$  particles  $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)$  in the  $3N$ -configuration space and from the  $N$  positions  $X_i(t)$  of these particles calculated with the pilot-wave theory, Norsen defines the  $N$  individual waves*

$$\Phi_i(\mathbf{x}_i, t) = \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t) \Big|_{\substack{\mathbf{x}_j = X_j(t) \\ j \neq i}} \quad (30)$$

*These wave functions  $\Phi_i(\mathbf{x}, t)$  are proportional to the individual time-dependent matter waves. Indeed, if  $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N, t)$  satisfies equation (19), we deduce :*

$$\Phi_i(\mathbf{x}_i, t) = \varphi^i(\mathbf{x}_i, t) \prod_{j \neq i} \varphi^j(X_j(t), t) = \varphi^i(\mathbf{x}_i, t) g_i(t), \quad (31)$$

#### 4 Louis de Broglie statistical-wave functions

In section 2, we introduced two types of wave function for each particle in a molecule, one external and one internal. In section 3, we studied the internal wave function associated with each matter wave. In

this section, we study the external wave function associated with the center-of-mass.

To interpret this wave function, let us study its convergence to classical mechanics when we tends the Planck's constant towards zero.

Let us consider a center-of-mass wave function verifying the Schrödinger equations (8) and (9) and make the semi-classical change of variables  $\psi(\mathbf{x}_G, t) = \sqrt{\rho^{\hbar}(\mathbf{x}_G, t)} \exp\left(i \frac{S^{\hbar}(\mathbf{x}_G, t)}{\hbar}\right)$ . The density  $\rho^{\hbar}(\mathbf{x}_G, t)$  and the action  $S^{\hbar}(\mathbf{x}_G, t)$  then verify the Madelung equations [27] (1926):

$$\begin{aligned} \forall(\mathbf{x}_G, t) \in \mathbb{R}^3 \times \mathbb{R}^+, \\ \frac{\partial S^{\hbar}(\mathbf{x}_G, t)}{\partial t} + \frac{(\nabla S^{\hbar}(\mathbf{x}_G, t))^2}{2M} + V(\mathbf{x}_G) - \frac{\hbar^2}{2M} \frac{\Delta \sqrt{\rho^{\hbar}(\mathbf{x}_G, t)}}{\sqrt{\rho^{\hbar}(\mathbf{x}_G, t)}} = 0 \end{aligned} \quad (32)$$

$$\begin{aligned} \forall(\mathbf{x}_G, t) \in \mathbb{R}^3 \times \mathbb{R}^+, \\ \frac{\partial \rho^{\hbar}(\mathbf{x}_G, t)}{\partial t} + \operatorname{div} \left( \rho^{\hbar}(\mathbf{x}_G, t) \frac{\nabla S^{\hbar}(\mathbf{x}_G, t)}{m} \right) = 0 \end{aligned} \quad (33)$$

with the initial conditions :

$$\rho^{\hbar}(\mathbf{x}_G, 0) = \rho_0^{\hbar}(\mathbf{x}_G) \quad \text{and} \quad S^{\hbar}(\mathbf{x}_G, 0) = S_0^{\hbar}(\mathbf{x}_G) \quad \forall \mathbf{x}_G \in \mathbb{R}^3. \quad (34)$$

Here,  $V(\mathbf{x}_G) = MV_g(\mathbf{x}_G)$  is the potential of (8). Let us now study the convergence of the density  $\rho^{\hbar}(\mathbf{x}_G, t)$  and the action  $S^{\hbar}(\mathbf{x}_G, t)$  of the Madelung equations, when the Planck constant  $\hbar$  tends to 0. We will restrict ourselves to “**prepared non-discerned quantum systems**”.

**Definition 1** - *A quantum system, subjected to a potential  $V(\mathbf{x})$ , is said to be a **prepared non-discerned** quantum systems, if we know at the initial time, only the density of the initial probability  $\rho_0^{\hbar}(\mathbf{x}_G)$  and the initial action  $S_0^{\hbar}(\mathbf{x}_G)$  of its center-of-mass wave function, and that these are functions  $\rho_0(\mathbf{x}_G)$  and  $S_0(\mathbf{x}_G)$  which are independent of  $\hbar$ .*

It is the case of a set of particles without interaction between each other and prepared in the same way: sources of free particles or in a linear field as in the Shimizu [28] experiment with cold atoms, sources of fullerenes, neutrons, electrons or  $C_{60}$  in a Young's slit experiment. One can then demonstrate the following proposition :

**Proposition 5** [29] - When  $\hbar$  tends to 0, if the center-of-mass wave function is prepared non-discerned, the density  $\rho^{\hbar}(\mathbf{x}_G, t)$  and the action  $S^{\hbar}(\mathbf{x}_G, t)$ , solutions to Madelung equations (32-34), converge to  $\rho(\mathbf{x}_G, t)$  and  $S(\mathbf{x}_G, t)$ , solutions to Hamilton-Jacobi statistical equations:

$$\frac{\partial S(\mathbf{x}_G, t)}{\partial t} + \frac{(\nabla S(\mathbf{x}_G, t))^2}{2m} + V(\mathbf{x}_G) = 0 \quad (35)$$

$$S(\mathbf{x}_G, 0) = S_0(\mathbf{x}_G) \quad (36)$$

$$\frac{\partial \rho(\mathbf{x}_G, t)}{\partial t} + \text{div} \left( \rho(\mathbf{x}_G, t) \frac{\nabla S(\mathbf{x}_G, t)}{m} \right) = 0 \quad (37)$$

$$\rho(\mathbf{x}_G, 0) = \rho_0(\mathbf{x}_G) \quad (38)$$

To interpret the center-of-mass wave function, we use the mathematical continuity between the density and action of quantum mechanics and the density and action of classical mechanics, solutions to the Hamilton-Jacobi statistical equations. The **classical particles** which verify the Hamilton-Jacobi statistical equations have two properties:

- They are prepared non-discerned because we do not know at the initial time the position of their centers of mass, but only their initial distribution  $\rho_0(\mathbf{x}_G)$ . In classical mechanics, we remove the indeterminacy by adding the initial position of the center-of-mass  $X_G(0)$ .
- They are piloted by the gradient of the Hamilton-Jacobi action, which corresponds to a field that *pilots* the center-of-mass with the equation :

$$\mathbf{v}(\mathbf{x}_G, t) = \frac{\nabla S(\mathbf{x}_G, t)}{m} \quad (39)$$

**The mathematical continuity** incites a continuity of interpretation by extending these two properties to the centers of mass of quantum particles that verify the Schrödinger equations of the external wave function:

- They are prepared non-discerned because we do not know at the initial time the position of their centers of mass, but only their initial distribution  $\rho_0(\mathbf{x}_G)$ . As in classical mechanics, we remove the indeterminacy by giving the initial position of the center-of-mass  $X_G^{\hbar}(0)$ .
- The centers-of-mass of quantum systems are guided by the gradient of the quantum action, which corresponds to a field that

“pilots” the center-of-mass by the equation:

$$\mathbf{v}^h(\mathbf{x}_G, t) = \frac{1}{m} \nabla S^h(\mathbf{x}_G, t) \quad (40)$$

Thus, the external wave function *pilots* the centers of mass of the quantum systems and corresponds to the **De Broglie-Bohm interpretation** [14, 30] restricted to the center-of-mass wave function only. It is therefore necessary to add the initial position  $X_G^h(0)$  of the center-of-mass as well as the equation that gives its evolution:

$$\frac{dX_G^h(t)}{dt} = \frac{1}{m} \nabla S^h(\mathbf{x}_G, t)|_{\mathbf{x}_G=X_G^h(t)} \quad (41)$$

Thus, the wave function of a molecule can be written as a product of two types of wave function: the wave function of the molecule’s center-of-mass in the laboratory frame of reference, which we have called the statistical wave function, and two relative wave functions defined in the center-of-mass frame of reference, the usually used stationary solution and a time-dependent dynamic solution composed of the product of  $N$  individual mass wave functions associated with the  $N$  elementary particles composing and interacting in the molecule. It is as if the statistical wave function guides the center-of-mass of the quantum molecule in the same way as the Hamilton-Jacobi action guides the center-of-mass of the classical particle.

**Definition 2** *We will say that two particles are prepared in the same way if they have the same statistical wave function.*

In the reference frame of a molecule’s center of mass, we have defined the relative time-dependent wave function of each of its elementary particles, as well as the product of the molecule’s  $N$  individual particles.

**Definition 3** *If we denote  $\varphi_R^j(\mathbf{x}, t)$  the relative time-dependent wave function of elementary particle  $j$  in the reference frame of the molecule’s center of mass, which we denoted  $\varphi^j(\mathbf{x}, t)$  in the previous section, then this function is written in the laboratory reference frame:*

$$\varphi_L^j(\mathbf{x}, t) = \varphi_R^j(\mathbf{x} - \mathbf{x}_G(t), t). \quad (42)$$

The center-of-mass  $\mathbf{x}_L^j(t)$  of the particle  $j$  in the laboratory reference is written:

$$\mathbf{x}_L^j(t) = \int \mathbf{x} |\varphi_L^j(\mathbf{x}, t)|^2 d\mathbf{x} = \mathbf{x}_R^j(t) + \mathbf{x}_G(t) \quad (43)$$



where  $\int \mathbf{x} |\varphi_R^j(\mathbf{x}, t)|^2 d\mathbf{x} = \mathbf{x}_R^j(t)$  (the center-of-mass of the particle  $j$  in the center-of-mass reference frame of the molecule noted  $\mathbf{x}^j(t)$  in the last section).

We have also :

$$\mathbf{x}_G(t) = \sum_j \frac{m_j \mathbf{x}_L^j(t)}{M} \quad (44)$$

We therefore obtain statistical and mass wave functions in the laboratory reference frame.

## 5 Statistical and matter waves of an elementary particle

We generalize the results to molecules by assuming that any elementary particle, such as an electron, can also be associated with two waves in the laboratory reference frame : a probability wave  $\psi(\mathbf{x}, t)$  and a matter wave  $\varphi_L(\mathbf{x}, t)$ . The statistical wave corresponds to Louis de Broglie's wave "which would be only a representation of probabilities" [13]. As for the molecule, we assume that the statistical wave pilots the center of gravity of the matter wave by equation (40) and that we have the correspondence :

$$\mathbf{x}_L(t) = \int \mathbf{x} |\varphi_L^j(\mathbf{x}, t)|^2 d\mathbf{x} = \mathbf{x}_G(t) \quad (45)$$

. where  $\mathbf{x}_L(t)$  is the center-of-mass of the matter wave  $\varphi_L(\mathbf{x}, t)$  and where  $\mathbf{x}_G(t)$  is not the center of mass of the statistical wave function, but the position obtained from equation (39) and the initial position  $\mathbf{x}_G(0)$ .

It is important to note that these two wave functions are different, as they behave very differently. The support of the statistical wave function spreads out in time, while that of the matter wave function remains bounded. More precisely, if we denote  $\sigma_h^{sta}(\mathbf{x}, t)$  the density of the statistical wave function  $\psi(\mathbf{x}, t)$ , and  $\sigma_h^{mas}(\mathbf{x}, t)$  that of the matter wave function  $\varphi_L(\mathbf{x}, t)$ , then  $\sigma_h^{sta}(\mathbf{x}, t)$  tends towards 0 as  $t$  increases, while  $\sigma_h^{mas}(\mathbf{x}, t)$  remains bounded and can be considered a singularity if  $h$  is made to tend towards 0.

There are at least two reasons for this difference in densities. The first is presented in Remark 3, where we made Poincaré's assumption that the matter particle was subject to a confining force. The second reason is presented in Remark 1 where the initial conditions do not have

the same interpretation:  $\sigma_h^{sta}(\mathbf{x}, 0)$  is the probability density of the statistical wave function, i.e. is the probability of finding the center of mass in  $\mathbf{x}$  at the initial instant, whereas  $\sigma_h^{mat}(\mathbf{x}, 0)$  is the mass density of the particle in  $\mathbf{x}$  at the initial instant. This difference between the densities of these two wave functions is perhaps theoretically due to the fundamental difference in discernibility between these two types of wave function: the statistical wave function corresponds to indiscernible particles, while the matter wave function corresponds to discernible particles. This is reflected in the difference found in sections 3 and 4 for the convergence of these two wave functions when Planck's constant is mathematically tended towards 0: the statistical wave function converges to the action and density of classical particles verifying the Hamilton-Jacobi statistical equations (section 4), while the matter wave function (or functions) converges to a singularity (section 3).

**Remark 8** - *We have called the external wave function the statistical wave and the internal wave function the matter wave. This can lead to confusion due to the terminology; in fact the wave function usually used in quantum mechanics in the two-slit or Stern and Gerlach experiments is the external function, which is generally called the matter wave (of de Broglie) in the literature.*

**Remark 9** *The extended particle model we defined in [31] confirms the existence of these two wave functions, statistical and matter, for an elementary spin 1/2 particle like the electron. The model corresponds to the description of the particle extended by an elastic string that vibrates, alternating at each period a kind of creation process followed by an annihilation process. It corresponds to a process à la Nelson, but unlike the latter, it is a deterministic, non-point process that is pseudo-random, having the same statistical characteristics as his stochastic process. Thanks to a generalization of the principle of least action to extended particles, we have associated it, as de Broglie did in his thesis, with a wave function that verifies the Schrödinger equation and whose phase is a complex action verifying a complex second-order Hamilton-Jacobi equation. We then demonstrate that the center of gravity of this extended particle follows the trajectories of the de Broglie-Bohm interpretation [31].*

**Remark 10** *In the case of a free particle satisfying Schrödinger's equation  $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi$ , we can look for an explicit solution of the form*

$\Psi(\mathbf{r}, t) = f(\mathbf{r}, t)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)}$  where  $\mathbf{p} = m\mathbf{v}$  is the impulse and  $E$  is the total energy ( $E = mc^2 = m_0c^2 + \frac{\mathbf{p}^2}{2m_0}$ ). We find the two following equations [32, 33]:

$$\Delta f + 2\left(\frac{m_0c}{\hbar}\right)^2 f = 0, \quad \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_0} \cdot \nabla f = 0 \quad (46)$$

The first equation is a Helmholtz equation in which the Compton length gives the scale. The second equation shows that the function  $f$  is of the form  $f(\mathbf{r}, t) = f(\mathbf{r} - \mathbf{v}t)$  with  $\mathbf{v} = \mathbf{p}/m_0$ . In this product, we obtain a description both particle-like ( $f$  is the matter wave term) and wave-like ( $e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)}$  is the plane wave term). If at the initial instant we note  $f(\mathbf{r}, 0) = g(\mathbf{r} - \mathbf{r}_0)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r})}$ , then  $\Psi(\mathbf{r}, t) = g(\mathbf{r} - \mathbf{r}_0 - \mathbf{v}t)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)}$  is the solution at instant  $t$ .

## 6 Conservation of the matter wave and role of statistical waves

The individual matter waves of a quantum system, molecule or elementary particle, correspond to a matter wave that is real. We can therefore assume that such waves carry all the energy and momentum of the individual particle.

The individual matter wave function therefore appears to be the invariant that conserves itself in quantum reactions, leading to the hypothesis of energy and momentum conservation in elastic particle collisions. The impact of the matter wave on a measuring screen measures the position of the molecule's center of mass. It can therefore be said that it is the matter wave that drives the reduction of the (statistical) wave packet at the moment of measurement.

It is more difficult to understand the quantum phenomenon associated with the statistical wave function. Louis de Broglie assumes that the statistical wave function (which he calls the phase wave) is a fictitious wave. Its main property is that it drives the center of mass  $\mathbf{x}_G(t)$  of the quantum system. Knowledge of the internal structure is generally not necessary to understand interference. We first present three cases (interference, Stern and Gerlach, EPR-B) where the external (statistical) wave function is sufficient: the particles are prepared in the same way and have the same external (statistical) wave function, but different internal (matter) wave functions. We will then show examples of collisions where both wave types are necessary.

### 6.1 Two slits experiments with the statistical wave

Two slits experiments are examples where only the statistical wave function of a particle interferes with itself. The statistical wave function passes through both slits, while the matter function passes through a single slit that depends on the initial position of its center of mass. Figure 2 represents a simulation of two slits experiment with  $C_{60}$  fullerene molecules under conditions similar to those of Nairz, Arndt and Zeilinger [34]. For detailed explanations of this simulation, see [35].

During the impact measurement of the  $C_{60}$  molecule at 5  $mm$  after the slits, it is the matter wave function that interacts with the detection screen and produces the impact.

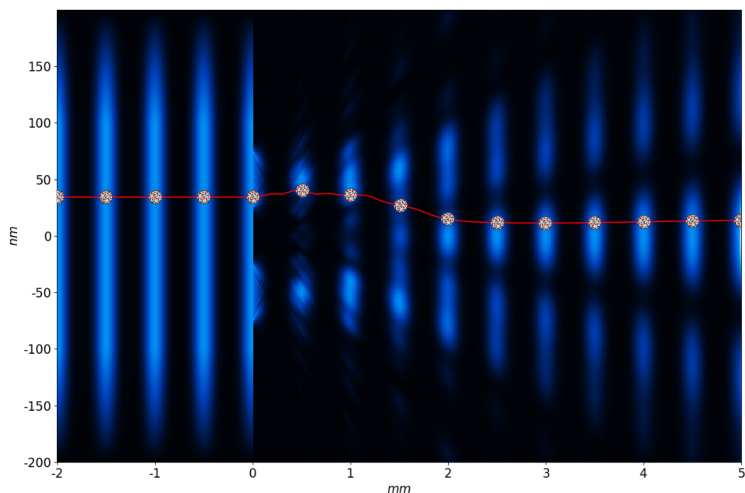


FIGURE 2 – Simulation of the evolution of the statistical (blue) and matter (white, magnified 13 times) wave functions of a  $C_{60}$  molecule under experimental [34] conditions at fifteen different times every  $2.5 \mu s$  (i.e. every  $0.5 mm$ ). The slits are placed at  $0 mm$  and are  $100 nm$  apart and  $55 nm$  wide. The average speed of the molecules is  $200 m/s$ , corresponding to a wavelength of  $2.8 pm$ . The red line corresponds to the trajectory of the center of gravity of a  $C_{60}$  molecule whose initial position (before the slits) was drawn at random in the initial wave packet [Videos of this experiment are available at: [vimeo.com/350139153](https://vimeo.com/350139153) and [vimeo.com/350132498](https://vimeo.com/350132498).]

For a statistical wave function "*prepared indiscerned*",  $\rho_0(\mathbf{x}_G)$  corresponds to the initial probability density of the center-of-mass and  $\rho^h(\mathbf{x}_G, t)$  to the probability density of the center-of-mass at time  $t$ . The statistical wave function *prepared indiscerned* can therefore be considered as a probability wave to which Born's statistical interpretation applies.

Double scale theory explains the existence of superposed states for the statistical wave function. But it is possible that this superposition does not exist a matter wave function.

In the study of the following two experiments (Stern and Gerlach and EPR-B), we will show how spin can be taken into account in double-scale theory with statistical functions.

### 6.2 Stern and Gerlach spin measurement with statistical wave

Let us consider Stern and Gerlach's experiment to measure the spin of a silver atom. This atom is prepared in  $E$  with a velocity  $\mathbf{v}$  parallel to ( $Oy$ ), see (Fig. 3).

The magnetic moments of these atoms have been prepared in a pure  $(\theta_0, \varphi_0)$  state, so that upon arrival in the the electromagnet  $\mathbf{A}_1$  at initial time  $t = 0$  the statistical wave function of each atom can be described by the Gaussian spinor in  $x$  and  $z$ :

$$\Psi^0(x, z) = (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z^2+x^2)}{4\sigma_0^2}} \begin{pmatrix} \cos \frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} \\ i \sin \frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} \end{pmatrix} \quad (47)$$

with  $\mathbf{r} = (x, z)$ .

The variable  $y$  is treated in the classical way with  $y = vt$ . For the silver atom [36], we have  $m = 1.8 \times 10^{-25}$  kg,  $v = 500$  m/s,  $\sigma_0 = 10^{-4}$  m. Thus, the spin of a molecule is a property carried by the molecule's statistical wave function, as for the center of mass. Here, we have a pure state and all the silver atoms (prepared in the same way) have the same magnetic moment orientation.

Most textbooks on quantum mechanics do not take spatial extension of the spinor into account ( $f(\mathbf{r}) = (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z^2+x^2)}{4\sigma_0^2}}$  in equation (47)) and simply take the wave function in the Hilbert space of dimension 2 generated by  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ :

$$|\psi^0\rangle = \cos \frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} |0\rangle + i \sin \frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} |1\rangle \quad (48)$$

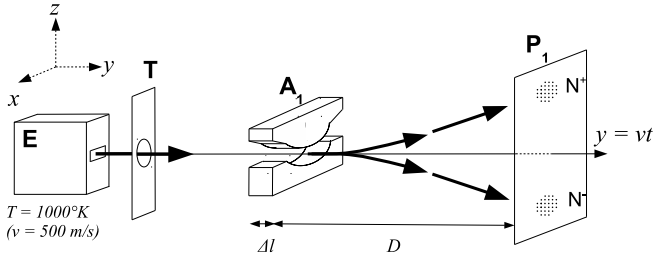


FIGURE 3 – Schematic diagram of Stern and Gerlach’s experiment : a jet of silver atoms, prepared in a pure  $(\varphi_0, \theta_0)$  state and emanating from the **E** enclosure, passes through an inhomogeneous magnetic field ( $\mathbf{A}_1$  magnet) before splitting into two distinct beams, giving two distinct spots  $N^+$  and  $N^-$  on the  $\mathbf{P}_1$  plate.

By retaining only the wave function (48) without the spinor’s spatial extension (47), we can’t take into account the initial position  $(x_0, z_0)$  of the molecule’s center-of-mass. It is this initial position that renders the evolution of the quantum system (external wave function + position) deterministic.

The evolution of the spinor  $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$  in a magnetic field  $\mathbf{B} = (B_x, B_y, B_z)$  is then given by the Pauli equation [36]:

$$i\hbar \begin{pmatrix} \frac{\partial \psi_+}{\partial t} \\ \frac{\partial \psi_-}{\partial t} \end{pmatrix} = -\frac{\hbar^2}{2m} \Delta \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} + \mu_B \mathbf{B} \boldsymbol{\sigma} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad (49)$$

where  $\mu_B = \frac{e\hbar}{2m_e}$  is the Bohr’s magneton,  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  corresponds to the Pauli matrices.

The silver atoms enter an electromagnetic field  $\mathbf{B}$  oriented mainly along the  $(Oz)$  axis,  $B_x = B'_0 x$ ;  $B_y = 0$ ;  $B_z = B_0 - B'_0 z$ , with  $B_0 = 5$  Tesla,  $B'_0 = \left| \frac{\partial B}{\partial z} \right| = 10^3$  Tesla/m over a length  $\Delta l = 1$  cm. At the exit of the magnetic field, we are dealing with particles to the plate  $P_1$  placed at a distance  $D = 20$  cm.

The particle remains in the magnetic field for  $\Delta t = \frac{\Delta l}{v} = 2 \times 10^{-5}$  s in the magnetic field. On leaving this field, we show [37, 38] that

at time  $t + \Delta t$  ( $t \geq 0$ ), the spinor of the statistical wave is equal to :

$$\begin{aligned} \Psi(x_G, z_G, t + \Delta t) &= \begin{pmatrix} R_+ e^{i \frac{s_+}{\hbar}} \\ R_- e^{i \frac{s_-}{\hbar}} \end{pmatrix} \\ &\simeq \begin{pmatrix} \cos \frac{\theta_0}{2} (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z_G - z_\Delta - ut)^2 + x_G^2}{4\sigma_0^2}} e^{i \frac{m u z_G + \hbar \varphi_+}{\hbar}} \\ i \sin \frac{\theta_0}{2} (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z_G + z_\Delta + ut)^2 + x_G^2}{4\sigma_0^2}} e^{i \frac{-m u z_G + \hbar \varphi_-}{\hbar}} \end{pmatrix} \end{aligned} \quad (50)$$

with :

$$z_\Delta = \frac{\mu_B B'_0 (\Delta t)^2}{2m} = 10^{-5} m, \quad u = \frac{\mu_B B'_0 (\Delta t)}{m} = 1 m/s. \quad (51)$$

This spinor will define the trajectory  $X_G(t) = (x_G(t), z_G(t))$  of the silver atom's center-of-mass from its initial position  $X_G(0) = (x_G(0), z_G(0))$  by the formula [39, 40]:

$$\frac{dX_G(t)}{dt} = \frac{\hbar}{2m\rho} \text{Im}(\Psi^\dagger \nabla \Psi)|_{\mathbf{x}=X_G(t)} \quad (52)$$

where  $\Psi^\dagger = (\Psi^{+*}, \Psi^{-*})$  et  $\rho = \Psi^\dagger \Psi$ . Bohm et al.[39] also define a field of spin vector  $\mathbf{s}$  as:

$$\mathbf{s}(\mathbf{x}, t) = \frac{\hbar}{2\rho} \Psi^\dagger(\mathbf{x}, t) \sigma \Psi(\mathbf{x}, t) = \frac{\hbar}{2} (\sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta). \quad (53)$$

The spin vector of an individual particle is evaluated along its trajectory as  $\mathbf{s} = \mathbf{s}(X_G(t), t)$ . This spin vector is totally defined by the spinor and the position of the atom's center-of-mass.

Figure 4 shows  $\rho(z_G, t) = \int \Psi^\dagger(x_G, z_G, t) \Psi(x_G, z_G, t) dx_G$ , the probability density of the presence of the silver atom for  $\theta_0 = \pi/3$  and  $\phi_0 = 0$ . The  $(Oy)$  axis, where the jet propagates, is on the x-axis ( $y = v_y t$ ) and the  $(Oz)$  axis on the y-axis (the  $x$  variable is not shown, as the wave remains Gaussian along this axis). The magnet  $A_1$  is represented on the figure by the two white rectangles. It is  $\Delta l = 1cm$  long and there is  $D = 20cm$  of free travel before the atom is measured on the  $P_1$  detection screen

A trajectory is also shown in the figure 4, along with its spin  $\mathbf{s}(X_G(t), t)$  along this trajectory. If the position of the particle's center-of-mass is at the top of the wave packet ( $z_0 \leq z^{\theta_0}$  with  $z^{\theta_0} = \sigma_0 F^{-1}(\sin^2 \frac{\theta}{2})$

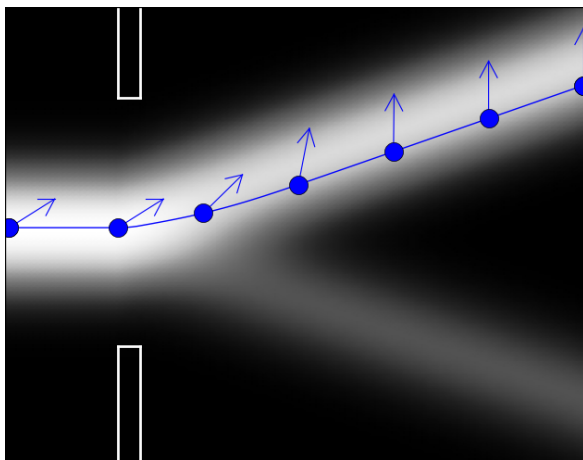


FIGURE 4 – The arrows indicate the orientation  $\theta$  of the spin vector  $\mathbf{s}$  (initially  $\theta_0 = \pi/3$ ) along the trajectory. The particle's position exists before the measurement; the particle then follows a deterministic trajectory, and the impact on the screen merely reveals its position.

where  $F$  is the repartition function of the normal centered-reduced law), as shown in the figure, the particle will be "measured" in spin UP; if the initial position is lower ( $z_0 \leq z^{\theta_0}$ ), it will be measured DOWN.

The statistical wave spinor uses only the solution to the Pauli equation and gives the same statistical results as the usual quantum mechanics for the Stern-Gerlach experiment. However, its resolution allows us to propose a clear interpretation of the spin measurement. The Stern-Gerlach experiment is not the measurement of the spin projection along the  $(Oz)$  axis, but the straightening of the spin orientation either in the direction of the magnetic field gradient, or in the opposite direction. The result depends on the initial position of the particle's center of mass in the statistical wave function. This is a simple explanation for the non-contextuality of spin measurement along different axes. The duration of the measurement is the time required for the particle to straighten its spin in the final direction. The "measured" value (the spin) is not a pre-existing value such as the particle's mass and charge, but a contextual value in accordance with Kochen and Specker's theorem [41]. The limit



case of this experiment with nano-objects is studied by Folman et al [42].

The Pauli spinor corresponds well to the statistical wave function of a molecule with spin 1/2. The problem of the statistical wave function becomes more complex in the case of two entangled particles, as we shall see in the EPR-BB experiment.

### 6.3 Simulation of the EPR-BB experiment with three statistical wave functions

Fig.5 presents the Einstein-Podolsky-Rosen-Bohm-Bell experiment. A source  $S$  created in  $O$  pairs of identical atoms  $A$  and  $B$ , but with opposite spins. The atoms  $A$  and  $B$  split following the  $Oy$  axis in opposite directions, and head towards two identical Stern-Gerlach apparatus  $\mathcal{A}$  and  $\mathcal{B}$ .

The electromagnet  $\mathcal{A}$  "measures" the  $A$  spin in the direction of the  $Oz$ -axis and the electromagnet  $\mathcal{B}$  "measures" the  $B$  spin in the direction of the  $Oz'$ -axis, which is obtained after a rotation of an angle  $\delta$  around the  $Oy$ -axis.

When creating the two intricate particles  $A$  and  $B$ , each particle has an initial statistical wave function  $\Psi_0^A(\mathbf{x}_G^A, \theta_0^A, \varphi_0^A)$  and  $\Psi_0^B(\mathbf{x}_G^B, \theta_0^B, \varphi_0^B)$  of type (47) with opposite spins:  $\theta_0^B = \pi - \theta_0^A$ ,  $\varphi_0^B = \varphi_0^A - \pi$ .

If we simulate the evolution of these two particles independently, each verifying a Pauli equation, without taking into account that they are linked by an initial singlet spinor, but only by opposite initial spins, we obtain spin measurements of  $A$  and  $B$  that do not violate Bell's inequalities.

On the other hand, if we simulate the evolution of these two particles, taking into account that they are also bound by an initial singlet spinor (47), we have shown [38] that the spin measurements of  $A$  and  $B$  verify Bell's inequalities [6].

To obtain these results, we consider, as in Stern and Gerlach's experiment, that the initial singlet wave function has a spatial extension :

$$\Psi_0(\mathbf{x}_G^A, \mathbf{x}_G^B) = \frac{1}{\sqrt{2}} f(\mathbf{x}_G^A) f(\mathbf{x}_G^B) (|+A\rangle|-B\rangle - |-A\rangle|+B\rangle) \quad (54)$$

where  $\mathbf{x}_G^A$  et  $\mathbf{x}_G^B$  are the center-of-mass of particles  $A$  et  $B$  and where  $f(\mathbf{x}) = (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{\mathbf{x}^2}{4\sigma_0^2}}$ .

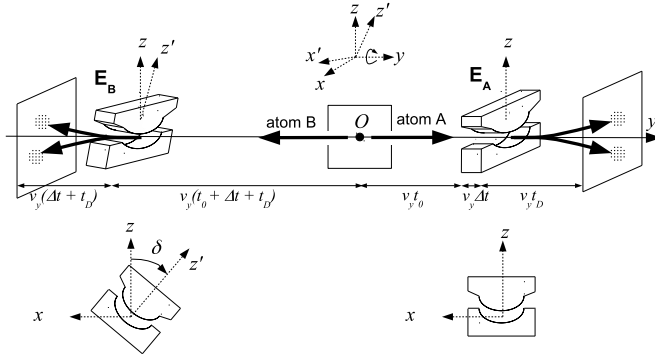


FIGURE 5 – Schematic configuration of EPR-B experiment.

This singlet function (54) can be recovered from Pauli's principle. Indeed, if we apply Pauli's principle, which stipulates that the wave function of a two-body object must be antisymmetric, to the instant of creation of the two intricate particles  $A$  and  $B$ , we must write :

$$\begin{aligned} \Psi_0(\mathbf{r}_A, \theta_A, \varphi_A, \mathbf{r}_B, \theta_B, \varphi_B) &= \Psi_A^0(\mathbf{r}_A, \theta_A, \varphi_A) \Psi_B^0(\mathbf{r}_B, \theta_B, \varphi_B) \\ &\quad - \Psi_A^0(\mathbf{r}_B, \theta_B, \varphi_B) \Psi_B^0(\mathbf{r}_A, \theta_A, \varphi_A) \quad (55) \\ &= -e^{i\varphi_A} f(\mathbf{r}_A) f(\mathbf{r}_B) (|+A\rangle |-B\rangle - |-A\rangle |+B\rangle) \end{aligned}$$

which is the singlet state with spatial extension (54) to within a factor. Again, this spatial extension is essential to correctly solve the Pauli equation in space, because it is necessary to take into account the atom's position in its external wave function. We then *measure* the spin of the two particles one after the other [38].

It is shown mathematically [38] that the first particle measured, the  $A$  particle, behaves in the first Stern-Gerlach apparatus in the same way as if it were not intricued.  $A$  is measured by first solving the Pauli equation with the intricate wave function (55) as it passes through the first Stern-Gerlach device. During the *measurement* of  $A$ , the density of the  $B$  particle also evolves as if it were unintrigued, and its spin straightens out to be always in opposition to that of the  $A$  particle [38]. The second measurement is a Stern-Gerlach-type measurement with particular orientations. The results of quantum mechanics and the violation of Bell's inequalities are thus perfectly obtained.

As with Stern and Gerlach, the external spinor (the internal wave) of the entangled state, [which uses only the solution of the Pauli equation on the external variables ( $\mathbf{r}$ ,  $\theta$ ,  $\phi$ ) of the two particles], gives the same statistical results for EPR-B as usual quantum mechanics. Quantum particles have a local position like a classical particle, but also possess non-local behavior due to the statistical wave function.

We refer you to our article *Replacing the Singlet Spinor of the EPR-B Experiment in the Configuration Space with two Single-particle Spinors in Physical Space* [38] where we show precisely how these three external spinors interfere, the singlet spinor with spatial extension that verifies the Pauli equation, and the spinors of the two entangled particles with their spatial extensions.

We therefore needed three statistical wave functions to model and solve the EPR-B experiment: two individual statistical wave functions, one for A and the other for B, plus an entangled statistical wave function for the pair (A, B). Note that the embedded wave function corresponds to the non-local hidden variable that helps explain Bell's theorem.

**Remark 11** *We have clearly demonstrated the existence of two wave functions associated with a molecule. But we have not explained the mechanism that unites them through the position of the centre of mass that they share. Is it a resonance phenomenon linked to the equality of the internal phase with that of the external phase as de Broglie wrote in 1913 [43] ? Is it linked to the existence of a reference frame with the non-local property of propagating the vibrations of the external wave function, like the non-local field of the walking droplets of Couder et al [44, 45] ?*

**Remark 12** *Yves Couder and his team [44, 45] have shown that there are strong similarities between the theory of walking droplets and the dBB theory. This analogy is transformed into a true bijection with the double scale theory.*

*The external wave function corresponds to Couder and Fort surface waves and pilots the centre of mass of the particle. The internal wave function corresponds to an extended particle like Couder's walking droplets. The link between these two waves is made, as with Couder, by the position of the particle's centre of mass.*

## 6.4 Collisions

The simultaneous existence of statistical wave functions and matter wave functions poses the problem of their status and interactions.

We have seen that individual matter wavefunctions correspond to real particles, as they are considered in classical mechanics. They carry the energy and momentum that will be conserved during elastic collisions between particles.

In the following examples, we explore the relationship between these two types of wave function.

**Example 1:** Let's start with the isolated hydrogen atom. It will be represented by two individual matter functions, which are real, that of the proton  $\varphi_L(\mathbf{x}_p, t)$  and that of the electron  $\varphi_L(\mathbf{x}_e, t)$ , and by a statistical wave function  $\psi(\mathbf{x}_G, t)$  which is fictitious and represents the wave function of the atom's center-of-mass.

**Example 2:** Let's consider a hydrogen molecule that we're going to split into two hydrogen atoms (with a laser shot, for example). Initially, we have 4 individual physical matter wavefunctions (2 protons and 2 electrons) and a fictitious statistical (or epistemic) wavefunction associated with these 4 individual particles, representing the wavefunction of the molecule's center-of-mass. After dissociation of the hydrogen molecule, these 4 individual particles are combined (physically) into 2 pairs of individual mass functions (proton + electron). We end up with 2 atoms, each with 2 individual matter waves, and each with a center-of-mass and a statistical wave function representing the wave function of this center-of-mass.

**Example 3:** Consider an atom that arrives in an ion trap and is stored, as is done experimentally. The atom becomes an ion when an electron is stripped from it. Initially, we have  $N$  individual matter wave functions corresponding to the  $N$  particles of the atom and a fictitious statistical wave function representing the wave function of the atom's center of mass. After ionization, a real electron (individual matter wavefunction) escapes from the ion box and acquires a statistical wave function representing the center-of-mass wave function of that electron. The  $N-1$  individual matter functions are reconfigured in the box (or in the potential that creates it). They acquire a statistical wave function that represents the wave function of the ion's center-of-mass.

**Example 4:** Let us consider the case of several hydrogen atoms. Can they be considered independent, as in a dilute gas? Or are there interactions between the atoms that prevent this conclusion? We realize that the choice of model for the quantum system will depend on the configurations and approximations the modeler makes based on the information

there are about the system, and not just on the observed ensemble. The description of the external wave function therefore depends on the user and their information about the system. It may be epistemic, but not entirely ontological. In the case of a dilute gas of atoms, the epistemic and ontological representations may be almost identical: an individual matter wave function for each proton and electron, and a single statistical wave function for all the hydrogen atoms, if these atoms can be considered to have been prepared in the same way. In the case of a gas of  $N$  undiluted atoms, there will be an individual mass wave function for each of the  $N$  protons and  $N$  electrons. On the other hand, the modeler's statistical wave function will not be unique, as it will depend on the choice of groupings made by the modeler.

**Remark 13** - *Møller [46] and Bhabha [47] elastic collisions can then be simply explained if we assume that the calculation of Feynman diagrams is done using the conservation of energy and momentum of individual matter wave functions. On the other hand, as we have shown, the computation of the statistical wave function is done using Feynman integrals. Elastic Compton scattering can also be explained simply if we generalize double-scale theory to electromagnetic fields, as we do in [22].*

## 7 Conclusion

Before concluding, let us summarise the double-scale theory: for each quantum system of  $N$  particles (atom nuclei or electrons), we associate  $N$  **individual internal wave functions**, one for each particle. These will be the **invariants**, which carry the mass, energy, electric charge, momentum and spin as long as there are only elastic collisions. These are the **beables** that Bell was looking for and the "elementary bricks" of our quantum theory.

We then associate an internal wave function with the quantum system, which will be the product of the individual internal wave functions.

These internal and individual wave functions are **physical realities** and local as in classical mechanics, the square of the modulus of which corresponds to a physical density. They correspond to a microscopic scale and become a point if we tend  $h$  towards 0.

On the other hand, the **external wave functions** are above all non-local **mathematical realities** which allow us to describe the interactions between the beables on our macroscopic scale in a hierarchical

construction. Each elementary beable is associated with an elementary external wave function whose square modulus corresponds to the probability of finding the beable's centre of mass. We define the external wave function for a molecule, and then for an aggregate, in a hierarchical manner, like in a Lego set. The square of the modulus of the molecule (resp. of the aggregate) gives the probability of finding the center-of-mass of the molecule (resp. of the aggregate).

In his 1923 articles, de Broglie spoke of fictitious wave functions.

We believe we have partly answered Louis de Broglie's program on his double solution, as well as Einstein's remarks. In the case of a molecule, the external wave function, our statistical wave function, corresponds to Schrödinger's  $\psi$ -wave, which is only a representation of probabilities, and the internal wave function, our matter wave function, corresponds to the u-wave representing the molecule. And as Einstein remarked, "it was not necessary that the whole should be represented by a single function, but by a set of several components". In particular, the dynamical matter wave function is also the product of the  $N$  individual wave functions of the  $N$  particles. These individual matter waves do not change in the elastic collisions. However, the external wave function depends to the preparation. In the EPR-B experiment, we use some functions to represent the external function: the spinor of the two particles and the singlet spinor.

This scale-dependent dual interpretation is a framework for reading quantum mechanics in a simple and realistic way.

**The wave-corpuscule duality** corresponds to the simultaneous existence of two wave functions linked together, at each instant  $t$ , by the position of the center-of-mass  $X_G^h(t)$ : while the statistical wave function corresponds to the wave (field) and the matter wave function corresponds to the corpuscule.

**Theoretically**, we therefore have field AND corpuscule.

**Experimentally**, a coherent source of particles is a set of particles which all have the same statistical wave function, but whose matter wave functions are different. Thus, in the slit diffraction, double slit interference, Stern and Gerlach spin measurement, tunneling effect and EPR-B experiments, the preparation of the quantum system is represented by the statistical wave function alone. The matter wave function, which is unknown, explains the statistical results of Born's interpretation.

**Quantum mechanics is complete** if the state of a quantum system corresponds to the simultaneous existence of both wave functions, incomplete otherwise. The position of the center-of-mass in the statistical wave function that must be added in the de Broglie-Bohm interpretation is obtained from the matter wave function by the equations (42) and (43). We can thus interpret Everett's multiple worlds as the set of matter wave functions compatible with the statistical wave function.

**The measurement corresponds to the impact of the matter wave** on the detection screen. It is thus the matter wave function that is involved in the collapse of the wave packet. The Copenhagen interpretation, which does not differentiate between the statistical and matter waves, cannot therefore see that the reduction of the wave packet concerns mainly the matter wave function. For the same reason, the GRW interpretation [48] requires an objective non-linear perturbation. This perturbation exists because the particle is stopped by the detection screen following a non-linear absorption phenomenon. In the double slit experiment, **the statistical wave function passes through both slits** while **the matter wave function passes through only one slit**.

**Heisenberg's inequalities** correspond, for the statistical wave function, to **uncertainty relations** on the positions and velocities of the centers of mass of a set of molecules admitting this same wave function. For the matter wave function, the Heisenberg inequalities correspond to **indeterminacy relations** on the different positions and velocities of elements of this extended particle [29].

**The non-local hidden variable** of Bell's theorem and the EPR-B experiment concerns the statistical wave functions [29] while the position of the impacts is a **local measured variable** corresponding to the matter wave function.

Our results were obtained using a heuristic approach based on the Born Oppenheimer approximation. They need to be confirmed by further studies.

This scale-dependent dual solution can be tested experimentally by an asymmetric double slit experiment such as those we proposed a few years ago [49] and are currently pursuing [31].

We believe that this new analysis us to extend this realistic interpretation to all quantum mechanics as well as to relativity. We are indeed preparing an extension of this double scale theory to the relativistic case by considering the Gordon decomposition of the Dirac equation as a

convection current corresponding to the statistical wave function (large components) and a spin current corresponding to the matter wave function (small components) [50]. We have also proposed a semi-classical gravity converging to the Newton gravity when  $\hbar \rightarrow 0$  [35].

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