Pilot-wave theory and the search for new physics

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ABSTRACT.

We show how pilot-wave theory points to new physics, beyond quantum mechanics, in three distinct ways. First, generalised cosmological initial conditions, departing from the Born rule, can lead to observable anomalies in the cosmic microwave background and in relic cosmological particles. Second, a breakdown of the Born rule in the deep quantum-gravity regime, with gravitational corrections that render the Born rule semiclassically unstable, can create anomalies in Hawking radiation from evaporating black holes. Third, a regularised equation of motion that remains finite at nodes of the wave function generates corrections to the Born rule at short distances, while a natural time-dependent generalisation implies an instability of quantum equilibrium at short times, effects which may be observable in high-energy collisions.

1   Introduction

'*We have to find a new view of the world that has to agree with everything that is known, but disagree in its predictions somewhere ... .*' (ref. [1], p. 171)

In the search for new physics, it can be useful to have a theory that agrees with all experiments carried out so far – and that also suggests new areas where novel predictions could be found. The pilot-wave approach to quantum mechanics, pioneered by Louis de Broglie in the 1920s and elaborated by David Bohm in the 1950s [2–7], provides such

a theory. It agrees with quantum mechanics in a broad regime, but offers three natural points of departure for new physics: (1) in the early universe, with non-standard initial conditions that violate the Born rule, (2) in the quantum-gravitational regime, where standard quantum probabilities are ill-defined, and (3) near nodes of the wave function, where pilot-wave dynamics breaks down and requires regularisation. These themes lead to specific predictions. In this paper we outline the basic motivations and key results.

Pilot-wave theory was first proposed in complete form in 1927 by de Broglie [2, 3] for a low-energy system of spinless particles whose wave function $\psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N, t)$ obeys the N-body Schrödinger equation (units $\hbar = 1$)

$$i\frac{\partial \psi}{\partial t} = -\sum_{n=1}^{N} \frac{1}{2m_n} \nabla_n^2 \psi + V\psi \ . \tag{1}$$

According to de Broglie, the particles move along a definite trajectory $(\mathbf{x}_1(t), ..., \mathbf{x}_N(t))$ determined by the law of motion

$$\frac{d\mathbf{x}_n}{dt} = \frac{1}{m_n} \boldsymbol{\nabla}_n S \ , \tag{2}$$

where $\psi = |\psi| e^{iS}$. This implies that, for an ensemble of systems with the same wave function ψ, an arbitrary distribution $\rho(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N, t)$ evolves by the continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{n=1}^{N} \boldsymbol{\nabla}_n \cdot (\rho \mathbf{v}_n) = 0 \ , \tag{3}$$

with a velocity field

$$\mathbf{v}_n = \frac{1}{m_n} \boldsymbol{\nabla}_n S \ . \tag{4}$$

From (1) it follows that $|\psi|^2$ obeys the same continuity equation,

$$\frac{\partial |\psi|^2}{\partial t} + \sum_{n=1}^{N} \boldsymbol{\nabla}_n \cdot \left(|\psi|^2 \, \mathbf{v}_n \right) = 0 \ . \tag{5}$$

We then have a simple theorem: if ρ and $|\psi|^2$ are equal at some initial time, they will remain equal for all time. This is the state of 'quantum equilibrium'

$$\rho(\mathbf{x}_1, ..., \mathbf{x}_N, t) = |\psi(\mathbf{x}_1, ..., \mathbf{x}_N, t)|^2 \ . \tag{6}$$

As shown in detail by Bohm in 1952, in this state we obtain agreement with the usual predictions of quantum mechanics, in particular for quantum 'measurements' of operator observables as usually understood [5].

Similar reasoning applies generally. For a system with configuration q and wave function $\psi(q, t)$, we have a Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi . \tag{7}$$

The system moves along a definite trajectory $q(t)$ determined by the law of motion

$$\frac{dq}{dt} = \frac{j(q, t)}{|\psi(q, t)|^2} , \tag{8}$$

where $j = j[\psi] = j(q, t)$ (with a form depending on $\hat{H}$ [8]) is the standard quantum current appearing in the configuration-space continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot j = 0 \tag{9}$$

derived from (7). This may be written as

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot \left(|\psi|^2 v\right) = 0 , \tag{10}$$

with a velocity field

$$v(q, t) = \frac{j(q, t)}{|\psi(q, t)|^2} . \tag{11}$$

For an ensemble of systems with the same wave function ψ, by construction an arbitrary distribution $\rho(q, t)$ evolves by the same continuity equation

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho v) = 0 . \tag{12}$$

And so again we have a state of quantum equilibrium

$$\rho(q, t) = |\psi(q, t)|^2 , \tag{13}$$

whose predictions agree with quantum mechanics. This construction applies, for example, to field theory on a background spacetime (flat or curved), where q is identified with the field configuration on a spacelike slice at global time t.[1]

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[1]Because the dynamics is nonlocal for entangled degrees of freedom, the dynamics is specified with respect to a preferred slicing of spacetime with preferred time parameter t [9].

Now, as it stands, pilot-wave theory might appear as just another reformulation of known physics, with nothing measureably new to offer. But in fact, careful inspection reveals three natural mechanisms for new physics, which we briefly sketch here, with further details below.

First, the theory provides an obvious generalisation to 'quantum nonequilibrium' ensembles with [10–21]

$$\rho(q,t) \neq |\psi(q,t)|^2 \ .$$

Here the equations of motion – (7) for ψ and (8) for q – remain the same, it is only the initial conditions (over an ensemble) that differ from usual. There is, of course, a conceptual difference between immutable laws of motion and variable initial conditions. If we accept (7) and (8) as basic laws, it is clear that in pilot-wave theory the Born rule (13) does not have the same status: the Born rule is not a law but merely an initial condition.[2] Therefore, the theory naturally invites us to consider more general initial conditions with $\rho \neq |\psi|^2$. The potentially observable consequences of this simple step, in particular for cosmology, are outlined in Section 2.

Second, in the context of quantum gravity, pilot-wave theory suggests that there is no fundamental equilibrium state at the Planck scale and, furthermore, that at lower energies small quantum-gravitational corrections can render the Born rule unstable, enabling initial equilibrium $\rho = |\psi|^2$ to evolve into final nonequilibrium $\rho \neq |\psi|^2$ [22, 23]. While these effects are expected to be very tiny, they may have observable consequences in the radiation from exploding primordial black holes (Section 3).

Third, the theory predicts its own demise at nodes where $\psi = 0$ and the velocity field (11) diverges (for a broad class of Hamiltonians found in nature). This problem is usually ignored as nodal regions are of (Lebesgue- or $|\psi|^2$-) measure zero. But as a matter of principle, new physics must set in sufficiently close to nodes, so as to regularise the theory in some fashion [24]. A simple regularisation of the dynamics implies small corrections to the equilibrium state close to nodes. We show how the form of the corrections (though not their overall magnitude) may be predicted. Generalising to a time-dependent regularisation again enables initial equilibrium $\rho = |\psi|^2$ to evolve into final nonequilibrium $\rho \neq |\psi|^2$

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[2]For a thorough discussion see ref. [21].

[24]. Such effects might potentially appear in high-energy collision experiments, in the form of smeared cross sections, or as anomalous spin or polarisation probabilities (Section 4).

2 Generalised initial conditions in cosmology

The application of quantum mechanics to the early universe is a fairly mature science, at least for quantum systems on a background classical spacetime. We know that small temperature anisotropies in the cosmic microwave background (CMB) were seeded by small inhomogeneities in the early universe. According to inflationary cosmology, those early inhomogeneities were in turn seeded by primordial quantum fluctuations at very early times [25–27]. The 'primordial power spectrum' measured by the *Planck* satellite then provides a record of the Born rule at work in the very early universe. This means that measurements of the CMB can be employed to test quantum mechanics, in particular the Born rule, at the earliest moments in the history of our universe [19, 28].

2.1 Quantum relaxation

According to pilot-wave theory, the Born rule $\rho = |\psi|^2$ (applied for example to a field on a background spacetime) is not a law of nature but merely a state of statistical equilibrium, analogous to thermal equilibrium in classical physics [10–21]. It can be understood as having arisen by a past process of dynamical relaxation, whereby an initial nonequilibrium ensemble with $\rho \neq |\psi|^2$ evolves – via the continuity equation (12) – towards the equilibrium state $\rho = |\psi|^2$ (on a coarse-grained level). This can be quantified by a decrease of the coarse-grained H-function

$$\bar{H}(t) = \int dq \, \bar{\rho} \ln(\bar{\rho}/\overline{|\psi|^2}) \tag{14}$$

(for coarse-grained densities $\bar{\rho}$ and $\overline{|\psi|^2}$), which satisfies a coarse-graining H-theorem $\bar{H}(t) \leq \bar{H}(0)$ (assuming no initial fine-grained structure at $t = 0$) [10, 12, 14]. Extensive numerical simulations have illustrated quantum relaxation in a variety of circumstances, with an approximately exponential decay [17, 29, 30]

$$\bar{H}(t) \approx \bar{H}_0 e^{-t/\tau} \tag{15}$$

(for wave functions that are superpositions of multiple energy eigenstates), where $\bar{H} \to 0$ implies $\bar{\rho} \to \overline{|\psi|^2}$. We may then understand the

Born rule observed today as having arisen from relaxation processes taking place in the remote past, presumably close to the big bang.

Such studies have been extended to field theory on a background expanding space. Consider a massless scalar field ϕ on flat expanding space with metric

$$d\tau^2 = dt^2 - a^2\delta_{ij}dx^i dx^j \tag{16}$$

and scale factor $a = a(t)$ (conventionally taking $a_0 = 1$ at time t_0 today). For an unentangled field mode with Fourier component

$$\phi_{\mathbf{k}} = \frac{\sqrt{V}}{(2\pi)^{3/2}}\left(q_{\mathbf{k}1} + iq_{\mathbf{k}2}\right) \tag{17}$$

(where V is a normalisation volume and $q_{\mathbf{k}1}$, $q_{\mathbf{k}2}$ are real), the wave function $\psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ satisfies the two-dimensional Schrödinger equation

$$i\frac{\partial\psi_{\mathbf{k}}}{\partial t} = -\frac{1}{2a^3}\left(\frac{\partial^2}{\partial q_{\mathbf{k}1}^2} + \frac{\partial^2}{\partial q_{\mathbf{k}2}^2}\right)\psi_{\mathbf{k}} + \frac{1}{2}ak^2\left(q_{\mathbf{k}1}^2 + q_{\mathbf{k}2}^2\right)\psi_{\mathbf{k}} , \tag{18}$$

with de Broglie velocities

$$\dot{q}_{\mathbf{k}1} = \frac{1}{a^3}\frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}1}}, \qquad \dot{q}_{\mathbf{k}2} = \frac{1}{a^3}\frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}2}} \tag{19}$$

(where $\psi_{\mathbf{k}} = |\psi_{\mathbf{k}}|\,e^{is_{\mathbf{k}}}$) [19, 28]. These match the usual equations for a two-dimensional harmonic oscillator with time-dependent mass $m = a^3$ and time-dependent angular frequency $\omega = k/a$ (units $\hbar = c = 1$). An arbitrary ensemble distribution $\rho_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ evolves by the continuity equation

$$\frac{\partial\rho_{\mathbf{k}}}{\partial t} + \frac{\partial}{\partial q_{\mathbf{k}1}}\left(\rho_{\mathbf{k}}\dot{q}_{\mathbf{k}1}\right) + \frac{\partial}{\partial q_{\mathbf{k}2}}\left(\rho_{\mathbf{k}}\dot{q}_{\mathbf{k}2}\right) = 0 , \tag{20}$$

which may be integrated forwards in time for given initial conditions.

2.2　Large-scale power deficit in the CMB

At short physical wavelengths $\lambda_{\mathrm{phys}} = a\lambda << H^{-1}$, where $H^{-1} = a/\dot{a}$ is the Hubble radius, we find the usual rapid and efficient coarse-grained relaxation $\bar{\rho}_{\mathbf{k}} \to \overline{|\psi_{\mathbf{k}}|^2}$ indicated above. However, at longer wavelengths $\lambda_{\mathrm{phys}} \gtrsim H^{-1}$ we find that relaxation is retarded or suppressed [19, 31–34]. We can then expect quantum noise to be suppressed at large cosmological scales, as appears to be observed (though not without controversy, as the data are especially noisy at large scales) [35].

The degree of suppression can be quantified by a nonequilibrium mean square

$$\left\langle |\phi_{\mathbf{k}}|^2 \right\rangle = \left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\mathrm{QT}} \xi(k) \ , \tag{21}$$

where $\langle ... \rangle_{\mathrm{QT}}$ is the quantum-theoretical equilibrium value and $\xi(k) < 1$ measures the 'squeezing' of the Born rule as a function of k. Extensive numerical simulations support an approximate form (neglecting small oscillations)

$$\xi(k) = \tan^{-1}(c_1 \frac{k}{\pi} + c_2) - \frac{\pi}{2} + c_3 \ , \tag{22}$$

with an increasing deficit at small k, where the unknown parameters c_1, c_2, c_3 depend on the initial state and on the time interval [33, 34]. Such large-scale suppression could have taken place during a (radiation-dominated) pre-inflationary era. Assuming for simplicity that the spectrum is unaffected by the transition to inflation, we may then expect the usual inflationary power spectrum [25]

$$\mathcal{P}_{\mathcal{R}}^{\mathrm{QT}}(k) = \frac{4\pi k^3}{V} \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle_{\mathrm{QT}} \propto \frac{4\pi k^3}{V} \left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\mathrm{QT}} \tag{23}$$

for primordial 'curvature perturbations' $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$ to take the corrected form [19, 28]

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{\mathrm{QT}}(k)\xi(k) \ , \tag{24}$$

with a large-scale deficit. This implies corrections to the angular power spectrum observed in the CMB [19, 28, 32, 33].

The corrected spectrum (24) has been fit to CMB data with some success, though at the time of writing there is no clear statistical preference for a large-scale deficit of the form (22) [36]. To confirm or rule out cosmological models with quantum relaxation will require more detailed predictions. Such models imply small oscillations in $\xi(k)$ [33]. We may also expect to find statistical anisotropy, with ξ showing some dependence on the direction of the wave vector $\mathbf{k}$ (at least at large scales) [37]. Further comparison with data is a task for future work.

2.3 Nonequilibrium relic particles

According to inflationary cosmology, the matter in our universe was created at early times by inflaton decay [27]. As the early inflationary expansion comes to an end, the energy of the inflaton field is converted into matter and radiation, initiating a hot big bang. If the early inflaton

field violates the Born rule, then so will its decay products [31, 38]. By this mechanism, the early hot universe could be filled with nonequilibrium particles that break the Born rule.

We may ask if such particles could survive to the present day and still be in nonequilibrium. To avoid relaxation, the particles would need to decouple at very early times so as to stream freely with negligible interaction.[3] Prominent theories of high-energy physics suggest that dark matter is largely composed of relic particles that decoupled very early – for example, gravitinos. It then seems conceivable that, even today, dark-matter particles could violate the Born rule [14, 19, 31, 38].

Today it is not possible to experiment with dark-matter particles directly, since they interact extremely weakly with ordinary matter. However, we may be able to observe their annihilation or decay products. For example, pairs of gravitinos are expected to annihilate into pairs of gamma-ray photons. In other models, dark-matter particles can decay to produce X-rays. If the parent particles violate the Born rule, so will the outgoing photons. Among other phenomena, the resulting spectral lines could have anomalous profiles [39].

We may then envisage experiments testing the Born rule for X-rays or gamma rays produced by disintegrating dark matter. The *Fermi Gamma-ray Space Telescope* has detected a puzzling excess of gamma rays from the Galactic centre [40], which on some models could originate from the annihilation or decay of dark-matter particles [41, 42]. We then have a case for testing the Galactic excess gamma-ray photons for deviations from the Born rule. The incoming photons could impinge on an appropriate two-slit screen, or perhaps a diffracting crystal or grating, where the resulting interference or diffraction pattern may be monitored for anomalies. The incoming photons could also be tested for anomalies in the angular dependence of their differential scattering cross sections in high-energy collisions or, more simply and cleanly, the photons could be tested directly for anomalies in their polarisation probabilities (cf. Section 4.4) [19, 43]. Such tests would have to be done aboard a dedicated satellite in space, to avoid absorption of the incoming gamma rays by the earth's atmosphere. The forthcoming QUICK[3] satellite mission includes an interferometer designed to test the Born rule in space [44]. Once such a system is fully operational, it could be deployed to test Galactic excess

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[3]The particles would also need to have effective wave functions with very few energy modes superposed [38].

gamma rays – or photons from some other exotic source – for deviations from the Born rule.

3 Quantum gravity and quantum probability

So far we have considered quantum systems on a background classical spacetime. In quantum gravity, spacetime itself is quantised. This turns out to have profound implications for the Born rule.

3.1 Pilot-wave quantum gravity

In the usual 'canonical' approach to quantum gravity, spacetime is foliated by spacelike slices with 3-metric g_{ij}, which is then subject to quantisation. In the presence of a matter field ϕ we obtain the Wheeler-DeWitt equation ($\hbar = c = 16\pi G = 1$) [45, 46]

$$\left(-G_{ijkl} \frac{\delta^2}{\delta g_{ij} \delta g_{kl}} - \sqrt{g} R + \hat{\mathcal{H}}_\phi \right) \Psi = 0 \tag{25}$$

for the wave functional $\Psi = \Psi[g_{ij}, \phi]$, where

$$G_{ijkl} = \frac{1}{2} g^{-1/2} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}) \tag{26}$$

and $\hat{\mathcal{H}}_\phi$ is an appropriate contribution from ϕ. There is also a constraint

$$-2D_j \frac{\delta \Psi}{\delta g_{ij}} + \partial^i \phi \frac{\delta \Psi}{\delta \phi} = 0 \tag{27}$$

(where D_j is a spatial covariant derivative) ensuring that Ψ is a function of the coordinate-independent 3-geometry.

It is noteworthy that Ψ has no explicit dependence on time t (where at the classical level t labels the foliation by spacelike slices). In the standard quantum-mechanical formulation, this presents numerous difficulties which are generally known as the 'problem of time' [47–53].

In the pilot-wave formulation, in addition to equations (25) and (27) for Ψ, we also have de Broglie guidance equations [54]

$$\frac{\partial g_{ij}}{\partial t} = 2N G_{ijkl} \frac{\delta S}{\delta g_{kl}} \ , \qquad \frac{\partial \phi}{\partial t} = \frac{N}{\sqrt{g}} \frac{\delta S}{\delta \phi} \tag{28}$$

for evolving fields g_{ij} and ϕ, where as usual $\Psi = |\Psi| e^{iS}$ and N is the 'lapse function' associated with the foliation.[4] Even though Ψ is static, there are time-dependent trajectories for g_{ij} and ϕ.

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[4]For simplicity and without loss of generality we take the 'shift vector' $N^i = 0$.

3.2   Problem of probability and its solution

Pilot-wave theory has been extensively applied to the simplified version of quantum gravity known as quantum cosmology [55]. These applications include bouncing cosmological models and demonstrations of singularity avoidance [56–58]. These works discuss and calculate de Broglie-Bohm trajectories, for example for the cosmological scale factor $a(t)$, without however discussing probability. There is a technical reason for this: solutions Ψ of the Wheeler-DeWitt equation are non-normalisable and cannot define a Born-rule probability density $|\Psi|^2$. This is because the Wheeler-DeWitt equation for Ψ on configuration space is mathematically analogous to a Klein-Gordon equation for a field ϕ on spacetime: the integral $\int dq\ |\Psi|^2$ diverges as does the integral $\int d^3x \int dt\ |\phi|^2$ [49, 50]. In pilot-wave theory, this problem can be solved by abandoning the Born rule at the fundamental level [22, 23].

We can illustrate the problem, and its solution, with a simple model of quantum cosmology. We consider a flat expanding universe with metric (16). The universe contains a homogenous matter field ϕ with potential $\mathcal{V}(\phi)$. We then have a Wheeler-DeWitt equation [59]

$$\frac{1}{2m_{\mathrm{P}}^2}\frac{1}{a}\frac{\partial}{\partial a}\left(a\frac{\partial\Psi}{\partial a}\right) - \frac{1}{2a^2}\frac{\partial^2\Psi}{\partial\phi^2} + a^4\mathcal{V}\Psi = 0 \tag{29}$$

for $\Psi(a,\phi)$, where $m_{\mathrm{P}}^2 = 3/4\pi G$ is the (rescaled) Planck mass squared. In terms of $\alpha = \ln a$ this can be rewritten as a two-dimensional Klein-Gordon equation

$$\frac{1}{m_{\mathrm{P}}^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \frac{\partial^2\Psi}{\partial\phi^2} + 2e^{6\alpha}\mathcal{V}\Psi = 0 \tag{30}$$

with a potential term. The free part has the general solution

$$\Psi = f(\phi - m_{\mathrm{P}}\alpha) + g(\phi + m_{\mathrm{P}}\alpha)\ , \tag{31}$$

where f and g are packets travelling in the two-dimensional 'spacetime' (α, ϕ). Clearly, the integral $\int\int d\alpha d\phi\ |\Psi|^2$ necessarily diverges. This problem can be solved by denying that $|\Psi|^2$ is a measure of probability. In pilot-wave theory, in addition to the Wheeler-DeWitt equation (29), we also have the de Broglie guidance equations [22, 23]

$$\dot{a} = -\frac{1}{m_{\mathrm{P}}^2}\frac{1}{a}\frac{\partial S}{\partial a}\ , \quad \dot{\phi} = \frac{1}{a^3}\frac{\partial S}{\partial\phi}\ . \tag{32}$$

For a theoretical ensemble with wave function $\Psi(a,\phi)$, we can consider an arbitrary (by definition normalisable) probability density $P(a,\phi,t)$ whose time evolution follows the continuity equation

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial a}\left(P\dot{a}\right) + \frac{\partial}{\partial \phi}\left(P\dot{\phi}\right) = 0 \qquad (33)$$

with the velocity field (32). This can be integrated forwards in time, given an arbitrary initial condition $P(a,\phi,t_0)$ at $t = t_0$. By construction, at all times $\int\int da d\phi\ P = 1$ whereas $\int\int da d\phi\ |\Psi|^2 = \infty$. Thus, necessarily, the system is in a perpetual state of quantum nonequilibrium[5]

$$P \neq |\Psi|^2 \ . \qquad (34)$$

If we accept that there is no equilibrium state, and no Born rule, in the deep quantum-gravity regime, we can consistently discuss ensembles with a normalised probability density P.

We might ask what happens to P as it evolves in time. Clearly, it cannot relax towards $|\Psi|^2$ as occurs in conventional systems (on a coarse-grained level), since P remains normalised. It has been suggested that, nevertheless, P might relax towards $|\Psi|^2$ in a local region of configuration space [60]. However, numerical simulations show that relaxation does not occur even locally [61].

3.3 Semiclassical emergence of the Born rule

At this point it is natural to ask how the theory can account for the observed Born rule in the semiclassical regime – that is, for a quantum system on a background classical spacetime, with an effective time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi \qquad (35)$$

for a standard (normalisable) wave function $\psi(q,t)$.

As usually understood this regime emerges, for example for a quantum field ϕ, when the Wheeler-DeWitt wave functional takes the form [62]

$$\Psi[g_{ij},\phi] \approx \Psi_{\text{WKB}}[g_{ij}]\psi[\phi,g_{ij}] \ , \qquad (36)$$

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[5]While this does not materially affect the discussion, in fact with our choice of configuration space the natural (non-normalisable) quantum density is equal to $a^2|\Psi|^2$ and not $|\Psi|^2$ [22, 23]. Strictly speaking we then have a perpetual nonequilibrium $P \neq a^2|\Psi|^2$.

where $\Psi_{\mathrm{WKB}}[g_{ij}]$ is a WKB state for the background. The effective time-dependent wave functional $\psi[\phi, t]$ arises simply by evaluating $\psi[\phi, g_{ij}(t)]$ along a classical background trajectory $g_{ij}(t)$. We then obtain the usual, much-studied regime, where we expect to find (coarse-grained) relaxation $\rho[\phi, t] \to |\psi[\phi, t]|^2$ to quantum equilibrium.

Thus, despite the absence of a fundamental Born-rule state, nevertheless the Born rule emerges as required in the semiclassical regime, and so the theory can still agree with observation.

3.4   Gravitational instability of quantum equilibrium

In the semiclassical approximation, the effective Hamiltonian $\hat{H}$ has tiny quantum-gravitational corrections which have been evaluated [59, 62]. Remarkably, there are both Hermitian and non-Hermitian terms. The non-Hermitian terms are usually discarded by hand, since in standard quantum mechanics they generate a small violation of probability conservation. However, we can make sense of those terms in pilot-wave theory: in their presence, probability is still conserved but the Born rule becomes unstable [22, 23]. To see this note first that, if we evaluate the de Broglie velocity for ϕ to the same approximation, we find that it takes the usual form $\dot{\phi} = (N/\sqrt{g})(\delta S/\delta \phi)$ but with S now equal to the phase of ψ. We then have the usual continuity equation

$$\frac{\partial \rho}{\partial t} + \int d^3x \, \frac{\delta}{\delta \phi}\left(\rho \dot{\phi}\right) = 0 \tag{37}$$

for $\rho[\phi, t]$. As for $|\psi[\phi, t]|^2$, writing $\hat{H} = \hat{H}_1 + i\hat{H}_2$ (with $\hat{H}_1$, $\hat{H}_2$ Hermitian), we find that the Schrödinger equation (35) implies a continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \int d^3x \, \frac{\delta}{\delta \phi}\left(|\psi|^2 \, \dot{\phi}\right) = s \, , \tag{38}$$

with a source term

$$s = 2 \, \mathrm{Re}\left(\psi^* \hat{H}_2 \psi\right) \, . \tag{39}$$

Because the continuity equations for ρ and $|\psi|^2$ no longer match, an initial distribution $\rho = |\psi|^2$ can evolve into a final distribution $\rho \neq |\psi|^2$. Small quantum-gravitational corrections can generate nonequilibrium from initial equilibrium. The timescale for this process has been esti-

mated:[6]

$$\tau_{\text{noneq}} \approx \frac{1}{2\left|\left\langle \hat{H}_2 \right\rangle\right|} \ . \tag{40}$$

In the presence of an evaporating Schwarzchild black hole of mass $M(t)$, the Hamiltonian $\hat{H}_{\mathbf{k}}$ of a scalar field mode $\mathbf{k}$ has a non-Hermitian correction $i\hat{H}_2$ where [63]

$$\hat{H}_2 \simeq -\frac{1}{12}\kappa \left(\frac{m_{\text{P}}}{M}\right)^4 \hat{H}_{\mathbf{k}} \ , \tag{41}$$

with κ a numerical factor and $m_{\text{P}} = \sqrt{\hbar c/G} \simeq 10^{-5}$ g (the standard Planck mass). If we take $\left\langle \hat{H}_{\mathbf{k}} \right\rangle$ to be of order the Hawking temperature

$$k_{\text{B}}T_{\text{H}} = \frac{\hbar c^3}{G}\frac{1}{8\pi M} = \frac{1}{8\pi}m_{\text{P}}c^2\frac{m_{\text{P}}}{M} \ ,$$

we find an estimated timescale [22, 23]

$$\tau_{\text{noneq}} \sim \frac{48\pi}{\kappa}t_{\text{P}}\left(\frac{M}{m_{\text{P}}}\right)^5 \ , \tag{42}$$

where t_{P} is the usual Planck time. The effect is significant only in the final stages of evaporation when M is close to m_{P}. We may then expect the very final burst of Hawking radiation to show deviations from the Born rule.[7]

In practice, we may reasonably hope to observe γ-ray Hawking radiation from exploding primordial black holes [66]. If such γ-rays were discovered, they could be tested for possible violations of the Born rule, such as anomalous polarisation probabilities [19, 43]. Again, such tests would have to be done aboard a dedicated satellite in space.

4   Regularised pilot-wave theory

The effects discussed so far require delicate measurements of radiation from the early universe, from disintegrating relic particles, or from exploding primordial black holes. One might ask if there are any worthwhile experiments that might be conducted in the laboratory, under

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[6]From the time derivative of the H-function $H(t) = \int D\phi \ \rho \ln(\rho/|\psi|^2)$ [22, 23].
[7]Related effects could also play a role in resolving the information-loss puzzle [19, 64, 65].

controlled conditions, and for which there may be plausible prospects of observing deviations from quantum mechanics.

As it stands, pilot-wave theory tells us that we are trapped in a state of quantum equilibrium – a state of 'quantum death' broadly analogous to the state of 'heat death' predicted by classical statistical thermodynamics [10–21]. As we saw in Section 2, we might find evidence for an early state of quantum nonequilibrium from careful measurements of the CMB, and residual nonequilibrium could in principle survive to the present day in certains kinds of relic cosmological particles. But once the equilibrium state is reached, there appears to be no realistic means of escape, except for the tiny quantum-gravitational effects which we described in Section 3 – and which seem beyond any hope of laboratory detection.[8] This somewhat discouraging conclusion is modified, however, if we carefully re-examine the fundamental equations of pilot-wave dynamics.

4.1   New physics at nodes ($\psi = 0$)

The Hamiltonians found in nature are often quadratic in the momenta. The de Broglie velocity (11) is then proportional to a phase gradient

$$\partial_q S = \operatorname{Im} \frac{\partial_q \psi}{\psi} \ , \tag{43}$$

which generally diverges at nodes where $\psi = 0$. This problem is usually ignored, perhaps because nodal regions are of measure zero and the trajectories tend to avoid them. But physically speaking, a diverging velocity signals a breakdown of pilot-wave dynamics at $\psi = 0$ – just as a diverging acceleration signals a breakdown of Newtonian gravity (for a point mass) at $r = 0$. In our view, there must be new physics close to points where $\psi = 0$ – just as in Newtonian gravity there is new physics close to $r = 0$ where general relativity becomes important [24].

It should be emphasised that (single-component) wave functions generically do have nodes: in n dimensions the simultaneous equations $\operatorname{Re} \psi = 0$, $\operatorname{Im} \psi = 0$ generally have solutions yielding nodal regions of dimension $n - 2$. Such points are often referred to as 'phase singularities', where the phase gradient (43) diverges. They might also be compared with the spacetime singularities of classical general relativity

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[8]We may discount as completely unrealistic attempts to escape from quantum death simply by waiting for rare fluctuations away from equilibrium [12].

(for example at the centre of a black hole). Unlike their gravitational analogues, however, phase singularities do not attract or absorb trajectories. Instead, trajectories tend to avoid them or circle around them – as we might expect since nodal regions have zero equilibrium probability (that is, zero $|\psi|^2$ measure). We can then understand why phase singularities are usually disregarded as unproblematic in applications of de Broglie-Bohm theory. Nevertheless, in principle, new physics must set in sufficiently close to such points.

Consider the simple example of a hydrogen-like atom in a stationary state

$$\psi(r,\theta,\phi,t) = \psi_{nlm}(r,\theta,\phi)e^{-iE_nt} \tag{44}$$

of energy E_n and with the usual quantum numbers nlm (in spherical coordinates). For $m \neq 0$ there is a nodal line $\psi_{nlm} = 0$ along the z-axis (where $\sin\theta = 0$). We have a deBroglie velocity

$$\mathbf{v} = \frac{1}{m_0}\boldsymbol{\nabla}S$$

(denoting the particle mass by m_0), where

$$S = m\phi - E_nt \ . \tag{45}$$

The trajectories circle around the z-axis at a fixed distance $d = r\sin\theta$. The speed (ref. [6], p. 150)

$$|\mathbf{v}| = \frac{m}{m_0}\frac{1}{d} \tag{46}$$

diverges as $d \to 0$.

We might ask if this problem could be merely an artifact of the low-energy theory. Perhaps such divergences are absent in the pilot-wave theory of high-energy physics.[9] Support for this appears to come from the pilot-wave trajectory theory of fermions, with a negative-energy 'Dirac sea' and a many-body Dirac equation [67–70]. Taking for simplicity the free one-body case, we have a Dirac equation

$$i\frac{\partial\psi}{\partial t} = -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + m\beta\psi \ , \tag{47}$$

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[9]For a recent review see ref. [7].

where ψ is a four-component wave function and $\boldsymbol{\alpha}$, β are appropriate 4×4 matrices. This implies a continuity equation

$$\frac{\partial(\psi^\dagger \psi)}{\partial t} + \boldsymbol{\nabla} \cdot (\psi^\dagger \boldsymbol{\alpha} \psi) = 0 \tag{48}$$

and a de Broglie guidance equation

$$\frac{d\mathbf{x}}{dt} = \frac{\psi^\dagger \boldsymbol{\alpha} \psi}{\psi^\dagger \psi} \ . \tag{49}$$

A general density ρ evolves by

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \ , \tag{50}$$

where $\mathbf{v} = \psi^\dagger \boldsymbol{\alpha} \psi / \psi^\dagger \psi$, and we have a quantum equilibrium distribution $\rho = \psi^\dagger \psi$.[10] If we assume that $\psi \neq 0$, elementary manipulations show that (ref. [6], p. 505)

$$|\mathbf{v}| \leq 1 \tag{51}$$

and the de Broglie velocity is bounded by the speed of light. If instead $\psi = 0$ – that is, if the four components vanish simultaneously for some $\mathbf{x}$, t – then (49) is strictly speaking undefined since both the numerator and the denominator vanish. For example, we might have an initial wave function $\psi(\mathbf{x}, 0)$ where the four components happen to equal the same function $f(\mathbf{x})$ where f has one or more nodal lines. Thus in principle nodes can occur, where (49) is undefined. However, the bound (51) applies arbitrarily closely to such points. There is no sign of any divergence close to nodes, and we may reasonably define (49) at nodes by a simple limiting process from neighbouring points where $\psi \neq 0$. For high-energy Dirac particles, then, the de Broglie velocity is well-behaved close to nodes and our argument for new physics does not apply.

Even so, our problem cannot be dismissed as an artifact of the low-energy theory, since diverging velocities still arise in high-energy bosonic field theory where the de Broglie velocity is again given by a phase gradient – as we saw in equations (19) and (28) for a scalar field. Referring to our example of an unentangled field mode $\mathbf{k}$ (Section 2.1), now

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[10]Numerical simulations show that quantum relaxation $\rho \to \psi^\dagger \psi$ takes place (on a coarse-grained level) along similar lines to the low-energy theory [71].

taken to be on Minkowski spacetime (so that $a = 1$), the wave function $\psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ satisfies [19]

$$i\frac{\partial \psi_{\mathbf{k}}}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial q_{\mathbf{k}1}^2} + \frac{\partial^2}{\partial q_{\mathbf{k}2}^2}\right)\psi_{\mathbf{k}} + \frac{1}{2}k^2\left(q_{\mathbf{k}1}^2 + q_{\mathbf{k}2}^2\right)\psi_{\mathbf{k}} \ , \qquad (52)$$

with de Broglie velocities

$$\dot{q}_{\mathbf{k}1} = \frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}1}}, \qquad \dot{q}_{\mathbf{k}2} = \frac{\partial s_{\mathbf{k}}}{\partial q_{\mathbf{k}2}} \ . \qquad (53)$$

These match the low-energy equations for a two-dimensional oscillator with mass $m = 1$ and angular frequency $\omega = k$. As in the low-energy particle theory, the velocities $\dot{q}_{\mathbf{k}1}$, $\dot{q}_{\mathbf{k}2}$ generally diverge close to nodes ($\psi_{\mathbf{k}} = 0$). This means that, at certain points in the field configuration space, the field velocity $\partial \phi / \partial t$ diverges, which seems physically unacceptable – just as a diverging particle velocity at certain points in 3-space seems unacceptable. Similar conclusions apply to the electromagnetic field, whose de Broglie guidance equation takes the form (in the temporal gauge [7, 12, 13])

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\delta S}{\delta \mathbf{A}} \ . \qquad (54)$$

The phase gradient also appears in the guidance equations for other (non-Abelian) gauge fields [7]. We seem obliged to accept that the pilot-wave theory of high-energy physics breaks down in certain regions of configuration space, where mathematical regularisation is needed, and close to which some sort of new physics must set in.

According to our argument, then, pilot-wave dynamics predicts its own demise at phase singularities where $\psi = 0$. We must encounter new physics as we approach such points. What that new physics might be is a matter for future research. Pending the development of an improved theory, however, we can explore simple regularisations of pilot-wave theory and attempt some confrontation with experiment.

4.2 Corrections to the Born rule at short distances

A regularisation of pilot-wave theory was once briefly remarked on (in a footnote) by Bell (ref. [72], p. 138). For a general system with wave function $\psi(q, t)$, we can regularise the de Broglie velocity $v = j/|\psi|^2$ by smearing j and $|\psi|^2$ with a narrowly-peaked function so that the denominator never vanishes. In this simple model, the smeared $|\psi|^2$ becomes the new equilibrium distribution.

Let us define the regularised quantities [24]

$$j(q,t)_{\text{reg}} = \int dq' \, \mu(q'-q)j(q',t) \, , \qquad (55)$$

$$\left(|\psi(q,t)|^2\right)_{\text{reg}} = \int dq' \, \mu(q'-q)|\psi(q',t)|^2 \, , \qquad (56)$$

where $\mu(q'-q)$ is positive and narrowly-peaked around $q'-q=0$ (for example a Gaussian) and $\int dq' \, \mu(q'-q) = 1$. We may then define a regularised de Broglie velocity

$$v(q,t)_{\text{reg}} = \frac{j(q,t)_{\text{reg}}}{(|\psi(q,t)|^2)_{\text{reg}}} \, . \qquad (57)$$

The usual theory is recovered when $\mu(q'-q) \to \delta(q'-q)$.

For an ensemble with the same wave function ψ, an arbitrary distribution $\rho(q,t)$ then satisfies

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho v_{\text{reg}}) = 0 \, . \qquad (58)$$

Now ψ still satisfies the Schrödinger equation, hence $|\psi|^2$ still satisfies (9), from which it is readily shown that

$$\frac{\partial(|\psi|^2)_{\text{reg}}}{\partial t} + \partial_q \cdot \left((|\psi|^2)_{\text{reg}} v_{\text{reg}}\right) = 0 \, . \qquad (59)$$

This takes the same form as (58). Thus, if $\rho = (|\psi|^2)_{\text{reg}}$ initially, we have $\rho = (|\psi|^2)_{\text{reg}}$ at later times. We then have a modified equilibrium distribution with a smeared Born rule,

$$\rho(q,t) = (|\psi(q,t)|^2)_{\text{reg}} \, , \qquad (60)$$

where $(|\psi|^2)_{\text{reg}}$ is positive. If instead $\rho \neq (|\psi|^2)_{\text{reg}}$ initially, in appropriate circumstances we can expect relaxation $\rho \to (|\psi|^2)_{\text{reg}}$ (on a coarse-grained level) as quantified by a decrease of

$$\bar{H}_{\text{reg}}(t) = \int dq \, \bar{\rho} \ln(\bar{\rho}/\overline{(|\psi|^2)_{\text{reg}}}) \qquad (61)$$

(cf. Section 2.1).

Such a regularised theory is of course not intended to be fundamental, but merely an effective description of some as-yet-unknown physics. This

model predicts a simple smearing of the usual Born distribution at short distances in configuration space (set by the width of μ). In particular, the new equilibrium density (60) does not vanish at nodes of ψ, in contrast with the usual prediction $\rho = |\psi|^2$ of quantum mechanics. We may then suggest searching for this effect experimentally.[11]

We can calculate the lowest order correction to the Born rule, in the limit of a very narrow regularising function μ. Consider smearing a general function $f(q)$, on n-dimensional configuration space, with a narrow normalised measure

$$\mu(q' - q) = \delta_a^n(q' - q) = \prod_{i=1}^{n} \delta_a(q_i' - q_i) \qquad (62)$$

of width a with respect to each degree of freedom q_i. Writing

$$I_a(q) = \int d^n q' \, \delta_a^n(q' - q) f(q') = \int d^n s \, \delta_a^n(s) f(q + s) \ , \qquad (63)$$

we can expand

$$f(q + s) = f(q) + s_i \partial_i f(q) + \frac{1}{2} s_i s_j \partial_i \partial_j f(q) + \dots \ . \qquad (64)$$

Taking

$$\int d^n s \, \delta_a^n(s) = 1 \ , \quad \int d^n s \, \delta_a^n(s) s_i = 0 \ , \quad \int d^n s \, \delta_a^n(s) s_i s_j = \delta_{ij} a^2 \ , \quad (65)$$

we find

$$I_a(q) = f(q) + \frac{1}{2} a^2 \nabla^2 f(q) + \dots \ , \qquad (66)$$

where $\nabla^2 = \sum_{i=1}^{n} \partial_i^2$. For $f(q) = |\psi(q, t)|^2$ (at fixed t) we then have the general result

$$\left(|\psi|^2\right)_{\text{reg}} = |\psi|^2 + \frac{1}{2} a^2 \nabla^2 (|\psi|^2) + \dots \ . \qquad (67)$$

The lowest order correction to the Born rule is proportional to the Laplacian of $|\psi|^2$. We might hope to set experimental bounds on the (presumably very small) parameter a.

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[11]If such an effect were found, it would not be 'nonequilibrium' but rather a modification of equilibrium.

As a simple example, consider a one-dimensional oscillator in the first excited state

$$\psi_1(x) = \left(\frac{4}{\pi}\right)^{1/4} (m\omega)^{3/4} x \exp\left(-\frac{m\omega x^2}{2}\right) \; . \tag{68}$$

The node at $x = 0$ is replaced by a minimum

$$\min\left(|\psi|^2\right)_{\text{reg}} = a^2 \frac{2}{\pi^{1/2}} (m\omega)^{3/2} + \dots \; , \tag{69}$$

with a specific frequency and mass dependence (to lowest order in a). Similarly, for the hydrogen-like state

$$\psi_{211}(r, \theta, \phi) = -\frac{1}{8\sqrt{\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} (\sin\theta) e^{i\phi} \; , \tag{70}$$

on the nodal line (z-axis) we find

$$\left(|\psi_{211}|^2\right)_{\text{reg}} = a^2 \frac{1}{32\pi a_0^5} e^{-r/a_0} + \dots \; , \tag{71}$$

with a specific dependence on r. The second example is unrealistic as the physics is superseded by the high-energy Dirac theory. However, the first example does apply realistically to high-energy bosonic field theory, where as noted a Fourier mode $\mathbf{k}$ of a massless scalar field (on Minkowski spacetime) corresponds mathematically to a two-dimensional oscillator with $m = 1$ and $\omega = k$. In principle, such effects could be tested by careful measurements of the electromagnetic field (perhaps in optical cavities).

4.3   Instability of quantum equilibrium at short times

We may also briefly discuss a natural generalisation of the above model to a time-dependent regularising function $\mu(q' - q, t)$ [24]. If there is new physics at nodes of ψ, it seems plausible that the function μ could be time dependent – for example, during high-energy collisions taking place over very short times.

Let us again define a regularised de Broglie velocity (57), with j_{reg} and $(|\psi|^2)_{\text{reg}}$ as in (55) and (56), but with $\mu(q' - q)$ replaced by $\mu(q' - q, t)$. We now find that, instead of (59), $(|\psi|^2)_{\text{reg}}$ satisfies

$$\frac{\partial(|\psi|^2)_{\text{reg}}}{\partial t} + \partial_q \cdot \left((|\psi|^2)_{\text{reg}} v_{\text{reg}}\right) = s \; , \tag{72}$$

with a 'source'

$$s(q,t) = \int dq' \, \frac{\partial \mu(q'-q,t)}{\partial t} |\psi(q',t)|^2 \ . \tag{73}$$

There is now a mismatch between the continuity equation (58) for ρ (which of course still holds) and the continuity equation (72) for $(|\psi|^2)_{\text{reg}}$. As in semiclassical gravity with small quantum-gravitational corrections (Section 3.4), an initial equilibrium distribution $\rho = (|\psi|^2)_{\text{reg}}$ can evolve into a final nonequilibrium distribution $\rho \neq (|\psi|^2)_{\text{reg}}$. In the extended regularised theory, with a time-dependent smearing function, the Born rule is unstable.

The extended model suggests that quantum nonequilibrium could be created in a high-energy collider experiment. If a collision occurs during an approximate time interval (t_i, t_f), we might take μ to be static ($\partial \mu / \partial t = 0$) at times $t < t_i$ and $t > t_f$, so that we have an equilibrium distribution $(|\psi|^2)_{\text{reg}}$ before and after the collision (where $(|\psi|^2)_{\text{reg}} \simeq |\psi|^2$ if μ has negligible width). If $\partial \mu / \partial t \neq 0$ during (t_i, t_f), an incoming equilibrium state $\rho_{\text{in}} = (|\psi|^2)_{\text{reg}}$ can evolve into an outgoing nonequilibrium state $\rho_{\text{out}} \neq (|\psi|^2)_{\text{reg}}$.[12]

4.4 Testing the Born rule in high-energy collisions

We have seen that the need for regularisation does not apply directly to trajectories of Dirac fermions. However, if these are coupled to regularised bosonic fields, where the latter have a modified or unstable Born rule, we can expect all particle species to display corrections to the Born rule. Such corrections could of course manifest in many ways. How might such corrections be most conveniently observed? We have two suggestions. The first involves testing for smeared zeros of differential scattering cross sections. The second involves testing for anomalous spin or polarisation probabilities. We briefly review these.

In high-energy physics we typically calculate S-matrix elements from initial states $|i\rangle$ to final states $|f\rangle$. These take the schematic form (omitting overall normalisation factors)

$$\langle f | \, \hat{S} \, | i \rangle \sim \delta^4(p_f - p_i)\mathcal{M} \ , \tag{74}$$

where $\mathcal{M}$ is the Feynman amplitude (obtained from Feynman rules) and $\delta^4(p_f - p_i)$ is an energy-momentum conserving delta-function. Applying

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[12]For a worked example in a cosmological context see ref. [24].

the usual Born rule, we may then derive differential cross sections of the quantum-theoretical form

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{QT}} \sim |\mathcal{M}|^2 \; . \tag{75}$$

Now, in a de Broglie-Bohm formulation, the outcome of a high-energy collision experiment at a final time t_f is determined by the initial conditions $\psi(q, t_i)$, $q(t_i)$ at time t_i, where the configuration q includes whatever particles and fields may be relevant.[13] For an ensemble of experiments with a given $\psi(q, t_i)$, the distribution of outcomes is determined by the initial distribution $\rho(q, t_i)$ of configurations. Thus, for a given initial quantum state, $d\sigma/d\Omega$ is determined by $\rho(q, t_i)$. Should $\rho(q, t_i)$ be unequal to the usual Born distribution $|\psi(q, t_i)|^2$, we can expect to find a departure

$$\frac{d\sigma}{d\Omega} \neq \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{QT}} \tag{76}$$

from the usual differential cross section. In particular, we may expect to find a smearing of the angular dependence and (perhaps) an erasure of zeros. Experimentally, then, we suggest carefully probing the angular distribution of outcomes close to points where $|\mathcal{M}|^2$ vanishes, in order to discover if the distribution truly goes to zero – or if instead it is smeared out at small angular scales.

Should anomalies in cross sections be observed, it will be natural to ask if they are really caused by corrections to the Born rule or if instead they are caused by some unexpected correction to the Hamiltonian. For this reason, spin or polarisation probabilities provide a cleaner test, as follows.

Any two-state quantum system has observables $\hat{\sigma} \equiv \mathbf{m} \cdot \hat{\boldsymbol{\sigma}}$ with eigenvalues $\sigma = \pm 1$, where $\mathbf{m}$ is a unit vector on the Bloch sphere and $\hat{\boldsymbol{\sigma}}$ is a Pauli spin operator. According to quantum theory, if $\hat{\sigma}$ is measured over an ensemble with density operator $\hat{\rho}$, we will find an expectation value

$$E_{\mathrm{QT}}(\mathbf{m}) \equiv \langle \mathbf{m} \cdot \hat{\boldsymbol{\sigma}} \rangle = \mathrm{Tr}\left[\hat{\rho}\left(\mathbf{m} \cdot \hat{\boldsymbol{\sigma}}\right)\right] = \mathbf{m} \cdot \mathbf{P} \; , \tag{77}$$

where $\mathbf{P} = \langle \hat{\boldsymbol{\sigma}} \rangle = \mathrm{Tr}\left[\hat{\rho}\hat{\boldsymbol{\sigma}}\right]$ is the average polarisation. The outcome $\sigma = +1$ then has a Born-rule probability

$$p_{\mathrm{QT}}^+(\mathbf{m}) = \frac{1}{2}\left(1 + E_{\mathrm{QT}}(\mathbf{m})\right) = \frac{1}{2}\left(1 + P\cos\theta\right) \; , \tag{78}$$

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[13]These may include degrees of freedom in the measuring apparatus.

where θ is the angle between $\mathbf{m}$ and $\mathbf{P}$. For spin-1/2 particles, values $\sigma = \pm 1$ represent spin (in units $\hbar/2$) up or down a spatial axis $\mathbf{m}$, while for photons they represent polarisation parallel or perpendicular to a spatial axis $\mathbf{M}$ (with θ corresponding to an angle $\Theta = \theta/2$ in 3-space). If the Born rule is violated, we can expect to observe deviations from the sinusoidal modulation (78).

For example, if single photons pass through a linear polariser, the emerging beam will be fully polarised ($P = 1$). If the beam then impinges on a second linear polariser, at angle Θ with respect to the first, (78) predicts a transmission probability

$$p_{\mathrm{QT}}^{+}(\Theta) = \cos^2 \Theta \,. \tag{79}$$

If we vary Θ over an ensemble of measurements, deviations from (79) would signal an unambigous violation of the Born rule.

It should be emphasised that the linearity in $\mathbf{m}$ of the quantum expectation value (77) is equivalent to additive expectation values for incompatible or non-commuting observables (such as spins along different axes) [43]. The latter is a deep and remarkable property of quantum states [73], which is violated in quantum nonequilibrium (for general deterministic hidden-variables theories) [43]. A nonlinear expectation value

$$E(\mathbf{m}) = \epsilon + P_i m_i + Q_{ij} m_i m_j + R_{ijk} m_i m_j m_k + \; ... \tag{80}$$

(summing over repeated indices, where in Bloch space ϵ is a constant scalar, P_i is a constant vector, and Q_{ij}, R_{ijk}, ... are constant tensors) then provides a simple and universal signature of quantum nonequilibrium [19], with an anomalous outcome probability

$$p^{+}(\mathbf{m}) = \frac{1}{2}\left(1 + E(\mathbf{m})\right) \,. \tag{81}$$

Note that the vector P_i in (80) can differ from the quantum-theoretical polarisation vector $\mathbf{P}^{\mathrm{QT}}$ expected from the usual quantum state preparation.

For a two-state quantum system deviations from the Born rule can also be parameterised by a spherical harmonic expansion

$$p^{+}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} b_{lm} Y_{lm}(\theta, \phi) \,, \tag{82}$$

where **m** is specified by angular coordinates (θ, ϕ) on the Bloch sphere and we may take $\mathbf{P}^{\mathrm{QT}}$ to point along $+z$.[14] In quantum equilibrium $p^+ = p^+_{\mathrm{QT}}$ and the only non-vanishing coefficients are

$$b^{\mathrm{QT}}_{00} = \sqrt{\pi} \ , \ b^{\mathrm{QT}}_{10} = P\sqrt{\pi/3} \ . \tag{83}$$

Anomalous coefficients can in principle be calculated from a hidden-variables theory (such as pilot-wave theory) with corrections to the Born rule.

Experimentally, we would hope to set upper bounds on the magnitudes of the parameters ϵ, Q_{ij}, R_{ijk}, ... , or equivalently, upper bounds on the magnitudes of the coefficients b_{lm} $(lm \neq 00, 10)$. This could involve careful analysis of spin or polarisation data from high-energy collisions.

In recent years, high-energy experiments have been repurposed as novel tests of entanglement and Bell's inequality [74]. Such experiments may also be repurposed as tests of the Born rule at high energies and at short times. Strong interactions take place over timescales of order 10^{-23} s. To probe the Born rule at such short times, we might consider photons emerging from a strong-interaction process (where photons can be radiated by quarks). By monitoring their polarisation probabilities, we may search for nonlinear expectation values (80) and associated violations of the Born rule.

5   Conclusion

In the spirit of our opening quote from Feynman's 1964 lectures on *The Character of Physical Law* [1], we have seen that de Broglie's pilot-wave theory is richly suggestive of new physics. We have summarised how the theory can naturally depart from quantum mechanics in three specific domains: the early universe, quantum gravity, and high-energy physics. In all three cases, we are able to predict new effects that are potentially observable or at least amenable to constraint by experiment.

Our quote from Feynman was taken from the final chapter of those lectures, entitled 'Seeking New Laws', which attempts to outline the various ways in which new laws (hence new physics) are most likely to be found. According to Feynman, it is often underestimated just how difficult it is to find new laws that account for what is known while at the same time making new predictions:

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[14]The reality of p^+ requires $b^*_{lm} = (-1)^m b_{l(-m)}$.

'If you can find any other view of the world which agrees over the entire range where things have already been observed, but disagrees somewhere else, you have made a great discovery. It is very nearly impossible, but not quite, to find any theory which agrees with experiments over the entire range in which all theories have been checked, and yet gives different consequences in some other range ... .' (ref. [1], pp. 171–2)

The often haphazard and serendipitous history of science suggests that we be wary of making hard and fast rules about how science should proceed. Nor should we be overly confident about which tactics are most likely to succeed. However, it seems fair to suggest that the de Broglie-Bohm or pilot-wave approach to quantum physics deserves more attention than it has generally received, in particular as regards the search for new physics. As Feynman rightly emphasised, to have a theory that agrees with experiment in domains thus-far explored, and which naturally suggests new physics in domains yet-to-be explored, is something of a rarity – a valuable asset whose implications deserve to be taken seriously. Whether or not that suggested new physics will ever be observed is, of course, a matter for further research and experiment.

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