Revisiting de Broglie's double solution program.

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RÉSUMÉ. *Dans un volume précédent des annales, nous avons proposé une équation de Schrödinger non-linéaire qui, couplée à un ansatz dit de factorisation, permettait de construire des solitons qui suivent les trajectoires de de Broglie-Bohm associées à l'onde pilote, elle même solution de l'équation de Schrödinger linéaire. Ici nous entendons exposer une version synthétique de ce modèle, et nous nous intéressons aussi aux questions suivantes: en quoi notre modèle conforte t'il les idées originales de de Broglie sur la double solution? et en quoi s'en démarque t'il?*

ABSTRACT. *In a previous issue of the annales, we proposed a non-linear Schrödinger equation which, coupled to the so-called factorisation ansatz, allowed us to build solitons which follow the de Broglie-Bohm trajectories associated to the pilot wave, the latter being solution of the linear Schrödinger equation. Here we intend to present a synthetcic version of this model, and we also tackle the following questions: whereby does our model comfort de Broglie's original assumptions regarding the double solution? and whereby does it differ from it?*

Key words double solution, soliton, non-linear Schrödinger equation, wave monism, trajectories in configuration space.

1 Introduction: a solution of de Broglie's double solution program.

1.1 Double solution program.

Louis de Broglie proposed in 1927 [1] a realistic interpretation of the quantum theory in which particles are guided by the solution of the linear Schrödinger equation (Ψ_L). The theory was generalised by David Bohm in 1952 [2, 3]. Certain ingredients of de Broglie's original idea disappeared in Bohm's formulation, in particular the double solution program, according to which the particle is associated to a wave u distinct from the pilot-wave Ψ_L, u being sometimes treated as a moving singularity [4], and sometimes as a solution ϕ_{NL} of a non-linear equation of very high amplitude, a "hump" ([5, 6, 7]). In a previous paper [8] we focused on the second alternative ("hump") that would be associated to a non-linear wave equation about which de Broglie wrote [6]

" ... a set of two coupled solutions of the wave equation: one, the Ψ wave, definite in phase, but, because of the continuous character of its amplitude, having only a statistical and subjective meaning; the other, the u wave of the same phase as the Ψ wave but with an amplitude having very large values around a point in space and which $(\cdots)$ can be used to describe the particle objectively."...

We are thus looking for a solitonic solution of a non-linear self-focusing equation, represented here by ϕ_{NL}, which supposedly has a very small size, which is reminiscent of Bohm's description of particles as material points.

The pilot wave interpretation (also commonly called de Broglie-Bohm (dBB) interpretation or simply Bohm interpretation [2, 3]) which is the backbone of the double solution program postulates[1] that

-(i) particles follow trajectories which obey the guidance equation (or the quantum potential in Bohm's approach);

-(ii) the distribution of positions at a certain time t_0 obeys the Born rule

-(iii) each measurement is in the last resort a measurement of position.

As the guidance equation is derived from the equation of conservation associated to Schrödinger's equation, combining postulates (i) and (ii)

[1] Mathematical details and precise definitions can be found in appendix.

ensures that the Born rule is satisfied at any time, at least when the observable that we consider is the position of the particle (this is the so-called equivariance property).

Taken together, (i), (ii) and (iii) ensure that the dBB interpretation leads to exactly the same predictions as the orthodox quantum theory.

1.2 Single particle case

In our model [8], the particles are represented by solitons (the ϕ_{NL} wave-functions) which are supposedly localized over tiny regions of space. In accordance with de Broglie's double solution program, the solitonic trajectories obey the guidance equation (see appendix for more details about these concepts). This means that if we write the pilot wave[2] in polar form:

$$\Psi_L(\mathbf{x},t) = R_L(\mathbf{x},t)e^{i\,\varphi_L(\mathbf{x},t)}, \tag{1}$$

where $R_L(\mathbf{x},t)$ and $\varphi_L(\mathbf{x},t)$ are two real functions, then the velocity of the solitons obeys at all times the guidance equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_L(\mathbf{x},t)\Big|_{\mathbf{x}=\mathbf{x}(t)} \tag{2}$$

In the single particle case, our model is based on three conditions (which admit a straightforward generalisation to the configuration space when several particles are present):

1. **Factorisation ansatz:** it is assumed that the "full" wave function denoted Ψ is the **product** of the pilot wave Ψ_L with the particle wave ϕ_{NL}

$$\Psi(t,\mathbf{x}) = \Psi_L(t,\mathbf{x}) \cdot \phi_{NL}(t,\mathbf{x}), \tag{3}$$

where Ψ_L, the so-called "pilot" wave, is a solution of the linear Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_L(t,\mathbf{x}) = (\frac{-\hbar^2}{2m}\nabla^2 + V^L(x,y,z,t))\Psi_L(t,\mathbf{x}), \tag{4}$$

while $\phi_{NL}(t,\mathbf{x})$ is supposed to be localized over a very small region of space, and V^L represents usual interactions (e.g. electro-magnetic).

[2]The pilot wave is a particular solution of Schrödinger's linear equation associated to the quantum system of interest; in more conventional descriptions it is called the wave function or quantum state of the system.

2. **Phase harmony:** it is assumed that the particle (soliton) is represented by a purely real function: $Im.\phi_{NL}(t,\mathbf{x}) = 0$. Taking the factorization ansatz into account, this implies that the phase of the full wave function Ψ is equal to the phase of the pilot wave Ψ_L, which is reminiscent of a condition baptised by de Broglie as the "phase harmony" condition.

3. Finally, $\Psi(t,\mathbf{x})$ is assumed to obey a **non-linear evolution equation** which reads

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m} + V^L(t,\mathbf{x})\Psi(t,\mathbf{x}) + V^{NL}(\Psi)\Psi(t,\mathbf{x}), \quad (5)$$

where V^L represents an arbitrary linear potential, of the type commonly considered when solving the linear Schrödinger equation (for instance an electro-magnetic potential) while V^{NL} represents a non-linear potential which we assume to be equal to the difference between the so-called quantum potential evaluated[3] at the level of the pilot wave Ψ_L with its counterpart evaluated at the level of the "full" wave function Ψ[4]:

$$V^{NL}(\Psi) = V_L^Q(\Psi_L) - V^Q(\Psi) = \frac{-\hbar^2}{2m}\frac{\Delta \mid \Psi_L(t,\mathbf{x}) \mid}{\mid \Psi_L(t,\mathbf{x}) \mid} + \frac{\hbar^2}{2m}\frac{\Delta \mid \Psi(t,\mathbf{x}) \mid}{\mid \Psi(t,\mathbf{x}) \mid} \quad (6)$$

Writing the soliton in polar form:

$$\phi_{NL}(\mathbf{x},t) = R_{NL}(\mathbf{x},t)e^{i\,\varphi_{NL}(\mathbf{x},t)}, \quad (7)$$

where the amplitude of ϕ_{NL} is denoted R_{NL} and its phase φ_{NL}, and making use of the factorization ansatz, one can check easily that the non-linear potential can also be expressed as follows:

$$V^{NL}(\Psi) = \frac{\hbar^2}{2m}\cdot(\frac{\Delta R_{NL}(\mathbf{x},t)}{R_{NL}(\mathbf{x},t)} + 2\frac{\nabla R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)}\cdot\frac{\nabla R_{NL}(\mathbf{x},t)}{R_{NL}(\mathbf{x},t)}) \quad (8)$$

As we have shown previously [8], this choice for V^{NL} has three consequences:

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[3]Let us denote $V^Q(f)$ the "quantum potential evaluated at the level of the function $f(t,\mathbf{x})$". By definition, $V^Q(f) = \frac{-\hbar^2}{2m}\frac{\Delta|f(t,\mathbf{x})|}{|f(t,\mathbf{x})|}$. See appendix for more details.

[4]Obviously this potential is NOT linear because $V^{NL}(\Psi_1+\Psi_2)\cdot(\Psi_1+\Psi_2) \neq V^{NL}(\Psi_1)\cdot(\Psi_1) + V^{NL}(\Psi_2)\cdot(\Psi_2)$.

i) it concentrates the wave function of the particle over a tiny region of space, in accordance with de Broglie's double solution program.

ii) if phase harmony is realized at an arbitrary time $t = 0$, it is still true at any time t. In other words, our choice of potential guarantees that the soliton ϕ_{NL} remains a pure real function at all times, provided it is real at a given time t_0.

iii) the velocity of the (barycentre of) $\phi_{NL}(t, \mathbf{x})$ obeys the guidance equation (2), which completes the fulfillment of de Broglie's program.

1.3 Main features of our model.

Combining equations (3,4,5), expressing $\Psi_L(\mathbf{x}, t)$ in function of its modulus and its phase through $R_L(\mathbf{x}, t)e^{i\varphi_L(\mathbf{x},t)}$, and also making use of the identity $\bigtriangledown\Psi_L(\mathbf{x}, t) = (\bigtriangledown R_L(\mathbf{x}, t))e^{i\varphi_L(\mathbf{x},t)} + \Psi_L(\mathbf{x}, t)i\bigtriangledown\varphi_L(\mathbf{x}, t)$, it is straightforward to show that ϕ_{NL} obeys the non-linear equation

$$\begin{aligned} i\hbar \cdot \frac{\partial \phi_{NL}(\mathbf{x}, t)}{\partial t} &= \\ -\frac{\hbar^2}{2m} \cdot \Delta\phi_{NL}(\mathbf{x}, t) &- \frac{\hbar^2}{m} \cdot (i\bigtriangledown\varphi_L(\mathbf{x}, t) \cdot \bigtriangledown\phi_{NL}(\mathbf{x}, t) \\ + \frac{\bigtriangledown R_L(\mathbf{x}, t)}{R_L(\mathbf{x}, t)} &\cdot \bigtriangledown\phi_{NL}(\mathbf{x}, t)) + V^{NL}(\Psi)\phi_{NL}(\mathbf{x}, t). \end{aligned} \tag{9}$$

This allows us to replace equation (5) by equation (9).

Let us consider the temporal evolution of the imaginary part of the soliton.

$$\begin{aligned} -\frac{\partial Im.(\phi_{NL}(\mathbf{x}, t))}{\partial t} &= \frac{1}{\hbar}Re.(i\hbar \cdot \frac{\partial \phi_{NL}(\mathbf{x}, t)}{\partial t}) = \\ \frac{1}{\hbar}Re.(-\frac{\hbar^2}{2m} \cdot \Delta\phi_{NL}(\mathbf{x}, t) &- \frac{\hbar^2}{m} \cdot (i\bigtriangledown\varphi_L(\mathbf{x}, t) \cdot \bigtriangledown\phi_{NL}(\mathbf{x}, t) \\ + \frac{\bigtriangledown R_L(\mathbf{x}, t)}{R_L(\mathbf{x}, t)} &\cdot \bigtriangledown\phi_{NL}(\mathbf{x}, t)) + V^{NL}(\Psi)\phi_{NL}(\mathbf{x}, t)) \end{aligned} \tag{10}$$

Making use of the equation (8), we find that if at any time the soliton is a purely real function of space, then

$$-\frac{\partial Im.(\phi_{NL}(\mathbf{x}, t))}{\partial t} = 0. \tag{11}$$

so that it remains so at all times. We shall always assume here that it is well so so that $\phi_{NL}(\mathbf{x},t) = R_{NL}(\mathbf{x},t)$. This establishes phase harmony.

It is worth noting at this level that in previous attempts to realize the double solution program phase harmony had to be postulated independently of the dynamics [9]. Here it appears to be a consequence of the dynamics.

Then, equation (9) reduces to a system of two equations

$$Im.(\phi^{NL}) = 0 \quad (12)$$

$$\frac{\partial Re.(\phi^{NL}(\mathbf{x},t))}{\partial t} = \frac{\partial R_{NL}(\mathbf{x},t)}{\partial t} = -\frac{\hbar}{m}\bigtriangledown\varphi_L(\mathbf{x},t)\cdot\bigtriangledown R_{NL}(\mathbf{x},t) \quad (13)$$

Assuming, in accordance with de Broglie's program, that the width of the peaked soliton is quite smaller than the typical scales of variation of $\varphi_L(\mathbf{x},t)$, equation (13) is nothing else than the guidance equation (2). Indeed, imposing that $\bigtriangledown\varphi_L(\mathbf{x},t) = \bigtriangledown\varphi_L(\mathbf{x}_0(t))$, with $\mathbf{x}_0(t) = \mathbf{x}_0(t = 0) + \int_0^t dt\mathbf{v}_{dB}(t)$ and $\mathbf{v}_{dB}(t) = \frac{\hbar}{m}\bigtriangledown\varphi_L(\mathbf{x}_0(t))$, it is straightforward to check that the solution of (13) is a solitary wave (we call in the present context a solitary wave a wave that keeps the same shape at all times, what is sometimes called a soliton in other contexts):

$R_{NL}(\mathbf{x},t) = R_{NL}(\mathbf{x} - (\mathbf{x}_0(t) - \mathbf{x}_0(t = 0)), t = 0)$.

Remarkably, there is no constraint, at this level, concerning the shape of the soliton $R_{NL}(\mathbf{x}, t = 0)$, excepted that $\varphi_L(\mathbf{x},t)$ varies very slowly over the region where the soliton is concentrated. In our previous paper [8], we studied two particular cases where the guidance equation is exactly fulfilled, which happens when $\bigtriangledown\varphi_L(\mathbf{x},t)$ does not depend on the position. This occurs when the pilot wave is a plane wave, in the free particle case, or when it is a coherent state of a harmonic oscillator. In that case, the Laplacian of $\varphi_L(\mathbf{x},t)$ cancels everywhere and the pilot wave also behaves as a solitary wave, in virtue of the conservation equation which then reads $\frac{\partial R_L(\mathbf{x},t)}{\partial t} = -\frac{\hbar}{m}\bigtriangledown\varphi_L(t)\cdot\bigtriangledown R_L(\mathbf{x},t)$. It also admits solutions of the type $R_L(\mathbf{x},t) = R_L(\mathbf{x}-(\mathbf{x}_0(t)-\mathbf{x}_0(t = 0))), t = 0)$ where $\mathbf{x}_0(t)$ represents the barycentre of the pilot wave.

1.4 Analysis of the 2nd order in time dynamics.

At this level, it is worth to put into evidence some qualitative features of the non-linear potential which help to understand in a less formal way how at the end we succeed in realizing de Broglie's double solution program. To do so, let us consider a particular case: the pilot wave is a gaussian packet of which the size is supposedly quite larger than the size of the soliton which is also supposed to be gaussian (such a soliton is a so-called "gausson"). For the sake of simplicity we shall limit ourselves here to 1 dimension of space, but the generalisation to 3 dimensions is straightforward.

The quantum potential in the case of a gaussian wave function $exp^{-(Ax^2+Bx+C)}$ is equal, up to a constant factor, to

$4(Re.A)^2(x- < x >)^2$.

Therefore, up to an irrelevant additive constant, $V_L^Q(\Psi_L) = -\frac{k}{2}(x - x_0^L)^2$, and $-V^Q(\Psi) = +\frac{\tilde{k}}{2}(x - x_0)^2$ where x_0^L represents the peak of the gaussian pilot wave while x_0 represents the peak/barycentre of Ψ; $\tilde{k}$ is an effective spring constant associated to Ψ. Because of the factorisation ansatz, it is the sum of k with the spring constant associated to the soliton (gausson-R_{NL}). $\tilde{k}$ is quite larger than k because the gausson is supposedly quite narrower than the pilot wave so that in good approximation the barycentre of the gausson is not distinguishable from the barycentre of Ψ.

$V_L^Q(\Psi_L)$ plays relatively to the gausson the same role here as an external time-dependent potential because it does not depend on the wave function of the gausson. It is accelerating but not self-focusing. In virtue of Ehrenfest's theorem it contributes to the acceleration of Ψ just in the same way as the quantum potential of the pilot wave estimated at the location of the gausson. It is self-antifocusing in the present case because it is a repulsive quadratic potential centered around the peak of the pilot wave.

On the contrary, $-V^Q(\Psi)$ is not self-accelerating:

$\frac{1}{m} < \Psi \mid \nabla V^Q(x,t) \mid \Psi >= 0$, because when we integrate the gradient of a gaussian function symetrically around its peak, it cancels by symmetry[5].

[5] The use of brackets here does not mean that we are estimating statistical averages as is usually done in standard quantum mechanics. Our idea is, as already explained previously [8], to estimate the localisation of the soliton by computing its barycentre, evaluated through the expression $\frac{<\Psi|x|\Psi>}{<\Psi|\Psi>}$. Now, the equation of evolution (5) preserves the norm and we chose to normalize Ψ to unity in the present paper, for the sake of simplicity, so that the barycentre is $< \Psi \mid x \mid \Psi >$ but other choices are possible [8]. In any case, if we would choose for instance a weight in $|\Psi|^4$ and not in $|\Psi|^2$ for estimating the position of the barycentre, there would be no notable difference because the soliton is peaked inside a small region of space.

This result can actually be generalised to all cases where the potential is an analytic function of the modulus of the wave function as we have shown in our previous paper [8].

It can also be established in full generality as has been shown recently by collaborators in Marseille [10]. The proof goes as follows

$< \Psi \mid \partial_x V^Q(x,t) \mid \Psi >= \frac{-\hbar^2}{2m} \int d^3x \mid \Psi(t,\mathbf{x}) \mid^2 \partial_x(\frac{\Delta|\Psi(t,\mathbf{x})|}{|\Psi(t,\mathbf{x})|})$

Integrating by parts, we find that $\frac{-\hbar^2}{2m} \int d^3x \mid \Psi(t,\mathbf{x}) \mid^2 \partial_x(\frac{\Delta|\Psi(t,\mathbf{x})|}{|\Psi(t,\mathbf{x})|})$

$= \frac{-\hbar^2}{2m} \int d^3x(-2) \mid \Psi(t,\mathbf{x}) \mid \partial_x(\mid \Psi(t,\mathbf{x}) \mid)(\frac{\Delta|\Psi(t,\mathbf{x})|}{|\Psi(t,\mathbf{x})|})$

$= \frac{-\hbar^2}{2m} \int d^3x(-2) \mid \Psi(t,\mathbf{x}) \mid \partial_x(\mid \Psi(t,\mathbf{x}) \mid)(\frac{(\partial_x^2+\partial_y^2+\partial_z^2)|\Psi(t,\mathbf{x})|}{|\Psi(t,\mathbf{x})|})$

$= \frac{\hbar^2}{m} \int d^3x(\partial_x(\mid \Psi(t,\mathbf{x}) \mid)(\partial_x^2 + \partial_y^2 + \partial_z^2) \mid \Psi(t,\mathbf{x}) \mid$

Now,

$\frac{\hbar^2}{m} \int d^3x(\partial_x(\mid \Psi(t,\mathbf{x}) \mid)(\partial_x^2 \mid \Psi(t,\mathbf{x}) \mid)= \frac{\hbar^2}{2m} \int d^3x\partial_x(\partial_x \mid \Psi(t,\mathbf{x}) \mid)^2$

$\frac{\hbar^2}{m} \int d^3x(\partial_x(\mid \Psi(t,\mathbf{x}) \mid)(\partial_y^2 \mid \Psi(t,\mathbf{x}) \mid)=$

$=-\frac{\hbar^2}{m} \int d^3x(\partial_y\partial_x(\mid \Psi(t,\mathbf{x}) \mid)(\partial_y \mid \Psi(t,\mathbf{x}) \mid)$

$=-\frac{\hbar^2}{2m} \int d^3x\partial_x(\partial_y \mid \Psi(t,\mathbf{x}) \mid)^2.$

In the same fashion we get

$\frac{\hbar^2}{m} \int d^3x(\partial_x(\mid \Psi(t,\mathbf{x}) \mid)\partial_z^2 \mid \Psi(t,\mathbf{x}) \mid=-\frac{\hbar^2}{2m} \int d^3x\partial_x(\partial_z \mid \Psi(t,\mathbf{x}) \mid)^2.$

As the wave function cancels at infinity, $< \Psi \mid \partial_x V^Q(x,t) \mid \Psi >$ is identically equal to 0, independent of the expression of $\Psi(t,\mathbf{x})$. The generalisation to $< \Psi \mid \partial_y V^Q(x,t) \mid \Psi >$ and $< \Psi \mid \partial_z V^Q(x,t) \mid \Psi >$ is straightforward.

$-V^Q(\Psi)$ is however self-focusing, it is an attractive quadratic potential centered around the barycentre of Ψ which is itself very close to the barycentre of the gausson; its presence guarantees that the gausson keeps a quasi-gaussian shape throughout time.

When the soliton is quite smaller than the typical variation scale of the pilot wave, in the case of quasi-gaussian shapes for the soliton, the self-focusing of $-V^Q(\Psi)$ largely dominates the possible self-defocusing of $V_L^Q(\Psi_L)$ which explains qualitatively why our solitonic solutions are stable.

In virtue of Eherenfest's theorem, the peak (barycentre) of Ψ undergoes an acceleration resulting from three potentials, the classical, linear potential, and the two quantum potentials.

To begin with, let us take account of the presence of the linear potential V^L at the level of equation (5). Its contribution is equal to the local acceleration resulting from the classical potential, estimated at the level of the soliton.

-this is equal to the classical acceleration in good approximation:

$\frac{1}{m} < \Psi \mid (-)\nabla V^L(x,t) \mid \Psi >\approx \frac{-1}{m}\nabla V^L(x_0,t)$ where x_0 represents the peak/barycentre of Ψ;

-the second one is the self-acceleration of the gausson for which only $V_L^Q(\Psi_L)$ contributes:

$\frac{1}{m} < \Psi \mid (-)\nabla V^{NL}(x,t) \mid \Psi >= \frac{1}{m} < \Psi \mid (-)\nabla V_L^Q(x,t) \mid \Psi >\approx \frac{-1}{m}\nabla V_L^Q(x_0,t)$.

Remaining in the limit case where we may neglect the variation of the gradient of $V_L^Q(\Psi_L)$ over the size of the soliton, the resulting self-acceleration is equal in good approximation to the acceleration undergone by a particle of which the trajectory obeys the guidance equation (2) (this acceleration obeys indeed the constraint (18) as explained in appendix); again, the small size of the soliton is a key ingredient for guaranteeing that the dBB dynamics (2,18) is satisfied.

1.5 Composite systems

It has also been shown in the past [11] that when more than one particle is present, a straightforward generalisation of conditions 1,2,3 makes it possible to generalize the single particle model to a multiparticle model endowed with the same properties as in the single particle case (i,ii,iii): then, the particles are represented by a product of localized solitons, while the pilot wave is, as usually, defined over the configuration space.

For instance, in the case of a double Stern-Gerlach interferometer, the full wave function takes the form

$|\Psi\rangle = |\Psi_L\rangle \cdot |\phi_{NL}\rangle$,

where $|\Psi_L\rangle = \psi_{++}((t,\mathbf{x}_A,\mathbf{x'}_B)| +^A +^B\rangle + \psi_{+-}((t,\mathbf{x}_A,\mathbf{x'}_B)| +^A -^B\rangle$ $+ \psi_{-+}((t,\mathbf{x}_A,\mathbf{x'}_B)| -^A +^B\rangle + \psi_{--}((t,\mathbf{x}_A,\mathbf{x'}_B)| -^A -^B\rangle$,

while $|\phi_{NL}\rangle = \phi^A_{NL}(t,\mathbf{x_A}) \cdot \phi^B_{NL}(t,\mathbf{x'_B})$.

It is straightforward to extend this model to the N particles case. In the rest of the paper, we will assume that electrons, neutrons and protons are the elementary particles on which our model is based, and that the associated solitons have a size comparable to their Compton wavelength. We will also assume that their shape is gaussian (other choices are possible but we will stick to these choices for the sake of simplicity).

2 Whereby does our model comfort de Broglie's original assumptions regarding the double solution?

2.1 Phase harmony

Phase harmony is a key ingredient of our model: the soliton possesses no complex phase but the full wave (Ψ) is the product of the soliton with the pilot wave. Its phase is thus equal to the phase of the pilot wave.

2.2 Factorisability of the wave associated to a composite system into individual components

In the way we treat composite systems, the solitonic contributions are a product of all individual solitonic contributions. From this point of view, each component possesses its own individuality. Even in the case of undistinguishable particles, two electrons for instance, each electron is endowed with a well-defined position and at the solitonic level they are thus distinguishable [12].

3 Whereby does our model differ from de Broglie's original assumptions regarding the double solution?

3.1 Factorisability of the wave associated to a composite system into individual components

The pilot wave however is in general entangled. Non-separability and non-locality naturally appear in our model because solitonic trajectories belong to the configuration space. Even tough our model respects phase harmony, the phase of the pilot wave is defined at the level of the configuration space in the case of composite systems.

It is not sure that de Broglie would have appreciated our model for these reasons because, already in 1927 [1], but also later with Andrade e Silva [13] and actually for the rest of his life [14], de Broglie tried to get rid of the configuration space[6], without notable success.

[6] For instance, in the 50's, he wrote the following [6]: …*"Or, la méthode de Schrödinger implique nécessairement l'emploi de l'espace de configuration et ne permet pas de se représenter le phénomène physique constitué par le mouvement des*

3.2 Interplay between the soliton and the pilot wave.

The picture that is often given of the double solution is the following [6] : the pilot wave and the soliton (often denoted the "u" wave in the literature) bounce together in the tiny region where the soliton is located. The amplitude of the soliton is supposedly quite larger than the amplitude of the pilot wave in this region. Outside this region, the non-linearity is not activated, linearity is reestablished and the tails of the soliton merge into the pilot wave. In this picture it is implicitly taken for granted that the full wave is the sum of the pilot wave with the u (solitonic) wave. Obviously there is a major flaw in this approach: in the case of non-linear equation the superposition priciple is no longer valid. In our approach the picture is totally different: the tails of the full wave (Ψ) are, like everywhere in space, the product of the soliton with the pilot wave. The non-linearity does not disappear far away from the soliton. This reflects a well-known property of the quantum potential: $V^Q(f)$, the "quantum potential evaluated at the level of the function $f(t,\mathbf{x})$"being defined to be equal to $V^Q(f) = \frac{-\hbar^2}{2m}\frac{\Delta|f(t,\mathbf{x})|}{|f(t,\mathbf{x})|}$, it does not depend on the scaling of f (it is invariant under rescaling). The non-linearity, in our model, does not vanish far away from the place where the soliton is concentrated...

3.3 Relativistic invariance.

Finally it is not clear at all how one could generalize our model in order to comply with relativistic invariance. Here again, it is not sure that de Broglie would have appreciated our model because relativistic invariance was at the core of all his research [7]. Like Einstein, de Broglie remained during all his life what we call today a "local realist" [16]

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corpuscules dans le cadre de l'espace physique. Sans doute la Mécanique classique se servait-elle souvent, elle aussi, de l'espace de configuration, mais ce n'était pas pour elle une nécessité: elle pouvait raisonner en considérant le mouvement des points matériels du système dans l'espace à trois dimensions et elle n'employait l'espace de configuration que comme un artifice mathématique permettant de présenter plus élégamment ou d'effectuer plus aisément certains calculs. Dès l'apparition des Mémoires de Schrödinger, tout en reconnaissant l'exactitude des résultats obtenus par sa méthode, j'avais trouvé paradoxal le principe même de cette méthode"... In ref.[15] he added: *...the fictitious space has never satisfied me , and I have done great work on this. In particular, one of my students, Mr Andrade e Silva, did a doctoral thesis to show how one can interpret this with our ideas, that is to say that everything happens in a physical space and it is only a certain representation in the configuration space...*

4 Conclusions

As we have shown, it is possible to find a non-linear potential which, coupled with the factorization ansatz, ensures that one can find solitons, located in tiny regions of space which move according to the guidance equation. Together with recent results about quantum equilibrium (see appendix for more details), our model provides a possible explanation of the properties of quantum systems at least in the non-relativistic limit. It is not sure that de Broglie would have approved our approach however for several reasons:

-our model remains intrinsically non-local because we did not find our way out of the configuration space;

-it does not possess any relativistic generalisation;

-the non-linearity remains huge far away from the tiny region where the soliton is located.

Finally, as we noted previously [8], the model is not really a wave monistic model because the non-linearity clearly establishes a distinction, to begin with, between the pilot wave and the soliton. Particles do not spontaneously "emerge" from the waves in our approach. One could argue that, as in standard mechanics the pilot wave is in the last resort determined by the preparation process. while the position of the soliton is out of control for the experimentalist. However, the double nature of the quantum object, wave and particle, here wave and soliton, is already implicit from the beginning, at the level of the non-linear potential (6).

Appendix: the (deterministic) de Broglie-Bohm dynamics; equivariance, guidance equation and quantum potential.

Pilot wave interpretation.

The dBB interpretation [17] is a dynamical and deterministic formulation of quantum mechanics in which it is assumed that the positions of the particle exist at all times, i.e. independently of the observer. We will consider, in what follows, a single spinless and non-relativistic particle for which a quantum wave function (also called the pilot wave) $\Psi_L(\mathbf{x},t)$ solves the (linear) Schrödinger equation:

$$i\hbar\frac{\partial\Psi_L(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_L(\mathbf{x},t) + V^L(\mathbf{x},t)\Psi_L(\mathbf{x},t), \tag{14}$$

where $V^L(\mathbf{x},t)$ is an external potential[7]. In the standard formulation of quantum mechanics, the probability distribution of all particle positions $P_{dB}(\mathbf{x},t)$ obeys the Born rule $P_{dB}(\mathbf{x},t) = |\Psi_L(\mathbf{x},t)|^2$. For convenience, let us express the wave function in polar form:

$$\Psi_L(\mathbf{x},t) = R_L(\mathbf{x},t)e^{i\,\varphi_L(\mathbf{x},t)}, \tag{15}$$

where $R_L(\mathbf{x},t)$ and $\varphi_L(\mathbf{x},t)$ are two real functions. The probability distribution is then given by $P_{dB}(\mathbf{x},t) = R_L(\mathbf{x},t)^2$ and is conserved through the continuity equation:

$$\frac{\partial R_L(\mathbf{x},t)^2}{\partial t} + \boldsymbol{\nabla}\cdot\left(R_L(\mathbf{x},t)^2\frac{\hbar\boldsymbol{\nabla}\varphi_L}{m}\right) = 0, \tag{16}$$

By analogy with classical hydrodynamics, the phase-function $\varphi(\mathbf{x},t)$ is associated to a velocity field $\mathbf{v}(\mathbf{x},t)$ given by:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_L(\mathbf{x},t)\bigg|_{\mathbf{x}=\mathbf{x}(t)}, \tag{17}$$

which is also called the **guidance equation** of de Broglie. This equation expresses how the pilot wave guides the trajectories of the particles. After integration of (17), the deterministic dB trajectory $\mathbf{x}(t)$ is obtained.

In order to mimick the distribution of positions in $(R_L)^2$ predicted in the standard interpretation, that is to say in order to mimick the Born rule, it suffices to impose that at a certain time t_0, $P_{dB} = (R_L)^2$ everywhere. Then, in virtue of equations (16) and (17), $P_{dB} = (R_L)^2$ everywhere at any time, which is also called the **equivariance** property.

Another ingredient of the dBB interpretation is the idea that every measurement is, in the last resort, a measurement of position. Combining this with equivariance, no experiment makes it possible to distinguish dBB predictions from the standard ones.

Quantum equilibrium

Some years ago, important results were obtained by Antony Valentini and coworkers [18, 19, 20, 21, 22, 23, 24, 25], who established that the Born rule is the consequence of the guidance equation: after a sufficiently

[7] We attributed the label L to this potential because it is assumed that it does not depend on Ψ. It acts thus linearly on the wave function, due to the fact that complex multiplication is distributive relative to addition and commutative.

long time, for nearly any initial distribution of position, the chaotic nature of the dBB dynamics ensures that the distribution will converge to the distribution in $|\Psi|^2$, in accordance with the Born rule. This process is called the onset of quantum equilibrium [26].

The validity of this mechanism has been established for a very large class of hamiltonians and wave functions: the distributions of positions will converge in time to the Born distribution $P_{dB} = (R_L)^2$, even when initially they depart from it. Ultimately the onset of the quantum equilibrium is due to the chaotic nature of the dBB dynamics in the vicinity of zeros of the pilot wave [25].

In virtue of the equivariance property, it suffices that the guidance equation is fulfilled in order that all the predictions of the dBB interpretation or, more generally, of the double solution approach coincide with the standard predictions.

Note that the guidance equation (17) is of the first order in time because it deals with velocities. David Bohm considered the accelerations associated to these velocities and showed that they derive from a non-classical potential, the so called quantum potential [2, 3], here denoted V_L^Q in order the emphasize the fact that it is related to the pilot wave Ψ_L. As can indeed be shown by a lengthy but straightforward computation, combining equations (14), (15) and (17), implies that

$$m\frac{d^2\mathbf{x}(t)}{dt^2} = -\nabla(V^L(\mathbf{x},t) + V_L^Q(\mathbf{x},t)), \tag{18}$$

where

$$V_L^Q(\mathbf{x},t) = \frac{-\hbar^2}{2m}\frac{\Delta R_L(\mathbf{x},t)}{R_L(\mathbf{x},t)} \tag{19}$$

Remark: dB versus dBB dynamics.

As already noted, de Broglie's guidance equation is of the first order in time while Bohm's equation is of the second order. Following conventions introduced in our previous paper [8] we refer to this fine structure in the dynamics by labelling by the label dB "de Broglie" velocities and dynamics as encapsulated in the guidance equation (17).

It is worth noting at this level a very important result demonstrated by Colin and Valentini [27]: equilibrium is reached after a sufficiently long time ONLY when initial velocities obey equation (17). On the contrary, when velocities are distributed arbitrarily, while accelerations obey equation (18), quantum equilibrium does not occur and is even unstable in the sense that when the initial distribution of positions and velocities is out of equilibrium it will converge to equilibrium ONLY if the initial distribution of velocities obeys de Broglie's guidance equation.

Acknowledgments.

Support of the Flemish Fund for Scientific Research (FWO), the John Templeton foundation, the Foundational Questions Institute (FQXI) and in particular the Fondation de Broglie is acknowledged. The author personally thanks Ralph Willox, Samuel Colin and Mohamed Hatifi for enlightening discussions throughout the past two decades.

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(Manuscrit reçu le 9 juillet 2025)