Developing the

Theory of Everything

Third Edition

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Nomenclature

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(V_1, V_2, V_3, V_4) canonical basis in $M_2(\mathbb{C})$, see equation (1.5)

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$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ Dirac matrix (our choice), page 273

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Γ_5 , page 280

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κ constant and non zero integer number, page 292

λ magnetic quantum number, page 72

λ_μ generators of $Cl_{3,1}$, page 277

Λ_a generators of the $Cl_{3,3}$ algebra, page 284

$\mathbb{R}$ field of real numbers, page 33

$\mathbb{C}$ field of complex numbers, page 33

$\mathbb{H} = Cl_3^+$ even sub-algebra of Cl_3 (quaternion field), page 240

$\mathbf{a}$ function of r , page 297

$\mathbf{A} := \gamma_\mu A^\mu$ electromagnetic potential (in space-time algebra), see equation (B.21)

$\mathbf{b}$ function of r , page 297

$\mathbf{c}$ function of r , page 297

$\mathbf{D}$ invariant derivation (extended relativistic invariance), page 192

$\mathbf{d}$ function of r , page 297

$\mathbf{d} = \gamma^\mu \partial_\mu$ Dirac differential operator, page 277

$\mathbf{F} = \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix}$ electromagnetic field (in space-time algebra), see equation (B.22)

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$\mathbf{J}$ probability current, page 103

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$\mathbf{K}$ chiral current, page 103

$\mathbf{l}$ left mass term, see equation (1.149)

$\mathbf{m} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix}$ matrix mass term, see equation (1.149)

$\mathbf{n}$ principal quantum number, page 70

$\mathbf{n} = |\kappa| + n$ (n: degree of radial polynomial functions), page 72

$\mathbf{r}$ right mass term, see equation (1.149)

$\mathbf{u}$ function of θ , page 297

$\mathbf{v}$ function of θ , page 297

$\mathcal{F}(\mathbb{R}^4, \mathbb{C})$ set of all ψ (wave functions), page 26

$\mathcal{F}(\mathbb{R}^4, \mathbb{C}^4)$ set of all ψ (Dirac wave functions), page 27

$\mathcal{G}$ group of complex 2×2 M matrices such that $|\det(M)| = 1$, page 36

$\mathcal{L}$ Lagrangian density, see equation (1.135)

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$\mathcal{L}_q^+$; $\mathcal{L}_q^-$ Lagrangian density (quarks), page 169

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$\mathrm{D}_3 = \mathrm{K}$ second current, see equation (1.95)

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$\mathrm{D}_R^1 = R^1\widetilde{R}^1$ right current, page 47

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j electric current (space-time vector), see equation (A.150)

$\mathrm{J} = \mathrm{J}^\mu\sigma_\mu$ probability current, page 44

$\mathrm{J}^\mu = \overline{\psi}\gamma^\mu\psi$ densities, components of J, page 44

k magnetic current), page 258

$\mathrm{K}^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$ components of the K current, see equation (1.89)

K_l left minus right current, see equation (4.178)

$\mathrm{v} = \frac{1}{\rho}\mathrm{J}$ reduced velocity, page 57

$\mathrm{x} = \mathrm{x}^\mu\sigma_\mu$ general element in space-time, see equation (A.52)

$\nabla = \partial_0 - \vec{\partial}$ first differential operator in space-time, see equation (A.56)

$\nu = E/h$ frequency, see equation (1.215)

$\Omega = r^{-1}(\sin\theta)^{-\frac{1}{2}}S$ dilator, see equation (C.4)

$\Omega_1 = \overline{\psi}\psi$ relativistic invariant, page 45

$\Omega_2 = -i\overline{\psi}\gamma_5\psi$ second relativistic invariant, see equation (1.90)

$\overline{\psi} = \psi^\dagger\gamma_0$ Dirac conjugate, page 45

$\overline{A} = \widehat{A}^\dagger$ A barre, page 243

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ partial derivative, see equation (1.10)

$\phi = \sqrt{2}\begin{pmatrix}\xi_1^1 & -\eta_2^{1*}\\ \xi_2^1 & \eta_1^{1*}\end{pmatrix}$ electron wave, page 41

$\phi = \sqrt{2}\begin{pmatrix}\xi_1^1 & -\eta_2^{1*}\\ \xi_2^1 & \eta_1^{1*}\end{pmatrix}$ wave of the electron, page 41

$\phi_p = -\phi_e\sigma_1$ wave of the positron (in Cl_3), page 67

$\mathbf{\Lambda}_n,\ n = 1, ..., 8$ generator of the $SU(3)_c$ group, see equation (3.48)

$\mathbf{\Lambda}_n,\ n = 1, ..., 8$ generators of the $SU(3)_c$ group, page 157

$\boldsymbol{\partial} = \gamma^\mu\partial_\mu$ opérateur différentiel de Dirac, page 273

$\boldsymbol{\partial}_\nu = \frac{\partial}{\partial X^\nu} = \mathrm{D}^\mu_\nu\partial_\mu$ Dirac operator, page 185

$\Psi : \phi \mapsto \phi_e$ wave with value: operator on Cl_3, see equation (2.1)

$\psi = \psi(x, y, z, t)$ wave function (function of space and time with complex value), see equation (1.1)

Ψ_b wave d(blue)+ u(blue), see equation (2.7)

Ψ_g wave d(green)+ u(green), see equation (2.7)

Ψ_l wave electron + neutrino-monopole, see equation (2.7)

$\Psi_L = \Psi^1_L + \Psi^8_L$ left part of the lepton wave, page 117

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Ψ_r wave d(rot)+ u(rot), see equation (2.7)

ρ main relativistic invariant, see equation (1.93)

ρ_l generalization of ρ in the lepton case, see equation (2.34)

σ_μ Pauli matrices, see equation (1.4)

$\sigma_{21} = \sigma_2\sigma_1$ is a 2-vector in Cl_3, page 42

θ_W Weinberg–Salam angle, see equation (2.212)

θ_W angle de Weinberg-Salam, page 129

$\underline{D}$ gauge-invariant derivative, see equation (3.51)

$\underline{\mathrm{J}} := \frac{m}{k\mathbf{l}}\mathrm{D}^1_L + \frac{m}{k\mathbf{r}}\mathrm{D}^1_R$ weighted current, page 86

$\vec{\partial}'$ $(= \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi)$, see equation (C.4)

$\vec{\partial} = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}$ main differential operator in Cl_3, see equation (A.34)

$\vec{\partial} \cdot \vec{u}$ divergence of $\vec{u}$, see equation (A.38)

$\vec{\partial} \times \vec{u}$ rotational of $\vec{u}$, see equation (A.38)

$\vec{u} \cdot \vec{v}$ scalar product, page 238

$\vec{u} \times \vec{v}$ vector product (or cross product), page 241

$\vec{\text{grad}}\ a = \vec{\partial} a$ gradient of the scalar a, see equation (A.38)

$\vec{E}$ electric field, page 256

$\vec{H}$ magnetic field, page 256

$\vec{u} = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3$ $(\vec{u}^2 = 1)$, page 290

$\widehat{\nabla} = \partial_0 + \vec{\partial}$ second differential operator in space-time, see equation (A.56)

$\widehat{A} = A_1 - A_2$ A hat, see equation (A.29)

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$\widetilde{R}^{3+n} = \widetilde{\phi}^{3+n}\frac{1+\sigma_3}{2}$ $n = 2, 3, 4$, page 151

ξ right wave, see equation (1.3)

ξ^1 right wave of the electron, page 58

$A \mapsto \widetilde{A}$ reverse, see equation (A.8)

$A,\ B,\ C,\ D$ functions of r (radial variable), page 291

$A^\dagger = (A^*)^t$ adjoint(conjugate transposed), see equation (A.28)

$A_1 = a + i\vec{v}$ even part of $A = a + \vec{u} + i\vec{v} + ib$, page 243

$A_2 = \vec{u} + ib$ odd part of $A = a + \vec{u} + i\vec{v} + ib$, page 243

$a_n, n = 1, 2, ..., 6$ invariant densities, see equation (2.31)

B chiral potential (space-time pseudo-vector), page 257

$C_{\mu\nu}$ curvature field, page 202

Cl_2 Clifford algebra of the Euclidean plane, page 239

Cl_3 Clifford algebra of 3-dimensional space, page 240

Cl_3^* group of invertible elements in Cl_3, page 32

$Cl_{1,3}$ space-time, page 272

$Cl_{2,2}$ Clifford algebra with real 4×4 matrices, page 278

$Cl_{3,1}^+$ even subalgebra of space-time, page 278

$Cl_{3,3}$ Clifford algebra End(Cl_3), page 283

$d = \frac{\mathbf{1-r}}{2}$ difference between the two masses, page 55

D^* group of similitudes R, page 35

$d_{n\mu}^p$ vectors of the Lagrangian density (quarks), page 169

$D_{\text{x}} : X \mapsto \text{x} = \phi X \phi^\dagger$ induced similitude, page 83

$d_\mu := -i\partial_\mu + qA_\mu + m_g \text{v}_\mu$ covariant derivative, see equation (1.208)

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G_2 function of r and of θ ($\lambda < 0$ case), page 298

G_3 function of r and of θ ($\lambda < 0$ case), page 298

G_4 function of r and of θ ($\lambda < 0$ case), page 298

$G_{\mu\nu} := \partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu$ gravitational field, page 86

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L^8 right non differential term of the neutrino-monopole, page 103

l^8 left non differential term of the neutrino-monopole, page 103

$L^n = \phi^n \frac{1-\sigma_3}{2}$ $n = 2, 3, 4$, page 151

l_u unit of length, page 205

$m := \frac{m_0 c}{\hbar}$ m_0 is the proper mass, see equation (1.2)

M dilator (general element in Cl_3), page 34

M_ϕ $SL(2,\mathbb{C})$ part of the electron wave, see equation (1.159)

M_ϕ $SL(2,\mathbb{C})$ part of the wave with determinant 1, page 56

$m_a := \frac{\mathbf{l}+\mathbf{r}}{2}$ arithmetic mean, page 55

$m_g = \sqrt{\mathbf{lr}}$ geometric mean, see equation (1.187)

$M_n(\mathbb{C})$ set of $n \times n$ complex matrices

$N = s + v + B + p_v + p_s$ general element of space-time algebra, page 272

n degree of angular polynomials, page 295

$P : A \mapsto \widehat{A}$ parity transformation, see equation (A.29)

$P : M \mapsto \widehat{M}$ main automorphism in Cl_3 (parity), page 42

p^+ projector used with spherical coordinates, page 253

P_+, P_- projectors, see equation (2.54)

$P_\mu,\ \mu = 0, 1, 2, 3$ projectors, see equation (2.58)

p_l projector onto the left wave, page 253

p_l projector used with spherical coordinates, page 253

p_r projector onto the right wave, page 253

r ratio of the R similitude, page 34

$R : \mathrm{x} \mapsto \mathrm{x}' = M\mathrm{x}M^\dagger$ similitude, page 34

r^1 right non differential term of the electron, page 103

$R^1 = \sqrt{2}\begin{pmatrix} \xi_1^1 & 0 \\ \xi_2^1 & 0 \end{pmatrix}$ right part of the ϕ wave, page 41

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$R^n = \phi^n \frac{1+\sigma_3}{2}$ $n = 2, 3, 4$, page 151

R^μ_ν real 4×4 matrix of the similitude R, page 34

$S = e^{-\frac{\varphi}{2} i_3} e^{-\frac{\theta}{2} i_2}$ rotator, see equation (C.4)

$S^{\mu\nu} = i\overline{\psi}\gamma^\mu\gamma^\nu\psi$ electric-magnetic momentum densities, see equation (1.88)

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$SL(2, \mathbb{C})$ group of 2×2 M matrices such as $\det(M) = 1$, page 32

$SO(3)$ group of rotations in 3-dimensional space, page 32

$SU(2)$ subgroup of unitary elements in $SL(2, \mathbb{C})$, page 32

T; $T^{n\mu}_{L\lambda}\ T^{n\mu}_{R\lambda}\ T^{3+n\mu}_{L\lambda}\ T^{3+n\mu}_{R\lambda}$ momentum-energy tensors (quarks), see equation (3.141)

T^μ_ν energy–momentum tensor density (Tetrode's tensor), see equation (1.208)

T^μ_ν energy-impulse tensor density (Tétrode's tensor), page 64

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Y_μ Lagrangian term, see equation (4.136)

$Z'^0 := \frac{Z^0}{\sqrt{3}}$ boson Z'^0, see equation (2.225)

Introduction

During the last century, gravitation was understood as space-time curvature thanks to general relativity (GR), mainly developed by Einstein. A century ago, de Broglie understood this : to the movement of any material particle a "phase wave" was associated, whatever kind or mass the particle may be. Its phase is tuned to the internal phase of the little clock which is the particle. From this idea, but conserving only the wave, quantum physics was built as a field theory, ruled by the gauge invariance. Quantum physics has been built as a theory of gauge invariance. From these theoretical progresses and mostly from a lot of discoveries at particle accelerators, the Standard Model (SM) of quantum physics was gradually built. However, the aim of all theoretical physics, a true unification of these separated parts of physics, quantum physics and General Relativity, the **Theory of Everything**, has not yet been achieved. This ToE is also the end goal of our work.

Many attempts have been made in recent decades, usually beginning with quantum physics and aiming to include gravitation. "Developing the Theory of Everything" also starts from relativistic quantum mechanics, and uses all experimental results of the Standard Model of particle physics, but in an entirely new formulation of quantum mechanics, as explained in Chapter 1. This new approach do not focus on Hermitian fields nor on operators upon those Hermitian fields. Actually the quantum wave is a function of space-time with values in real Clifford algebras. And this new approach allows us to understand the true reason for the quantization of action, as well as the true nature of light and the electromagnetic field, which turns out to be simply momentum–energy components of the quantum wave. Moreover our approach allows us to integrate the spin of the quantum wave to a true fields theory, justifying the exclusion principle without the configuration spaces of non relativistic quantum mechanics. Strong constraints of relativistic physics give again to the phase wave its true nature of physical wave: it does not be a simple mathematical tool allowing us to compute probabilities, without giving up on those probabilities that are truly relevant.

Novelties also arise here from the same minimal mathematical tools used for both SM and GR: the Cl_3 algebra and its multiplicative group

Cl_3^*. They are described in Appendix A. The space-time $Cl_{1,3}$ allows us to describe electro-weak interactions. The End(Cl_3) group, included in the $Cl_{3,3}$ Clifford algebra which is its Lie algebra, contains the Cl_3^* subgroup. They are described in Appendix B. The necessity for this tool in GR comes from Whitney's theorem: space-time being a 4-dimensional manifold, an 8-dimensional linear space is sufficient to obtain an embedding, and moreover Cl_3 is $\mathbb{R}^8$ as linear space and as topological space. Next $Cl_{3,3}$, which contains $Cl_{1,3}$ as sub-algebra, has dimension $64 = 2^{3+3} = 4^3$, which is the number of Christoffel symbols of the space-time manifold.

The use in physics of these algebras began as soon as 1927 with the Pauli algebra Cl_3 and next the Dirac algebra in 1928 in the frame of what is now named the "first quantization". This step of quantization was followed by a second step, the quantification of the fields (electromagnetic field, boson fields of electro-weak and strong interactions). This second step includes all results got by the first one. This comes from the including of Cl_3^*, which is the group of endomorphisms in $\mathbf{C}^2$, in the End(Cl_3) group. This group includes the gauge group of the Standard Model.

From 1967 the first quantization D. Hestenes re-built with the mathematical tool of space-time algebra [77, 78, 79, 80, 83, 82]. The beginning of the present work [12] was made in this framework. The first novelty departing from the Cliffordian framework was the use of an improved wave equation, a nonlinear wave equation which will be presented in Chapter 1. Second novelty: the natural geometric framework of relativistic quantum mechanics did not be the Dirac matrix algebra, nor the space-time algebra a sub-algebra of this complex algebra, but the only Cl_3^* Lie group. This, unknown at the beginning of relativistic quantum mechanics, was a consequence of our preceding works [14, 15, 17, 18, 19].

The Cl_3 algebra, containing its multiplicative group, was first promoted by W. Baylis [3]. It is isomorphic to the even part of the Hestenes' $Cl_{1,3}$ space-time algebra. It is also the Lie algebra of its multiplicative Cl_3^* group, usually named $GL(2, \mathbb{C})$ in Lie group theory. And the new framework, very restricted (8-dimensional instead 16-dimensional), very strained, both algebraically and topologically, first allowed us to see the original Dirac theory as incomplete: We got numerous extra tensor densities from the relativistic link between spinors and tensors. Moreover those tensor densities introduce automatically a difference between left and right waves, and then even the bias for left waves.

The third novelty was the true understanding of the Cl_3^* Lie group as the invariance group of all electromagnetic laws, quantum wave of the electron with spin included [17, 18, 19, 21, 22]. It is this Cl_3^* Lie group, strictly included in its Lie algebra Cl_3, which is essential for the electron quantum wave: an application from space-time, a manifold included in Cl_3, whose scalar fields take values in the maximal commutative sub-ring of Cl_3.

These three steps allowed us to build a completely relativistic theory of electro-weak interactions from the generalization of our improved Dirac equation. This wave equation used a term of proper mass: we made precisely what was impossible with the first theory of weak interactions [114] which suppressed the proper mass. The improved equation was generalized to a wave equation for all fermions and anti-fermions of the first generation [24, 29, 30, 28, 25, 31, 33, 34, 32, 35, 36, 37, 48, 47, 49, 50, 39]. The exposition of all our previous works in [50] gave new results like the Lorentz force acting on the whole lepton wave (magnetic monopole included) and the full coloured quarks wave. We also got a wave equation for the complete wave of colored quarks. In the wave equations a vector described the local reduced velocity of the fluid of probability. This allowed us to partially mask the crossing between right and left spinors in the proper mass term. Consequently we were able to read the wave equations in a recursive manner giving the properties of the gauge fields. And above all we got the **quantization of the kinetic momentum** with the awaited $\hbar/2$ value, both for the electron and for the proton and the neutron, thanks to their three colored quarks. We also completely explained the double logical link between Lagrangian density and wave equation, justifying the least action principle. For the fermion part of the quantum wave, our wave equation ensues from a Lagrangian density which differs only from the one of the Standard Model by its nonlinear mass term. There is no necessity to postulate a Lagrangian density for the boson part of the quantum wave.

About the quarks which make up protons and neutrons of nucleus in atoms we must recall this: the charge of quarks, $+2/3$ of the charge of the positron for the u quark and $-1/3$ of the charge of the positron for the d quark are measured with an extraordinary precision. This is undoubtedly the best experimental result of all physics: if a very tiny difference was existing from these values then a non ionized atom could not be neutral and electrostatic forces between atoms should be much stronger than gravitational forces. The matter could not build stars and planets. It is then very important to satisfy these $+2/3$ and $-1/3$ will be completely explained in this book. And this result is the first and main experimental validation of our theoretical construction. The SM such as described here does not need a choice of the values of the charges, their values are a consequence of the constraints imposed by the relativistic and gauge invariances. About quarks the SM tells us several properties that must be explained: the existence of three and only three states of color for each quark. We explain this three. Fermions also exist in three “generations" very similar. Here also our explanation is simple, this three also comes from the dimension of space. Then we do not wait for a fourth generation, except for a fourth neutrino and we shall explain why [1]. This fourth neutrino may be stable and identical to its anti-particle.

1. But we do not have all answers to the many questions about these three generations, neither about the disparity in masses nor on the change of generation for neutrinos nor on the mixing of generations for quarks in hadrons. Anyway many parameters must be experimentally measured before the building of a complete explicative model.

Numerous attempts at unification were made by extending the restricted framework of the Minkowski space-time and expressing the Dirac wave as a function from curved space-time in the same set of values which remains $\mathbb{C}^4$. This is today considered a completely solved problem, both via the older formalism with the Dirac matrices and via the space-time algebra used by Hestenes and his Cliffordian school [5][6][9][10]. But all these attempts fundamentally confuse two Lie groups structurally different: That is fully explained in Chapter 1. ToE needing absolute mathematical rigor, it is impossible to build the theory on the quicksand of these previous undertakings. For instance, we find that we certainly need a separate space-time manifold; we also absolutely need the $SU(2)$ group in nonrelativistic quantum mechanics and the $SL(2,\mathbb{C})$ group in relativistic quantum mechanics. We use, as enlargement of the possibilities, only the extra dimensions of the invariance group of all physical laws. Various sub-domains of physics, particularly those using cross product, curl, chirality, are completely dependent of these only three dimensions.

What we propose here follows many previous attempts. For instance Weyl's unified theory was based on the idea of gauge that he introduced[113]. This attempt at a unified theory was equivalent to the use of a similitude group as a local invariance group. Weyl's theory was developed at a time when neither the gauge invariance of the quantum wave nor the chirality of weak interactions could be accounted for. Moreover, the one-parameter Lie group generated by the ratio of similitude is the multiplicative group of positive real numbers when a $U(1)$ group is needed for the electric gauge invariance.

Another major attempt was Penrose's theory of twistors [103][104]. Since this theory also began from the Dirac equation and from left and right spinors of the Dirac theory, there are numerous connections between twistor theory and what we study here. Nevertheless, we start in a very different direction. We do not start from the Hamiltonian form of nonrelativistic quantum mechanics: we begin with only the fermionic part of the Lagrangian density in the Standard Model. This is also due to the necessity for a total logical coherence between the Standard Model and general relativity.

The most important attempts at unification were worked out by Einstein from 1917 until his death [113]. He tried various possible paths to unify electromagnetism, gravitation and quantum physics. It is one of his paths that we develop here in Chapter 4, a space-time manifold with torsion. We also follow his idea about the incompleteness of quantum mechanics as built a century ago. We then explain why Einstein could not fully develop this approach himself, when quark properties and the importance of chiral waves were yet unknown. Moreover, Einstein could not foresee the strong link between space-time manifold and the Cl_3^* Lie group.

Many other attempts like string and super-symmetry theories, which were very popular thirty years ago, were based on the use of numerous supplementary dimensions of space-time. The starting point of these theories, aiming for a theory of everything, was a greater gauge group: a simple one unifying electromagnetism, weak interactions and strong interactions, like $SU(5)$ or $SO(10)$. We do not follow this still tempting path. The gauge group of the Standard Model, is actually a sub-group of another group: $\text{End}(Cl_3)$ which is the natural invariance group of the Cl_3 algebra, 64-dimensional on the real field. And this restriction is useful: it gives the reason for the difference between quarks and leptons which do not see strong interactions. That also does not need to be postulated; it is a necessary consequence of the kind of mathematics used in this version of the SM.

So our research path is new: The space-time manifold of General Relativity is parameterized due to the 8 dimensions of Cl_3^*. The electron wave is a function on this manifold, with value in the Cl_3^* Lie group. This multiplicative group contains the $SU(2)$ group of non relativistic quantum mechanics. This Cl_3^* group is also a ring, and a left and right $\mathbb{A}$-module on the sub-ring $\mathbb{A}$ of diagonal terms. Moreover, the $\text{End}(Cl_3)$ group is a 64-dimensional group. It is also a ring, containing Cl_3^* as sub-ring, extensive enough to describe quantum waves of all objects, particles and antiparticles, of each generation. Moreover, multiplication in $\text{End}(Cl_3)$ is a generalization of multiplication in $Cl_3^* = \text{End}(\mathbb{C}^2)$.

This research path does not need new particles, neither new ideas still unknown: we hence respect all of discoveries coming from high energy physics. We corroborate, for the Standard Model, its analysis of so many experimental researches accumulated since a century of quantum physics. The only possible objects that may be added to the ones known are right-handed neutrino waves. We thus study these complete neutrinos which may also be called magnetic monopoles.

The main reason why we are able to add gravitation to the three other kinds of interactions, is that we use a nonlinear term of proper mass in each wave equation. The Weinberg-Salam model could not obtain such mass terms.

Chapter 1 is devoted to the electron in the Cl_3 algebra framework. Most of our novelties are presented there, such as the true number of tensor densities, the improved wave equation, the extended relativistic invariance, two momentum-energy tensors and the link between these tensors and the electromagnetic field. That chapter is the only one which does not seem to be concerned by "second quantization". There we use the notation of first quantization, with experimental results on energy levels obtained in this framework. The main change from the first edition is that we use a mass term for each left and right part of the fermionic wave. Despite its non linearity our wave equation gets again, and enhances, all results obtained from the Dirac equation, especially those giving the electron states in atoms. The validity domain of the wave equation is extended to electron systems, without need of a configuration space. The inclusion of space-time in Cl_3 allows us to link the electromagnetic field to the momentum-energy tensors of the electron wave.

Chapter 2 explains how passing from the Cl_3 algebra to the End(Cl_3) algebra is equivalent to go from first to second quantization for the fermion part of the Standard Model. Through this extended algebra we satisfy the decomposition of the full wave function into sixteen parts, eight left ones and eight right ones. This second chapter also studies weak interactions mixing the electron with the electron neutrino. This neutrino wave is incorporated in the wave function of a leptonic magnetic monopole: this is the only possibility of extending the fermionic wave function as allowed by the Standard Model. We also fully explain the origin of the extremal principle and of the quantization of action. We present a soliton solution for a self-interacting electron, an edge case of negligible exterior electromagnetic potential.

The study is extended in Chapter 3 to weak and strong interactions of quarks. We generalize the form-invariant derivative. This derivative simplifies the part of weak interactions for the quark wave function. We justify the preference for left waves. We present the wave equation without any use of complex values.

Chapter 4 incorporates gravitation in the heart of quantum physics such as described in the preceding chapters. The formulation of general relativity as the equality between two tensors is extended to be an equality between two connections of the space-time manifold. The global structure of space-time both accounts for the EPR paradox and gives a cosmic expansion with the most recent estimate of the beginning of the acceleration.

Afterwards we present our conclusions. That chapter includes many items that we cannot elaborate further in this introduction. There we also explain why we changed the title from “a” to “the” theory of everything. A summary of our described novelties closes this text.

The most technical parts are placed in four appendices. The presentation of the tools of Clifford algebras comprises Appendices A and B. There we show in a detailed and as basic manner as possible the algebras used in previous chapters. The main change from previous editions is the exact resolution of the improved wave equation in the hydrogen atom case The resolution of the Dirac equation for the in Appendix C. That resolution separates the variables in spherical coordinates. We get back all results (quantum numbers, energy levels) which ensured the success of the Dirac equation, despite no use of angular momentum operator, no use of Hilbert space and despite the non linearity of our wave equation. Various calculations form Appendix D.

This text neither describes a final theory of all interactions nor *a fortiori* a final theory of all physics. Many deep questions remain, like emission and absorption of the momentum-energy of the quantum electron wave, which is light. Louis de Broglie thought that much work remained after his own work, for several generations of physicists, to clear new paths in the vast land of quantum physics that he began to discover, a century ago.

Chapter 1

The electron wave with spin 1/2

First we present the usual matrix framework, the quantum wave of the electron and its wave equation. We study the wave equation in the Clifford algebra of space and in the Clifford algebra of space-time. We study tensor densities of the linear wave equation (the Dirac equation). The form invariance of the wave equation is extended to the multiplicative group of the Clifford algebra of space. The relativistic invariance introduces left and right parts of the wave. We simplify the Lagrangian density from which the wave equation comes, and we study the gotten improved wave equation. A dual logical link exists between the wave equation and Lagrangian density. The electric gauge invariance is not changed. A second gauge invariance and a second conservative current appear. Gauge invariance and form invariance are compatible with mass terms. We coherently set out the normalization of the wave, the charge conjugation, the solutions for the hydrogen atom. We address the issue of three generations. We introduce the notions of numeric dimension and invariant space-time. We study the energy–momentum vector and the dynamics of two energy–momentum tensors. This gives the Lorentz force for the electron. We identify a direct link between the electromagnetic field and momentum–energy tensors of the electron wave. The Pauli exclusion principle is linked to the normalization of the wave and to the conservation of momentum-energy. We study the recursion of the improved wave equation and its consequences.

1.1 The wave equation of the electron

In 1926 two major breakthroughs were made about the electron: the discovery of the electron spin, which means that the electron is a little magnet, even at rest, and the formulation of a wave equation by Erwin Schrödinger. This equation reads with [56]:

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = H(\psi), \tag{1.1}$$

where $\psi = \psi(x, y, z, t)$ is a complex number, for each value of x, y, z, t, and h is the Planck constant. The wave of the Schrödinger equation is then a function with partial derivatives from $\mathbb{R}^4$ into $\mathbb{C}$. This wave equation is linear. Functions that are solutions of the wave equation form a linear subspace of the linear space $\mathcal{F}(\mathbb{R}^4, \mathbb{C})$. Most of the concepts of quantum physics so far have come from the study of the Hamiltonian H included in the wave equation. Since we will not use this wave equation, nor Hilbert spaces, we only remark now the very particular role played by time in that equation.

The electron is also a magnet; this is the origin of the properties of permanent magnets that we use daily. So we must account for this magnet. After Pauli's first attempt to explain these magnetic properties, Dirac made use of and carried forward Pauli's attempt by coming up with another wave equation, only a few months later: this wave equation was published as early as 1928 [64, 65]. Nearly a century later, we can present this wave equation (in semi-modern notation, and with the usual Einstein summation convention) as follows [1]:

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi;\ q := \frac{e}{\hbar c};\ \hbar := \frac{\mathrm{h}}{2\pi};\ m := \frac{m_0 c}{\hbar}. \tag{1.2}$$

The four A_μ are the components of the space-time vector called exterior electromagnetic potential [2] that is created by other charges; e is the charge of the electron and m_0 is the proper mass [3]. We must see the great difference between the ψ of the Schrödinger equation and the ψ of the Dirac equation expressed as:

$$\psi := \begin{pmatrix}\xi\\ \eta\end{pmatrix};\ \xi := \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix};\ \eta := \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \tag{1.3}$$

1. Most modern presentations use a system of "natural" units where $c = 1$ and $\hbar = 1$. We will see in 1.5.4 why we cannot here use the comfortable $\hbar = 1$ convention.

2. We will see in 1.10 how the electromagnetic field, then also the potential, is not exterior, but dependent on the wave.

3. The wave equation of the electron always includes a mass term and a charge term. This equation is too often presented without its charge term, as if the electric interaction could be removed and restored at will. Even when the interaction with other charges is negligible, the action of the electro-weak field issued from the wave itself cannot be suppressed nor neglected, we shall see this for the atomic electrons.

because now the $\xi_j = \xi_j(x,t)$ and the $\eta_j = \eta_j(x,t)$ play the same role as functions of space and time coordinates with value in the complex field. The Dirac wave is hence a function [4] with derivatives from $\mathbb{R}^4$ into $\mathbb{C}^4$. This wave equation is linear. The solutions are thus elements of a linear subspace of the set of functions $\mathcal{F}(\mathbb{R}^4, \mathbb{C}^4)$. The Dirac equation needs to choose four suitable γ^μ matrices. Our choice, single and permanent, is [5] :

$$\begin{aligned}
&\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \gamma_j := \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} , \\
&\sigma^j = -\widehat{\sigma}^j = \widehat{\sigma}_j := -\sigma_j ,\ j = 1,2,3, \qquad (1.4) \\
&\gamma_0 = \gamma^0 := \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \ ; \ I_2 = \sigma_0 = \sigma^0 = \widehat{\sigma}^0 = \widehat{\sigma}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,
\end{aligned}$$

The σ_μ matrices and their products generate an 8-dimensional algebra on $\mathbb{R}$ which is 4-dimensional on $\mathbb{C}$, named the Pauli algebra or the 2×2 matrix algebra: $M_2(\mathbb{C})$. The choice of the Pauli matrices is fixed by the intrinsic basis (V_1, V_2, V_3, V_4) of $M_2(\mathbb{C})$ where the projectors V_n satisfy:

$$\begin{aligned}
V_1 &:= \frac{1}{2}(\sigma_0 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \ V_2 := \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \\
V_3 &:= \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \ V_4 := \frac{1}{2}(\sigma_0 - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \qquad (1.5)
\end{aligned}$$

With the matrices of this canonical basis, ξ is the **right** part and η is the **left** part of the wave. With the Dirac theory indeed it is the γ_5 matrix that allows us the definition of projectors on the right and left parts of the wave:

$$\gamma_5 := i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} ; \ \frac{1+\gamma_5}{2}\psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix} ; \ \frac{1-\gamma_5}{2}\psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix} . \quad (1.6)$$

The Dirac matrices are not uniquely defined. The Dirac theory easily proves that any other choice satisfies:

$$\gamma'^\mu = M\gamma^\mu M^{-1}; \ \psi' = M\psi, \qquad (1.7)$$

where M is any 4×4 fixed invertible matrix. This always allows us to come back to our present choice (1.4). Moreover another choice for Dirac matrices also need a matching change of the wave itself. The present choice is convenient both for the resolution of the wave equation in the case of the hydrogen atom [13] [36] and for an electron at high velocity. On the contrary, for the study of an electron with low velocity and for deriving the

4. We may also consider the Dirac wave as a set of four functions from $\mathbb{R}^4$ into $\mathbb{C}$. Thus, even if the usual term for the wave of quantum mechanics is "wave function", here we use four functions, not only one, and thus the general and simpler term "wave" seems more appropriate in the Dirac theory.

5. The meaning of the notation $\widehat{\sigma}$ is explained in A.3.3.

Pauli equation, the initial choice of γ'_μ matrices was [12]: intended for the Pauli equation describing slow electrons[12], using:

$$M = M^{-1} := \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_5);\ \gamma'_0 = M\gamma_0 M = \gamma_5;\ \gamma'_j = M\gamma_j M = -\gamma_j, \quad (1.8)$$

for $j = 1, 2, 3.$ [6] We then have:

$$\psi' = M\psi = \begin{pmatrix}\psi'_1\\ \psi'_2\\ \psi'_3\\ \psi'_4\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_1+\eta_1\\ \xi_2+\eta_2\\ \xi_1-\eta_1\\ \xi_2-\eta_2\end{pmatrix};\ \chi := \begin{pmatrix}\psi'_1\\ \psi'_2\end{pmatrix};\ \omega := \begin{pmatrix}\psi'_3\\ \psi'_4\end{pmatrix}. \quad (1.9)$$

This choice thus mixes the right and left parts of the wave. This mixing is a disaster for all that concerns spin, magnetism and weak interactions. It is the very reason justifying that we never use the γ'^μ. Coming back to the (1.4) matrices, the Dirac equation (1.2) reads:

$$0 = \begin{pmatrix} im & \sigma^\mu(\partial_\mu + iqA_\mu)\\ \widehat{\sigma}^\mu(\partial_\mu + iqA_\mu) & im\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix};\ \partial_\mu = \frac{\partial}{\partial x^\mu};\ x^0 = ct. \quad (1.10)$$

It is equivalent to the following system:

$$\begin{aligned} 0 &= \sigma^\mu(\partial_\mu + iqA_\mu)\eta + im\xi,\\ 0 &= \widehat{\sigma}^\mu(\partial_\mu + iqA_\mu)\xi + im\eta. \end{aligned} \quad (1.11)$$

The mass term of the wave equation for the ξ right part of the wave contains the left η part, while the mass term of the wave equation for the left η part of the wave contains the right ξ. This crossing of terms forbids the separated use of either ξ or η alone. Next, to get the Pauli equation it is necessary to break the space-time symmetry of these equations by:

$$\begin{aligned} 0 &= \partial_0\eta + \vec{\partial}\eta + iq(A_0 - \vec{A})\eta + im\xi;\ \vec{\partial} = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3,\\ 0 &= \partial_0\xi - \vec{\partial}\xi + iq(A_0 + \vec{A})\xi + im\eta;\ \vec{A} = A^1\sigma_1 + A^2\sigma_2 + A^3\sigma_3. \end{aligned} \quad (1.12)$$

Multiplying by i we get the named Hamiltonian form:

$$\begin{aligned} -i\partial_0\eta &= i\vec{\partial}\eta - q(\mathrm{A}_0 - \vec{\mathrm{A}})\eta - m\xi,\\ -i\partial_0\xi &= -i\vec{\partial}\xi - q(\mathrm{A}_0 + \vec{\mathrm{A}})\xi - m\eta. \end{aligned} \quad (1.13)$$

Adding and subtracting these equations we get:

$$0 = (\partial_0 + iqA_0 + im)\chi + (\vec{\partial} - iq\vec{A})\omega, \quad (1.14)$$

$$0 = (\partial_0 + iqA_0 - im)\omega - (\vec{\partial} - iq\vec{A})\chi. \quad (1.15)$$

6. The γ_5 matrix does not belong to the $Cl_{1,3}$ sub-algebra of $Cl_{2,3} = M_4(\mathbb{C})$, thus the γ'^0 matrix does not belong to $Cl_{1,3}$.

The nonrelativistic approximation replaces (1.15) with another equation (De Broglie considered this as totally incorrect [60]), gotten by the drastic suppress of a time derivative:

$$\omega = \frac{i}{2m}(\vec{\partial} - iq\vec{A})\chi, \tag{1.16}$$

and next by substituting ω in (1.14) we arrive at the Pauli equation:

$$(\partial_0 + iq\mathrm{A}_0 + im)\chi = \frac{1}{2im}(\vec{\partial} - iq\vec{\mathrm{A}})^2\chi. \tag{1.17}$$

The substitution of ω is aimed to get the Schrödinger equation where we have $\hbar c\partial_0\psi = iE\psi$, and by the nonrelativistic approximation $E \approx m_0c^2$. Next we easily place the Pauli wave equation under the Hamiltonian form of the Schrödinger equation (1.1) [7] : [8]

$$i\hbar c\partial_0\chi = H\chi;\ H\chi = (eA_0 + m_0c^2)\chi + \frac{\hbar^2}{2m_0}(\vec{\partial} - i\frac{e}{\hbar c}\vec{A})^2\chi. \tag{1.18}$$

Now with the usual summation of upper and lower indices, we let:

$$\begin{aligned} &\nabla := \sigma^\mu\partial_\mu;\ \widehat{\nabla} := \widehat{\sigma}^\mu\partial_\mu;\ A := \sigma^\mu A_\mu;\ \widehat{A} := \widehat{\sigma}^\mu A_\mu, \\ &\mathbf{A} := \gamma^\mu A_\mu = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix};\ \boldsymbol{\partial} := \gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}. \end{aligned} \tag{1.19}$$

These calculations actually operate in Clifford algebra, more precisely in two algebras: the Pauli algebra which is also called Cl_3 (as Clifford algebra of $\mathbb{R}^3$), and the Clifford algebra of space-time $Cl_{1,3}$, which is the algebra used more particularly by Hestenes, Boudet and Casanova [9]. A detailed introduction to Clifford algebra is presented in [21, 28, 36] and also in this book, Appendix A. We next detail in Appendix B the properties of $Cl_{1,3}$

7. The Hamiltonian of the Schrödinger equation, coming from the Hamiltonian of classical mechanics, is then replaced by a Hamiltonian operator acting on the wave. To obtain a Hamiltonian formalism in the same relativistic framework as the Dirac wave itself, is the aim of I. Kanatchikov [84]. It is indeed a prerequisite step for a unification of the different Hamiltonian formalisms of various parts of physical theory.

8. The Hamiltonian form of the wave equation still contains anti-commuting Pauli or Dirac matrices. Quantum fields theory uses a exponential of this Hamiltonian without accounting for the non commutativity of matrix products. As long as the space direction is the 3 direction, the results of all calculations are exacts, since the σ_3 matrix is diagonal and since diagonal matrices commute. But as soon as another direction must be accounted for, difficulties arise with relativistic mechanics. Therefore we use here only the Dirac equation and its Lagrangian formalism, not the approximation by the Pauli equation nor the Hamiltonian form of the first Dirac equation. This Hamiltonian form uses a $\beta = \gamma'^0 = \gamma_5$ matrix. It does not belong to the $Cl_{1,3}$ algebra generated by the four γ^μ matrices. It really belongs to the $Cl_{2,3} = M_4(\mathbb{C})$ algebra generated by the five $\gamma^a,\ a = 5, 0, 1, 2, 3$. That changes relative time of the relativistic wave equation into a 2-dimensional time which cannot be our physical oriented time. The Hamiltonian form of the first Dirac equation cannot be fully relativistic.

which is isomorphic to a real subalgebra of the complex algebra $M_4(\mathbb{C})$. Actually $Cl_{1,3}$ is a left (and a right) modulus on the Cl_3 ring. In the calculations of the Dirac theory, this corresponds to the calculation by blocks of 2×2 matrices for the 4×4 matrices. The system (1.11), equivalent to the Dirac equation, is expressed as:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{i}{m} \begin{pmatrix} 0 & \nabla + iqA \\ \widehat{\nabla} + iq\widehat{A} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{1.20}$$

This has the recursive functional form

$$\psi = f(\psi);\ f(\psi) = \frac{i}{m}\gamma^\mu(\partial_\mu + iqA_\mu)\psi = \frac{i}{m}(\boldsymbol{\partial} + iq\mathbf{A})\psi. \tag{1.21}$$

This form is very useful in studying the second-order equation that we come to now.

1.1.1 Second-order equation

By iteration of the functional f we get $\psi = f[f(\psi)]$ which means:

$$\begin{aligned} \psi &= \frac{i}{m}(\boldsymbol{\partial} + iq\mathbf{A})[\frac{i}{m}(\boldsymbol{\partial} + iq\mathbf{A})]\psi, \\ &= -\frac{1}{m^2}[\Box\psi + iq\boldsymbol{\partial}(\mathbf{A}\psi) + iq\mathbf{A}\boldsymbol{\partial}\psi - q^2\mathbf{A}^2\psi], \\ \Box :&= \boldsymbol{\partial\partial} = \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3. \end{aligned} \tag{1.22}$$

where $\Box$ is the D'Alembertian. Multiplying by m^2 this is equivalent to:

$$0 = (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\boldsymbol{\partial}(\mathbf{A}\psi) + \mathbf{A}\boldsymbol{\partial}\psi]. \tag{1.23}$$

And we have:

$$\boldsymbol{\partial}(\mathbf{A}\psi) = (\boldsymbol{\partial}\mathbf{A})\psi + 2A^\mu\partial_\mu\psi - \mathbf{A}\boldsymbol{\partial}\psi; \tag{1.24}$$

$$\mathbf{F} = \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix} = \boldsymbol{\partial}\mathbf{A} = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} = \begin{pmatrix} \nabla\widehat{A} & 0 \\ 0 & \widehat{\nabla}A \end{pmatrix}. \tag{1.25}$$

Then the electromagnetic field ($\mathbf{F}$ in space-time algebra, $F = \vec{E} + i\vec{H}$ in space algebra, where $\vec{E}$ is the electric field and $\vec{H}$ the magnetic field) allows us to obtain at the second order:

$$\begin{aligned} 0 &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[(\boldsymbol{\partial}\mathbf{A}) + 2A^\mu\partial_\mu - \mathbf{A}\boldsymbol{\partial} + \mathbf{A}\boldsymbol{\partial}]\psi, \\ 0 &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\mathbf{F} + 2A^\mu\partial_\mu]\psi. \end{aligned} \tag{1.26}$$

We may remark that the classical electromagnetic field $\mathbf{F}$ comes with a field of operators $2A^\mu\partial_\mu$. It is in accordance with quantum field theory, where the electromagnetic field is a field of operators. We may also see two things that seem strange in this wave equation: firstly the field of operators is a scalar field, acting on ξ and η in the same way while the classical part $\mathbf{F}$ is exactly a bivector field, which is well established experimentally: this is linked to the complete absence of longitudinal polarization in light. Secondly the squares $m^2-q^2\mathbf{A}^2$ are of opposite signs, while the energy–momentum of the electron is the sum of a mechanical energy–momentum $m\mathbf{v}$ and an electromagnetic energy–momentum $q\mathbf{A}$ [9], instead of a difference between these two energy–momentum vectors. We thus replace (1.24) with an equality similar to the Leibniz rule for the derivative of a product:

$$\boldsymbol{\partial}(\mathbf{A}\psi) = \boldsymbol{\mathcal{F}}(\psi) + \mathbf{A}\boldsymbol{\partial}\psi;\ \boldsymbol{\mathcal{F}}(\psi) = \boldsymbol{\partial}(\mathbf{A}\psi) - \mathbf{A}\boldsymbol{\partial}\psi, \tag{1.27}$$

$$\boldsymbol{\mathcal{F}}(\psi) := \begin{pmatrix} \mathcal{F}(\phi) & 0 \\ 0 & \widehat{\mathcal{F}(\phi)} \end{pmatrix};\ \mathcal{F}(\phi) := \nabla(\widehat{A}\phi) - A(\widehat{\nabla}\phi). \tag{1.28}$$

The second-order wave equation thus gives:

$$\begin{aligned} 0 &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\boldsymbol{\mathcal{F}}(\psi) + 2\mathbf{A}\boldsymbol{\partial}\psi] \\ &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\boldsymbol{\mathcal{F}}(\psi) - 2\mathbf{A}(iq\mathbf{A} + im)]\psi \qquad (1.29) \\ &= [\Box + (m + q\mathbf{A})^2]\psi + iq\boldsymbol{\mathcal{F}}(\psi). \end{aligned}$$

$$0 = [\Box + (m + q\mathbf{A})^2]\phi + iq\boldsymbol{\mathcal{F}}(\phi). \tag{1.30}$$

This yields both the expected sign for the energy–momentum term and an electromagnetic field that is actually a field of operators acting on and depending on the quantum ϕ wave of the electron. This wave equation is thus non-linear, so it is not a linear Klein-Gordon equation.

1.1.2 The form invariance of the Dirac equation

Attention please: it will be necessary to explain and correct a mistake made as early as the beginning of relativistic quantum physics, next used in all books on electron physics. Attention again: this form-invariance is very different from anything used in relativistic physics before quantum mechanics. First, space-time is considered in practice as a subset of the Cl_3 algebra (that was named the Pauli algebra) because, with Greek indices at 0, 1, 2, 3 and:

$$\mathrm{x}^0 = ct\ ;\ \ \vec{\mathrm{x}} = \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3\ ;\ \ \partial_\mu = \frac{\partial}{\partial \mathrm{x}^\mu}, \tag{1.31}$$

9. This characterization of the energy–momentum of the electron as a sum is explicit in the work of Hestenes (formula 6.22c) in [80]).

quantum physics as early as 1927 [10] wrote in the framework of Pauli's wave equation:

$$\vec{\mathrm{x}} = \begin{pmatrix} \mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1 + i\mathrm{x}^2 & -\mathrm{x}^3 \end{pmatrix}. \tag{1.32}$$

This includes the whole physical space in the Cl_3 algebra. It is necessary because the $SU(2)$ group replaces the $SO(3)$ rotation group in the Pauli theory. And $SU(2)$ is a subgroup of the Cl_3^* Lie group, which is the multiplicative group in the Cl_3 algebra. Next, this inclusion of the physical space was the starting point to extend the previous inclusion, by adding $\sigma_0 = I$ to the σ_j. This is linked to the representations of the Lorentz group [101][107] After Rose's book [106] explaining with modern terms the results exposed in de Broglie's book [56], all succeeding reports on this invariance introduced the Lorentz transformation of the x^μ using first an infinitesimal Lorentz transformation. They actually only use the mathematical tool of the Lie algebra of the $SL(2,\mathbb{C})$ Lie group, which is the set $sl(2,\mathbb{C})$ of the 2×2 complex matrices with a null trace. And using a general result about the $SO(n)$ group, they suppose, without any proof, that this result is also true in the $SO(1,3)$ case. They thus think that the Lie algebra of the $SO(1,3)$ Lie group is the same as the Lie algebra generated by the bivectors of the $Cl_{1,3}$ or $Cl_{3,1}$ algebras, or generated by the vectors and bivectors in Cl_3. And this algebra is also the Lie algebra of the $SL(2,\mathbb{C})$ Lie group. The algebraic calculations are hence formally exact, but the strangeness of the situation is hidden by the fact that there are not one but two exponential functions. With any Lie group, an exponential function exists which applies a vicinity of the null element of the Lie algebra (the null 2×2 complex matrix) into a vicinity of the unit element of the Lie group. And since here two different groups are working: the group of 2×2 complex matrices M such that $|\det(M)| = 1$. The second group is the group of Lorentz transformations, the unity of this group being the identity function id. It is nonsense to write both $\exp(0) = I_2$ and $\exp(0) = \mathrm{id}$, it should imply $I_2 = \mathrm{id}$. It is hence necessary to consider two exponential functions, that we note $\exp_1$ and $\exp_2$: $\exp_1$ applies all elements near zero in a sub-linear space of the $M_2(\mathbb{C})$ algebra, into a sub-group of the $GL(2,\mathbb{C})$ linear group, while $\exp_2$ applies the same elements near zero into the Lorentz group, where each transformation may be associated to a 4×4 real matrix. We have:

$$\mathrm{x} = \mathrm{x}^\mu \sigma_\mu = \mathrm{x}^0 + \vec{\mathrm{x}} = \begin{pmatrix} \mathrm{x}^0 + \mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1 + i\mathrm{x}^2 & \mathrm{x}^0 - \mathrm{x}^3 \end{pmatrix}. \tag{1.33}$$

And thus space-time is identified to the self-adjoint part of the Cl_3 algebra, which is the subset or this algebra whose M elements satisfy $M = M^\dagger$. This identification is the starting point of Chapter 4. The algebraic structure of

10. All that must be known about Cl_3 is included in Appendix A, with a minimal level of mathematics. Thus an informed lecturer may skip this Appendix. Nevertheless a review might be useful.

Cl_3 is richer than the complex field. Instead of a single conjugation we now have three: the $P : M \mapsto \widehat{M}$ transformation is the main automorphism of this algebra. This P, called **parity** in quantum mechanics, allows us to separate even and odd parts and satisfies $\widehat{AB} = \widehat{A}\widehat{B}$. Next the antimorphism $M \mapsto \widetilde{M} = M^\dagger$ is the **reversion** which satisfies $\widetilde{AB} = \widetilde{B}\widetilde{A}$. These conjugations generate a third one, the product of the previous conjugations: $M \mapsto \overline{M} = \widehat{M}^\dagger = \mathrm{tr}(M) - M$ is an anti-morphism since $\overline{AB} = \overline{B}\,\overline{A}$ (more details in A.3.3). And we get [11] :

$$\begin{aligned} \widehat{\mathrm{x}} &= \overline{\mathrm{x}} = \mathrm{x}^0 - \vec{\mathrm{x}}, \qquad (1.34)\\ ||\mathrm{x}||^2 &= \det(\mathrm{x}) = \mathrm{x}\widehat{\mathrm{x}} = \mathrm{x}\cdot\mathrm{x} = (\mathrm{x}^0)^2 - (\vec{\mathrm{x}})^2 = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \end{aligned}$$

The square $||\mathrm{x}||^2$ of the pseudo-norm of any space-time vector x is thus simply the determinant of this vector. Therefore the scalar product of two space-time vectors x and y reads:

$$\mathrm{x}\cdot\mathrm{y} = \frac{1}{2}(\mathrm{x}\widehat{\mathrm{y}} + \mathrm{y}\widehat{\mathrm{x}}) = \frac{1}{2}(\widehat{\mathrm{x}}\mathrm{y} + \widehat{\mathrm{y}}\mathrm{x}) = \mathrm{x}^0\mathrm{y}^0 - \mathrm{x}^1\mathrm{y}^1 - \mathrm{x}^2\mathrm{y}^2 - \mathrm{x}^3\mathrm{y}^3. \qquad (1.35)$$

The parity transformation $P : \mathrm{x} \mapsto \widehat{\mathrm{x}}$ is thus included in the geometric structure of space-time (see Chapter 4). Let $M = \exp_1(U)$ where U is any fixed element in Cl_3 (that means any fixed nonzero Pauli matrix) and let R be the transformation of space-time into itself such that for any x is associated x′ given by [12]:

$$\begin{aligned} &\mathrm{x}' = R(\mathrm{x}) = M\mathrm{x}M^\dagger = \exp_1(U)\mathrm{x}\exp_1(U)^\dagger; \quad M = \begin{pmatrix} a & -\bar{b} \\ c & \bar{d} \end{pmatrix}\\ &R = \exp_2(U); \quad \mathrm{x}'^\mu\sigma_\mu = R^\mu_\nu \mathrm{x}^\nu\sigma_\nu, \qquad (1.36)\\ &2R^\mu_\nu = \begin{pmatrix} a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d} & -ab+cd-\bar{a}\bar{b}+\bar{c}\bar{d} & i(ab+cd-\bar{a}\bar{b}+\bar{c}\bar{d}) & a\bar{a}-b\bar{b}+c\bar{c}-d\bar{d} \\ a\bar{c}-b\bar{d}+c\bar{a}-d\bar{b} & ad-bc+\bar{a}\bar{d}-\bar{b}\bar{c} & i(-ad+bc+\bar{a}\bar{d}-\bar{b}\bar{c}) & a\bar{c}+b\bar{d}+c\bar{a}+d\bar{b} \\ i(a\bar{c}+b\bar{d}-c\bar{a}-d\bar{b}) & i(ad+bc-\bar{a}\bar{d}-\bar{b}\bar{c}) & ad+bc+\bar{a}\bar{d}+\bar{b}\bar{c} & i(a\bar{c}-b\bar{d}-c\bar{a}+d\bar{b}) \\ a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d} & -ab-cd-\bar{a}\bar{b}-\bar{c}\bar{d} & i(ab+cd-\bar{a}\bar{b}-\bar{c}\bar{d}) & a\bar{a}-b\bar{b}-c\bar{c}+d\bar{d} \end{pmatrix}. \end{aligned}$$

The $M = \exp_1(U)$ matrix is a 2×2 matrix with complex values. The $R = \exp_2(U)$ transformation has a R^μ_ν matrix which is a 4×4 matrix with real values. It is obvious that these two objects cannot be equal. Hence the identification of $\exp_1$ and $\exp_2$, the identification of M and R is **nonsense**! Now look at the determinant of those matrices. We note:

$$\det(M) = re^{i\theta}\ , \quad r := |\det(M)|\ ; \quad \underline{M} := r^{-1/2}M. \qquad (1.37)$$

11. $\mathbb{R}$ is conventionally included in each Clifford algebra. With the Cl_3 case this is equivalent to the identification between numbers a and scalar matrices aI_2. This simplifies many calculations. This identification is often used in mathematics; for instance the $\mathbb{R}$ field is put into the $\mathbb{C}$ field.

12. Only one other possibility exists: $\vec{\mathrm{x}}\,' = R(\mathrm{x}) = M\mathrm{x}M^\dagger/\sqrt{r}$. With $N := \sqrt[4]{r}M$ we have $\vec{\mathrm{x}}\,' = R(\mathrm{x}) = N\mathrm{x}N^\dagger$, and we recover our simpler form.

So r is the modulus and θ is an argument of the determinant of M (r is only the modulus of the determinant, not the determinant itself). We get:

$$\begin{aligned}(\mathrm{x}'^0)^2 - (\mathrm{x}'^1)^2 - (\mathrm{x}'^2)^2 - (\mathrm{x}'^3)^2 &= \det(\mathrm{x}') = \det(M\mathrm{x}M^\dagger)\\ &= re^{i\theta}\det(\mathrm{x})re^{-i\theta} = r^2[(\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2]. \quad (1.38)\end{aligned}$$

Therefore R multiplies any space-time pseudo-distance by r. It is thus called similitude with ratio r. We call M the dilator of the R similitude. Even if, since 1928, most physicists confused similitudes and dilators, we shall use here two different words because **a similitude cannot be a dilator**. Consider now $\underline{R}$, transformation such as:

$$\underline{\mathrm{x}}' = \underline{R}(\mathrm{x}) = \underline{M}\mathrm{x}\underline{M}^\dagger. \quad (1.39)$$

We then get:

$$\mathrm{x}' = r^{1/2}\underline{M}\mathrm{x}r^{1/2}\underline{M}^\dagger = r\underline{M}\mathrm{x}\underline{M}^\dagger = r\underline{R}(\mathrm{x});\ R = r\underline{R}. \quad (1.40)$$

Hence R is the product, in any order, of $\underline{R}$ and of a homothety with ratio r. And since $\underline{M}$ was defined such as:

$$|\det(\underline{M})| = 1, \quad (1.41)$$

$\underline{M}$ was defined to belong to the $\mathcal{G}$ subgroup in $GL(2,\mathbb{C})$ made of the elements whose modulus of the determinant is 1. We thus get instead (1.38):

$$\begin{aligned}(\underline{\mathrm{x}}'^0)^2 - (\underline{\mathrm{x}}'^1)^2 - (\underline{\mathrm{x}}'^2)^2 - (\underline{\mathrm{x}}'^3)^2 &= \det(\underline{\mathrm{x}}') = \det(\underline{M}\mathrm{x}\underline{M}^\dagger)\\ &= |\det(\underline{M})|^2\det(\mathrm{x}) = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \quad (1.42)\end{aligned}$$

Thus $\underline{R}$ is a Lorentz transformation. With the usual summation convention of upper and lower indices, we let:

$$\mathrm{x}'^\mu = R^\mu_\nu \mathrm{x}^\nu;\ \underline{\mathrm{x}}'^\mu = \underline{R}^\mu_\nu \mathrm{x}^\nu. \quad (1.43)$$

(R^μ_ν) is the real 4×4 matrix of the R similitude and $(\underline{R}^\mu_\nu)$ is the real 4×4 matrix of the Lorentz transformation $\underline{R}$. We hence have the following (see A.4.2) for any dilator $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$:

$$2R^0_0 = |a|^2 + |b|^2 + |c|^2 + |d|^2 > 0, \quad (1.44)$$

and x'^0 then has the same sign as x^0 at the origin: the similitude R, and hence also $\underline{R}$, conserve the arrow of time. Moreover, for any dilator M in Cl_3, and even if the ratio is null, we obtain (detailed calculations in A.4.6) the following simple and nontrivial equality:

$$\det(R^\mu_\nu) = r^4. \quad (1.45)$$

And hence if r is nonzero then $r^4 > 0$: $\det(R) > 0$. Thus R conserves the orientation of space-time, and since the transformation conserves the orientation of time, R also conserves the orientation of space. Moreover we get:

$$\det(\underline{R}^{\mu}_{\nu}) = 1. \tag{1.46}$$

This concludes the demonstration that $\underline{R}$ is a transformation belonging to the Lorentz group (the transformation group conserving the space-time pseudo-metric). But this group is not the invariance group of the wave with spin 1/2. Only the restricted Lorentz group is obtained from (1.36). Now we consider the f function which associates to the dilator M the similitude $R = f(M)$. Let M' be any other dilator, with:

$$\det(M') = r'e^{i\theta'} \quad ; \quad R' = f(M') \quad ; \quad \mathrm{x}'' = M'\mathrm{x}'M'^{\dagger}. \tag{1.47}$$

We then get :

$$\begin{aligned}\mathrm{x}'' = M'\mathrm{x}'M'^{\dagger} = M'(M\mathrm{x}M^{\dagger})M'^{\dagger} = (M'M)\mathrm{x}(M'M)^{\dagger}\\ R' \circ R = f(M') \circ f(M) = f(M'M),\end{aligned} \tag{1.48}$$

and with $r \neq 0$, f becomes a homomorphism [13] from the $(Cl_3^*, \times)$ group into the $(D^*, \circ)$ group where D^* is the set of all similitudes with nonzero ratio. These two groups are Lie groups: $(Cl_3^*, \times)$ is the 8-dimensional $GL(2, \mathbb{C})$ group. The $(D^*, \circ)$ group is also a Lie group, but only a 7-dimensional group: one dimension disappears because the kernel of f is not reduced to the neutral element. Consequently f^{-1} cannot exist. Let θ be any real number and let M be a dilator such that:

$$M = e^{i\theta/2} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}; \ \det(M) = e^{i\theta}, \tag{1.49}$$

we then get:

$$\mathrm{x}' = M\mathrm{x}M^{\dagger} = e^{i\theta/2}\mathrm{x}e^{-i\theta/2} = \mathrm{x}. \tag{1.50}$$

$f(M)$ is thus the neutral element and M belongs to the kernel of f. Therefore the kernel is a one-parameter group and only seven parameters remain in D^*. Six of those define a proper Lorentz transformation and the seventh is the ratio of the similitude r. For instance if the dilator is

$$M = e^{a+b\sigma_1} = e^a[\cosh(b) + \sinh(b)\sigma_1], \tag{1.51}$$

thus the similitude R defined in (1.36) satisfies

$$\begin{aligned}\mathrm{x}' = M\mathrm{x}M^{\dagger} &= e^{a+b\sigma_1}(\mathrm{x}^0 + \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3)e^{a+b\sigma_1}\\ &= e^{2a}[e^{2b\sigma_1}(\mathrm{x}^0 + \mathrm{x}^1\sigma_1) + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3].\end{aligned} \tag{1.52}$$

13. Most quantum physicists use the name “representation” in place of homomorphism.

We hence get:

$$\begin{aligned}
\mathrm{x}'^0 + \mathrm{x}'^1\sigma_1 &= e^{2a}\big[\cosh(2b) + \sinh(2b)\sigma_1\big](\mathrm{x}^0 + \mathrm{x}^1\sigma_1),\\
\mathrm{x}'^0 &= e^{2a}\big[\cosh(2b)\mathrm{x}^0 + \sinh(2b)\mathrm{x}^1\big];\ \mathrm{x}'^2 = e^{2a}\mathrm{x}^2,\\
\mathrm{x}'^1 &= e^{2a}\big[\sinh(2b)\mathrm{x}^0 + \cosh(2b)\mathrm{x}^1\big];\ \mathrm{x}'^3 = e^{2a}\mathrm{x}^3.
\end{aligned} \tag{1.53}$$

We can see that the similitude R is the product, in any order, of a proper Lorentz transformation (boost) mixing the temporal component x^0 and the spatial component x^1 by a homothety with ratio $r = e^{2a}$. Next example, if:

$$M = e^{a+bi\sigma_1} = e^a[\cos(b) + \sin(b)i\sigma_1], \tag{1.54}$$

then the similitude R defined in (1.36) satisfies:

$$\begin{aligned}
\mathrm{x}' = M\mathrm{x}M^\dagger &= e^{a+bi\sigma_1}(\mathrm{x}^0 + \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3)e^{a-bi\sigma_1}\\
&= e^{2a}[\mathrm{x}^0 + \mathrm{x}^1\sigma_1 + e^{2bi\sigma_1}(\mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3)].
\end{aligned} \tag{1.55}$$

We thus have:

$$\begin{aligned}
\mathrm{x}'^2\sigma_2 + \mathrm{x}'^3\sigma_3 &= e^{2a}\big[\cos(2b) + \sin(2b)i\sigma_1\big](\mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3),\\
\mathrm{x}'^2 &= e^{2a}\big[\cos(2b)\mathrm{x}^2 + \sin(2b)\mathrm{x}^3\big];\ \mathrm{x}'^0 = e^{2a}\mathrm{x}^0,\\
\mathrm{x}'^3 &= e^{2a}\big[-\sin(2b)\mathrm{x}^2 + \cos(2b)\mathrm{x}^3\big];\ \mathrm{x}'^1 = e^{2a}\mathrm{x}^1.
\end{aligned} \tag{1.56}$$

And so R is the product of a rotation with axis $O\mathrm{x}^1$ and a $2b$ angle by the same homothety with ratio $r = e^{2a}$. The distinction between dilator M and similitude R is absolutely necessary. This distinction was unfortunately never made prior to our work: the Dirac theory confused M and R, so much so that the same name was given to these different objects! Here we will absolutely avoid calling M a Lorentz transformation since it is a very different object, even if each dilator $M = \exp_1(U)$ allows us to define the similitude $R = \exp_2(U)$. The Lie group of dilators, $Cl_3^* = GL(2,\mathbb{C})$, and the Lie group $\mathcal{D}^*$ of the similitudes are very different. They do not have the same topology, nor do they even have the same dimension. Thus they do not have the same set of infinitesimal elements that is the same Lie algebra. Hence they must not be confused, even in the neighborhood of the neutral element. The previous calculations are simple because we start from the dilator M to get the similitude R. The reverse calculation is impossible and it is actually nonsense, because the similitude R is the image of the dilator M by the function f, and this function is not invertible. None isomorphism may exist between the 8-dimensional group of the dilators and the 7-dimensional group of the similitudes.

Restricted Lorentz group

If we now add the condition $|\det(M)| = 1$, we identify M and $\underline{M}$. The set of the dilators $\underline{M}$ is $\mathcal{G}$, and (1.38) is reduced to:

$$(\mathrm{x}'^0)^2 - (\mathrm{x}'^1)^2 - (\mathrm{x}'^2)^2 - (\mathrm{x}'^3)^2 = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \tag{1.57}$$

The similitude R is then simply a Lorentz transformation and the set of R is called the restricted Lorentz group, usually denoted $\mathcal{L}_+^\uparrow$. The time orientation and space orientation are separately conserved. The determinant satisfies:

$$1 = |e^{i\theta}| \quad ; \quad \theta \in \mathbb{R}. \tag{1.58}$$

The 7-dimensional Lie algebra of $\mathcal{G}$ and the 6-dimensional Lie algebra of $\mathcal{L}_+^\uparrow$ cannot be confused. The first one is the Lie algebra of a 7-dimensional real Lie group, while the second is the Lie algebra of a 6-dimensional real Lie group. What happens is not only that dilation and similitude are mixed up, another big mistake is to confuse $\det(M) = 1$ (which means $M \in SL(2,\mathbb{C})$) with $|\det(M)| = 1$ (which meanss $M \in \mathcal{G}$). The reason of these big mistakes is the unquestioned replacement of real by complex numbers: the Pauli wave equation induced a mixing up of the $SO(3)$ and $SU(2)$ Lie groups, which have the same complex Lie algebra: the algebra $su(2)$ of the Hermitian matrix with a null trace. Since $\det[\exp(M)] = \exp(\mathrm{tr}(M))$, the element $\exp(M), M \in su(2)$ has a determinant 1. But if you actually work with $SO(3)$ and its 3×3 real matrices you have no reason to identify these real matrices to 2×2 complex matrices. You are not allowed to talk nonsense in mathematical physics.

The exponential function is general in Lie group theory. It is a function from a neighborhood (which may be small) of the zero in the Lie algebra, on a neighborhood of the unity in the Lie group (which is generally only a part of the full group). In the simplest case, the $GL(n,\mathbb{C})$ Lie group, which has the algebra of $n \times n$ complex matrices as complex Lie algebra, the exponential function is simply:

$$\exp(M) = e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}. \tag{1.59}$$

Non relativistic quantum mechanics uses two simple properties of $SU(2)$: first, any element M in $SU(2)$ satisfies:

$$M = \exp(ia^j\sigma_j), \ j = 1,2,3, \ a^j \in \mathbb{R}. \tag{1.60}$$

Second, for any rotation R in $SO(3)$, a M exists, defined up a sign, such that $R = f(M)$ where f is the homomorphism applying each dilator on the associated similitude. We now consider $M = -1 + \sigma_1 + i\sigma_2$, which is an element of $SL(2,\mathbb{C})$ since $\det(M) = 1$. And M only satisfies:

$$\begin{aligned}(\sigma_1 + i\sigma_2)^2 = 0; \ \exp(\sigma_1 + i\sigma_2) = 1 + (\sigma_1 + i\sigma_2) + 0,&\\ M = -[1 - (\sigma_1 + i\sigma_2)]; \ M = -\exp[-(\sigma_1 + i\sigma_2)],&\\ M = \exp(i\pi - \sigma_1 - i\sigma_2).&\end{aligned} \tag{1.61}$$

Hence the exponential function, in the $SL(2,\mathbb{C})$ case has several properties different from the exponential function in the $SU(2)$ case. The lack of understanding of this difference induced false theorems: Bacry [1] claimed (without proof!) that any element $M \in SL(2,\mathbb{C})$ necessary reads e^B with $tr(B) = 0$. A counterexample is $M = -1+\sigma_1+i\sigma_2$ for which $tr(B) = 2ik\pi$, $k \in \mathbb{Z}$. And it is the restriction $\underline{f}$ of f to the $\mathcal{G}$ group which is a homomorphism from $\mathcal{G}$ into $\mathcal{L}_+^\uparrow$, and with the same kernel, the 1-dimensional $U(1)$ group, thus $\underline{f}$ is not invertible : the calculation of M from R is nonsense.

Nevertheless the $\mathcal{G}$ group contains as a subgroup the $SU(2)$ group of the 2×2 unitary matrices with determinant 1. The restriction of $\underline{f}$ to this subgroup is a homomorphism from $SU(2)$ into the $SO(3)$ rotation group in space. The kernel of this homomorphism is now reduced to $\{\pm 1\}$. This is the basis of all calculations using the spin of a system of electrons or a system of particles. Of course all results of these calculations, like the 6j and 9j symbols, are exact since they properly use theorems on complex Lie groups, and they are calculated not by composing rotations but by actually multiplying unitary matrices. From false theorems you can sometimes get true results, with luck.

1.2 Extended invariance

The first important change that we now propose is the removal of the condition $|\det(M)| = 1$ and its replacement with $\det(M) \neq 0$ (this condition is used only to obtain the structure of multiplicative group). That is to say, we replace the 7-dimensional $\mathcal{G}$ Lie group by the 8-dimensional $GL(2,\mathbb{C}) = Cl_3^*$ Lie group itself. This group is also the multiplicative group in Cl_3, and Cl_3 is the Lie algebra of its subset Cl_3^*. We may put forward four reasons:

1. This is possible (and very surprising) because the properties (1.38), (1.44) and (1.45) are general and do not suppose that $\det(M) = 1$ nor $|\det(M)| = 1$ [17, 18, 19]. Nowhere do these restrictive equalities seem necessary for the wave of the electron. To see this it is enough to never use infinitesimal transformations, contrary to most course books, and simply to directly calculate in the Lie groups.

2. This value of the determinant of the dilator has no geometric origin in space-time, while gravitation is linked to the geometry of space-time. And Cl_3^* is obviously a geometric group since it is the multiplicative group of the algebra including: first the vectors of the physical space, with a multiplication assembling the scalar product and the cross-product of vectors, plus the mixt product of three vectors; secondly Cl_3 includes the space-time vectors which form the self-adjoint part of this algebra.

3. The Russian physicist V. Fock [75] rebuilt general relativity from the properties of electromagnetism and gravitation. His starting point was, as it was for Einstein, the invariance of the speed of light regarding any frame of reference, even in movement of translation. Since light is an electromagnetic wave, Fock considered an electromagnetic wavefront. He then proved

that the transformation linking the coordinates of an event was necessarily linear, and he afterwards proved that the R transformation was necessarily a Lorentz similitude, which means the product of a Lorentz transformation and a homothety. But he was working from electromagnetism (thus solely from the similitude R) and had no luck to introduce the dilator M that comes only with the quantum wave. Of course Fock was also a master of quantum mechanics but he no more accounted for the difference between M and R than did other physicists. Since he wanted to get only the Lorentz transformations he claimed that the ratio of the homothety was necessarily 1. Even though the invariance group of electromagnetic laws was known to be much larger than the Lorentz group and included the similitudes, Fock's error flooded all the Russian work in this domain of physics. Afterwards the success of Landau's books extended this error to the whole of QFT. We may also remark that, even if the homothety is reduced to the identity ($r = 1$), the group of the dilatorss should not reduce to $SL(2,\mathbb{C})$ but to the $\mathcal{G}$ subgroup of $GL(2,\mathbb{C})$, set of the M satisfying $|\det(M)| = 1$. This 7-dimensional Lie group cannot be equal nor isomorphic to the restricted Lorentz group, a 6-dimensional Lie group.

4. This extension of the relativistic invariance will allow us to understand the geometry of the four kinds of interactions in physics (electromagnetism, weak interactions, strong interactions and gravitation), the quantization of kinetic momentum and charges, the proper nature of the electromagnetic field. The power of this approach comes from the inclusion of parity transformation in the geometry of space-time resulting from $||\mathrm{x}||^2 = \mathrm{x}P(\mathrm{x})$. Above all, this extension shall allow us to understand that the quantum wave is a space-time function with value in that invariance group.

We now return to the Dirac equation and we look at how the wave with spin 1/2 comes to be, without imposing the condition $|\det(M)| = 1$. First the right wave ξ and the left wave η do not transform similarly:

$$\xi' = \xi'(\mathrm{x}') = M\xi = M\xi(\mathrm{x});\ \eta' = \eta'(\mathrm{x}') = \widehat{M}\eta = \widehat{M}\eta(\mathrm{x}). \tag{1.62}$$

This is actually the origin of the existence of right waves and left waves: they do not transform similarly in Lorentz transformations. The change is caused by the boost (1.51), not by the rotations, because we have, for instance:

$$\xi' = e^{a+b\sigma_1}\xi;\ \eta' = e^{a-b\sigma_1}\eta. \tag{1.63}$$

With transformations like (1.54), which are rotations, we have $\widehat{M} = M$, and so the right wave and left wave transform similarly. Consequently the theory of weak interactions may only start from the Dirac wave equation, which is relativistic, the only one able to distinguish between right waves and left waves (in contrast to the Pauli theory which only knows left-handed and right-handed parts of the wave). This is well known in the Standard Model. To see how the system (1.11) is changed (a system equivalent to the

Dirac equation), we need the following nontrivial relation (details in A.4.5):

$$\nabla = \overline{M}\nabla'\widehat{M};\ \nabla' := \sigma^\mu\partial'_\mu;\ \partial'_\mu := \frac{\partial}{\partial \mathrm{x}'^\mu}. \tag{1.64}$$

Hence we must separate (this is new, a supplementary constraint) the space-time vectors transforming like x which we call contravariant vectors, from the vectors transforming like ∇ which we call covariant vectors. The two supplementary dimensions of the invariance group thus induce new constraints which are added to the already strong constraints of relativistic invariance: in tensor calculus we now have no possibility of moving a tensor index up or down. We also cannot replace a contravariant n vector with the covariant ∇ like Lasenby did in [89]. Another constraint coming from (1.64): ∇ acts on $\widehat{\phi}$, never on ϕ: the wave equation uses $\nabla\widehat{\phi}$, never $\nabla\phi$. The reason is the $\widehat{M}$ in $\nabla = \overline{M}\nabla'\widehat{M}$, which must act on $\widehat{\phi}$. The previous constraints remain: for the transformations of the kind (1.51) as well as for those of the kind (1.54), when we have a θ angle with the transformation of ξ and η, we get a double angle 2θ with the transformation of x and ∇. This too is well known in quantum physics. The system (1.11) becomes, if qA is transformed like ∇ (which is necessary for the gauge invariance):

$$\begin{aligned}\xi' &= M\xi = M\frac{i}{m}(\overline{M}\nabla'\widehat{M} + iq'\overline{M}A'\widehat{M})\eta = M\overline{M}\frac{i}{m}(\nabla' + iq'A')\eta',\\ \eta' &= \widehat{M}\eta = \widehat{M}\frac{i}{m}(\widetilde{M}\widehat{\nabla}'M + iq'\widetilde{M}\widehat{A}'M)\xi = \widehat{M}\widetilde{M}\frac{i}{m}(\widehat{\nabla}' + iq'\widehat{A}')\xi'.\end{aligned} \tag{1.65}$$

And with $\det(M) = re^{i\theta}$ we have:

$$M\overline{M} = re^{i\theta};\ \widehat{M}\widetilde{M} = re^{-i\theta};\ \overline{M} = re^{i\theta}M^{-1}. \tag{1.66}$$

The system (1.65) can hence be expressed as:

$$\xi' = re^{i\theta}\frac{i}{m}(\nabla' + iq'A')\eta';\ \eta' = re^{-i\theta}\frac{i}{m}(\widehat{\nabla}' + iq'\widehat{A}')\xi'. \tag{1.67}$$

In the particular case where M belongs to $SL(2,\mathbb{C})$, this is reduced to:

$$\xi' = \frac{i}{m}(\nabla' + iqA')\eta';\ \eta' = \frac{i}{m}(\widehat{\nabla}' + iq\widehat{A}')\xi'. \tag{1.68}$$

So we are right in saying that the form of the wave equation is unchanged [14] .

For a complete use of the extended invariance group we shall first change the appearance of the wave equation, placing any calculation in the framework of the same algebra. This means that all elements of the wave equation

14. A lecturer in a hurry may not see what differs here from the conventional exposition of quantum mechanics. In fact, there is no difference except for the distinction between ∇ acting on η and $\widehat{\nabla}$ acting on ξ, and the distinction between dilator and similitude. But these distinctions (linked to parity) will prove essential for geometric properties of the space-time manifold in Chapter 4.

– differential operators, potentials, addition and multiplication, values of the wave, space and time – will be put into the same algebraic-geometrical structure. Next we shall change the wave equation itself by simplifying the Lagrangian density from whence the equation comes.

1.3 The Dirac equation in Cl_3

Instead the initial formalism of Dirac matrices which forms a 16-dimensional linear space on $\mathbb{C}$, and hence a 32-dimensional linear space on $\mathbb{R}$, instead the $Cl_{1,3}$ space-time algebra, a 16-dimensional linear space on $\mathbb{R}$ promoted by D. Hestenes [78]–[82] it is possible to find a simpler formalism, cheaper in dimension: we use now the Cl_3 algebra, a 8-dimensional linear space on $\mathbb{R}$.[15] The form used as early as 1928 for the relativistic invariance that we just studied, is the first reason for our choice. This invariance uses only a subgroup of Cl_3^*, the multiplicative group of Cl_3. Second, the Dirac wave of the electron has value only in Cl_3, not in the full space-time algebra. Third, which is the main reason, the use of Cl_3 was discovered by one of the present authors as sufficient for the description of the entire Dirac theory [14, 15]. It will allow us to obtain the link between left and right spinors in the simplest way, Finally and above all, Cl_3 is the perfect framework to describe the left wave and the right wave and all the chiral quantities that are obtained from them, like the energy-momentum tensors.

For the expression of the Dirac wave in Cl_3, it is enough to replace the column matrices ξ and η by 2×2 matrices with a null column. This changes nothing concerning the calculation because the product of matrices is a row-to-column multiplication which operates separately on each row of the left matrix and on each column of the right matrix in any product. For an easier calculation of tensor densities we use a $\sqrt{2}$ factor and let:

$$R^1 := \sqrt{2}\begin{pmatrix}\xi_1^1 & 0\\ \xi_2^1 & 0\end{pmatrix};\ \widehat{L}^1 := \sqrt{2}\begin{pmatrix}\eta_1^1 & 0\\ \eta_2^1 & 0\end{pmatrix},$$
$$\phi := R^1 + L^1 = \sqrt{2}(\xi^1\ \ \widehat{\eta}^1) = \sqrt{2}\begin{pmatrix}\xi_1^1 & -\eta_2^{1*}\\ \xi_2^1 & \eta_1^{1*}\end{pmatrix}. \tag{1.69}$$

We note that the complex conjugate of z is either $\overline{z}$, which is the usual notation in mathematics, or z^*, which is usual in the Dirac theory. We have:

$$\widehat{\phi} = \sqrt{2}\begin{pmatrix}\eta_1^1 & -\xi_2^{1*}\\ \eta_2^1 & \xi_1^{1*}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\eta^1 & \widehat{\xi}^1\end{pmatrix},$$
$$R^1 = \phi\frac{1+\sigma_3}{2} = \sqrt{2}(\xi^1\ \ 0);\ \widehat{L}^1 = \widehat{\phi}\frac{1+\sigma_3}{2} = \sqrt{2}(\eta^1\ \ 0). \tag{1.70}$$

15. The use of Cl_3 was first promoted by W. Baylis [3].

The link between ϕ, R^1 and L^1 is independent of the reference frame used because transformations in (1.62) are equivalent [16] to:

$$\phi' = \phi'(\mathrm{x}') = M\phi = M\phi(\mathrm{x});\ \widehat{\phi}' = \widehat{\phi}'(\mathrm{x}') = \widehat{M}\widehat{\phi} = \widehat{M}\widehat{\phi}(\mathrm{x}). \tag{1.71}$$

The system (1.11) (equivalent to the Dirac equation) is then expressed as:

$$\begin{aligned} 0 &= (\nabla + iqA)\widehat{\phi}\frac{1+\sigma_3}{2} + im\phi\frac{1+\sigma_3}{2}, \\ 0 &= (\widehat{\nabla} + iq\widehat{A})\phi\frac{1+\sigma_3}{2} + im\widehat{\phi}\frac{1+\sigma_3}{2}. \end{aligned} \tag{1.72}$$

Applying the parity operator $P : M \mapsto \widehat{M}$ [17] to the second equation, we get the equivalent system:

$$\begin{aligned} 0 &= (\nabla + iqA)\widehat{\phi}\frac{1+\sigma_3}{2} + im\phi\frac{1+\sigma_3}{2}, \\ 0 &= (\nabla - iqA)\widehat{\phi}\frac{1-\sigma_3}{2} - im\phi\frac{1-\sigma_3}{2}. \end{aligned} \tag{1.73}$$

This system is itself equivalent to the single equation via addition:

$$0 = \nabla\widehat{\phi} + qA\widehat{\phi}i\sigma_3 + m\phi i\sigma_3, \tag{1.74}$$

because each line of the system (1.73) is obtained by applying on (1.74) a projector on the right and left wave, that is, in the Pauli algebra the multiplication on the right side by $(1 \pm \sigma_3)/2$. We will finish the simplification of the Dirac equation by multiplying the right side with $-i\sigma_3 = \sigma_{21} = \sigma_2\sigma_1$. The Dirac equation is hence equivalent to:

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + m\phi, \tag{1.75}$$

and, using the parity transformation P, is equivalent to: Since $\widehat{\phi} = \sqrt{2}(\eta^1\ \widehat{\xi}^1)$, we can consider $\widehat{\phi}$ as comprising two left waves: η^1 and $\widehat{\xi}^1$. And, by the transformation parity, (1.75) is equivalent to:

$$0 = \widehat{\nabla}\phi\sigma_{21} + q\widehat{A}\phi + m\widehat{\phi}. \tag{1.76}$$

Once again and despite the very different look **these equations are exactly the Dirac equation**. The gauge invariance now has the form:

$$\phi \mapsto \phi' = \phi e^{ia\sigma_3}\ ;\ \ A \mapsto A' = A - \frac{1}{q}\nabla a \tag{1.77}$$

16. This equivalence is not trivial and comes from the fact that σ_2 is imaginary while σ_1 and σ_3 are real matrices. The result is, for any ϕ, that the P transformation $M \mapsto \widehat{M}$ exchanges ξ and η.

17. P is, from the mathematical point of view, the main automorphism in Cl_3 changing i into $-i$ and σ_j into $-\sigma_j$. From the physical point of view P stands for "parity" which exchanges right and left waves.

When quantum physics becomes relativistic, the multiplication by the imaginary number i, the generator of the electric gauge, that commutes with all, is replaced by a multiplication by the right side by $i_3 = i\sigma_3$. This i_3 is, from the geometrical point of view, a bivector or 2-vector, which means an oriented area (a cross product). The i which commutes with all other terms, in Cl_3, is the 3-vector $i = \sigma_1\sigma_2\sigma_3$ which is an oriented volume. This other i is the one used for instance in the expression for the electromagnetic field as a sum of an electric field and a magnetic field: $F = \vec{E} + i\vec{H}$. For anyone used to a single i, this is a major and difficult change, because the non commutativity of the multiplication is much less easy than the commutative product of complex numbers. However changing the status of i is absolutely necessary from the geometric point of view: a 2-vector is an oriented area, while a 3-vector is an oriented volume, and an area is not a volume.

All objects present in the wave equation are now in the same algebra. Calculations using 2×2 matrices are much simpler than using 4×4 Dirac matrices. Even if we should have no better reason, this simplicity would be enough to justify our new form of the Dirac equation. But we have other reasons, for instance the fact that, in Cl_3 the co-matrices giving the inverse of a matrix are reduced to numbers, or the particular properties of the exponential function, seen in A.3.8.

1.3.1 Plane wave

This section uses the simplest case where the interaction with the exterior and interior electromagnetic field is supposed negligible [18]. We thus let $A = 0$. The Dirac equation in the Cl_3 algebra is reduced to

$$\nabla\widehat{\phi}\sigma_{21} + m\phi = 0. \tag{1.78}$$

We consider a plane wave with a phase φ such that:

$$\phi = \phi_0 e^{\varphi\sigma_{21}} \ ; \quad \varphi = m\mathrm{v}_\mu \mathrm{x}^\mu. \tag{1.79}$$

We use the space-time vector called the reduced velocity:

$$\mathrm{v} = \sigma^\mu \mathrm{v}_\mu, \tag{1.80}$$

and ϕ_0 is a fixed term which gives

$$\nabla\widehat{\phi}\sigma_{21} = \sigma^\mu\partial_\mu(\widehat{\phi}_0 e^{\varphi\sigma_{21}})\sigma_{21} = -m\mathrm{v}\widehat{\phi}. \tag{1.81}$$

Therefore the Dirac equation is equivalent to

$$\phi = \mathrm{v}\widehat{\phi}. \tag{1.82}$$

18. We may be doubtful of the possibility of ignoring the charge of the electron and the electromagnetic field issued from its charge. All calculations made from plane waves must be considered only as a mathematical exercise, with the only aim to justify Fourier analysis

By using the P conjugation this is equivalent to

$$\widehat{\phi} = \widehat{\mathrm{v}}\phi. \tag{1.83}$$

Then combining the two previous equalities we have:

$$\phi = \mathrm{v}(\widehat{\mathrm{v}}\phi) = (\mathrm{v}\widehat{\mathrm{v}})\phi = (\mathrm{v}\cdot\mathrm{v})\phi. \tag{1.84}$$

If ϕ is invertible we then get:

$$1 = \mathrm{v}\cdot\mathrm{v} = \mathrm{v}\widehat{\mathrm{v}} = \mathrm{v}_0^2 - \vec{\mathrm{v}}^{\,2}, \tag{1.85}$$

$$\mathrm{v}_0^2 = 1 + \vec{\mathrm{v}}^{\,2}\ ;\ \ \mathrm{v}_0 = \pm\sqrt{1+\vec{\mathrm{v}}^{\,2}}, \tag{1.86}$$

with *a priori* two possibilities for the sign. The minus sign implied a negative energy for the particle – this was at the beginning a serious dissatisfaction for Dirac. He hoped to get rid of the nonphysical negative quantities that came from the Klein-Gordon equation which was the relativistic version of the Schrödinger equation. And it was impossible to suppress these negative energies [19]. They were necessary for the Fourier transformation, or for getting a small enough wave packet. Six years later the discovery of the positron, a particle which had the same mass as the electron yet an opposite charge, completely changed the problem: these plane waves with negative energies were associated to the positron. And this association was considered as the triumph of the Dirac theory. Nevertheless these waves with negative energies induced formidable problems when their effects on the emission or absorption of light were calculated. Moreover positrons seemed to have the same proper mass, not a mass opposite to the proper mass of the electron (we will see this later).

The calculation that we present here is **much simpler** than the calculation made in relativistic quantum physics books using complex 4×4 matrices. This is a sufficient reason, among many others, to prefer the Cl_3 algebra to the Dirac algebra.

1.3.2 Tensor densities without a derivative

The $\mathrm{J} = \mathrm{J}^\mu\sigma_\mu$ current is one of the tensor quantities of the Dirac theory such that the definition $\mathrm{J}^\mu = \overline{\psi}\gamma^\mu\psi$ of the four components is made from the spinor wave without a partial derivative. We may first remark with L. de Broglie [56] about the very strange character of these tensor densities

19. Plane waves, even if they are calculated here much more simply, are not the panacea often presented. De Broglie warned us against the abuse made with these waves: a wave unlimited in space or in time does not exist in nature. In an electronic microscope a train of waves is always limited in space and in time. We saw in 1.2 how Fock made use of an electromagnetic wavefront. Moreover this calculation neglects the charge term, as if we were able to remove or restore a charge at will. Hence plane waves are much too virtual and unreal to be very interesting from the physical point of view.

which had no true equivalent in physics before quantum theory. Several other similar quantities were quickly noted [56], first a scalar one:

$$\Omega_1 = \overline{\psi}\psi \ ; \quad \overline{\psi} = \psi^\dagger \gamma_0 = (\eta^{1\dagger} \ \xi^{1\dagger}), \tag{1.87}$$

where $M^\dagger$ is the adjoint matrix (transposed conjugate). Next the six:

$$S^{\mu\nu} := i\overline{\psi}\gamma^\mu\gamma^\nu\psi, \tag{1.88}$$

are considered as the components of an anti-symmetric tensor of rank two. The four K^μ:

$$\mathrm{K}^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \ ; \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{1.89}$$

are considered the components of a pseudovector in space-time, theoretically linked to an antisymmetric tensor of rank three, even if this link is never used. Finally:

$$\Omega_2 = -i\overline{\psi}\gamma_5\psi, \tag{1.90}$$

is a relativistic invariant and allows us to define the main invariant ρ and the Yvon-Takabayasi β angle:

$$\Omega_1 = \rho\cos\beta \ ; \quad \Omega_2 = \rho\sin\beta \ ; \quad \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \tag{1.91}$$

With the Weyl spinors (left and right waves) we get:

$$\begin{aligned} \Omega_1 &= \xi^{1\dagger}\eta^1 + \eta^{1\dagger}\xi^1 \ ; \quad \Omega_2 = i(\xi^{1\dagger}\eta^1 - \eta^{1\dagger}\xi^1), \\ \rho e^{i\beta} &= \Omega_1 + i\Omega_2 = 2\eta^{1\dagger}\xi^1 = 2(\eta_1^{1*}\xi_1^1 + \eta_2^{1*}\xi_2^1), \\ \rho e^{-i\beta} &= \Omega_1 - i\Omega_2 = 2\xi^{1\dagger}\eta^1 = 2(\eta_1^1\xi_1^{1*} + \eta_2^1\xi_2^{1*}). \end{aligned} \tag{1.92}$$

Very soon these tensor densities were intensively studied because physicists were eager to link these quantities to classical physics, where all studied quantities are vectors and tensors (but they are not tensor densities). Actually these 16 tensor densities that we previously detailed know nothing of the phase of the wave: they contain the product of $\overline{\psi} = \psi^\dagger\gamma_0$ by ψ and are then gauge-invariant under the electric gauge [20] . Thus we cannot substitute the dynamics of these densities for the dynamics of the ψ wave itself. Back

20. It is the same for any quantity of the kind $\psi\overline{\psi}$, which are thus not general, despite the opinion of many physicists [72]. The preferred argument, based on the dimension of the linear space of Dirac matrices, with dimension 16 on the complex field, has no reason to apply here since the tensor densities are real ones. This 16 is actually a difference between triangular numbers ($36 - 10 - 10$, where $36 = 9 \times 8/2$, $10 = 5 \times 4/2$). The numerous studies based on these only 16 densities [10][99][111] miss an essential point: the electron is also able of weak interactions. Moreover, the tensor densities without derivatives are not the only important densities in the Dirac theory. Some others with derivatives are used, while others were still misunderstood [53]. Worse yet, the list of the tensor densities which exist from the electron wave is infinite [15]. Hence we cannot know everything about these tensor densities.

to Cl_3 we can get a true simplification and a great generalization. First Ω_1 and Ω_2 are together linked, very simply if we identify real numbers and scalar matrices, by:

$$\overline{\phi} := \widehat{\phi}^\dagger;\ \det(\phi) = \phi\overline{\phi} = \overline{\phi}\phi = \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \tag{1.93}$$

So ρ is the modulus and the Yvon-Takabayasi β is an argument of the determinant of $\phi = \phi(\mathrm{x})$; hence they depend on x. Moreover, $\phi(\mathrm{x})$ is an invertible element in the ring Cl_3 if and only if $\rho \neq 0$. The detailed calculation of J_μ and K_μ (see A.4.2) using ξ^1 and η^1 gives

$$\mathrm{J} = \mathrm{J}^\mu\sigma_\mu = \phi\sigma_0\phi^\dagger = \phi\phi^\dagger\ ;\quad \mathrm{K} = \mathrm{K}^\mu\sigma_\mu = \phi\sigma_3\phi^\dagger. \tag{1.94}$$

Attention, please, as this is a major change in the Dirac theory, first obtained by D. Hestenes [79], we immediately see that these two space-time vectors, which were known to be orthogonal and with opposite scalar squares, are now part of a $(\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$ list, formed by four space-time vectors:

$$\mathrm{D}_0 = \mathrm{J}\ ;\quad \mathrm{D}_1 = \phi\sigma_1\phi^\dagger\ ;\quad \mathrm{D}_2 = \phi\sigma_2\phi^\dagger\ ;\quad \mathrm{D}_3 = \mathrm{K}. \tag{1.95}$$

The components of D_1 and D_2 (since not gauge-invariant) cannot be linear combinations of the 16 densities known through the old formalism of 4×4 complex matrices. We must then consider this formalism as seriously incomplete, weak, misleading. On the contrary, and still more helpful than just the simplification of calculations, the shift to Cl_3 allows us to discover new quantities which will prove very useful in the next chapters.

For any similitude R defined by a dilator M, the four vectors D_μ transform similarly:

$$\mathrm{D}'_\mu = \phi'\sigma_\mu\phi'^\dagger = (M\phi)\sigma_\mu(M\phi)^\dagger = M\phi\sigma_\mu\phi^\dagger M^\dagger = M\mathrm{D}_\mu M^\dagger. \tag{1.96}$$

We recall that in relativistic physics the tensors are defined and classed by the way that they transform in a Lorentz transformation. We then have no reason to be concerned by D_0 and D_3, and no reason to deny the possibility of the existence of D_1 and D_2. The four D_μ vectors transform like the space-time vector x. We then say they are **contravariant**. They are also vectors of the same length. Moreover, they are orthogonal to each other and form a **mobile basis** of space-time because we have:

$$\begin{aligned}2\mathrm{D}_\mu\cdot\mathrm{D}_\nu &= \mathrm{D}_\mu\widehat{\mathrm{D}}_\nu + \mathrm{D}_\nu\widehat{\mathrm{D}}_\mu = \phi\sigma_\mu\phi^\dagger\widehat{\phi}\widehat{\sigma}_\nu\overline{\phi} + \phi\sigma_\nu\phi^\dagger\widehat{\phi}\widehat{\sigma}_\mu\overline{\phi}\\ &= \phi\sigma_\mu\rho e^{-i\beta}\widehat{\sigma}_\nu\overline{\phi} + \phi\sigma_\nu\rho e^{-i\beta}\widehat{\sigma}_\mu\overline{\phi} = \rho e^{-i\beta}\phi(\sigma_\mu\widehat{\sigma}_\nu + \sigma_\nu\widehat{\sigma}_\mu)\overline{\phi} = \rho e^{-i\beta}\phi 2\delta_{\mu\nu}\overline{\phi}\\ &= 2\delta_{\mu\nu}\rho e^{-i\beta}\phi\overline{\phi} = 2\delta_{\mu\nu}\rho e^{-i\beta}\rho e^{i\beta};\ \mathrm{D}_\mu\cdot\mathrm{D}_\nu = \delta_{\mu\nu}\rho^2.\end{aligned} \tag{1.97}$$

Of course since here we use the pseudo-metric in space-time of special relativity with the choice of a + sign for time, we have:

$$\delta_{00} = 1\ ;\quad \delta_{11} = \delta_{22} = \delta_{33} = -1\ ;\quad \delta_{\mu\nu} = 0\ ,\ \mu \neq \nu. \tag{1.98}$$

Among these ten relations (1.97), only three were known from the old formalism:

$$\mathrm{J}\cdot\mathrm{J}=\rho^2\ ;\quad \mathrm{K}\cdot\mathrm{K}=-\rho^2\ ;\quad \mathrm{J}\cdot\mathrm{K}=0. \tag{1.99}$$

Now for the tensor densities $S^{\mu\nu}$, we let:

$$S_3 := S^{23}\sigma_1+S^{31}\sigma_2+S^{12}\sigma_3+S^{10}i\sigma_1+S^{20}i\sigma_2+S^{30}i\sigma_3. \tag{1.100}$$

And we proved (see details in A.4.3) that:

$$S_3=\phi\sigma_3\overline{\phi}. \tag{1.101}$$

We can see immediately that S_3 is one of the four:

$$S_\mu := \phi\sigma_\mu\overline{\phi},\ \ \mu=0,1,2,3. \tag{1.102}$$

So now we have met S_0, which will be called a_1, since we have (see A.4.1):

$$a_1=S_0=\phi\sigma_0\overline{\phi}=\phi\overline{\phi}=\rho e^{i\beta}=\det(\phi). \tag{1.103}$$

With the four contravariant vectors D_μ which each have four components, together with S_0 which has two, and the three S_j, $j=1,2,3$ which each have six components, we count 36 tensor densities without a derivative. This is much more than the 16 known from the old Dirac theory, and is evident proof of the incomplete character of the old formalism. We may notice that this 36 is, like 16, a square, but this is a numerical coincidence because 36 is actually a triangular number: in Clifford algebras the triangular numbers $n(n-1)/2$ often appear. Since the right and left spinors forming the electron wave are the fundamental quantities (and we will see this by the study of weak interactions in the following chapter), the true counting is as follows: with each spinor, the right one R^1 and the left one $\widehat{L}^1$ in (1.70), we obtain $4\times 5/2=10=4+6$ densities (this 4 is the number of the numerical functions of each spinor). Four of them form a space-time vector, the $6=3\times 4/2$ others form a space-time bivector. With the right spinor, the vector D_R^1 and the bivector S_R^1 satisfy [29]:

$$\mathrm{D}_R^1 := R^1\widetilde{R}^1;\ S_R^1 := R^1\sigma_1\overline{R}^1. \tag{1.104}$$

With the left spinor of the electron, the vector D_L^1 and the bivector S_L^1 satisfy

$$\mathrm{D}_L^1 := L^1\widetilde{L}^1=(\eta^{1\dagger}\sigma^\mu\eta^1)\sigma_\mu;\ S_L^1 := L^1\sigma_1\overline{L}^1. \tag{1.105}$$

In his theory of the magnetic monopole, G. Lochak was the first to notice the fundamental role of the left and right currents [90, 91, 92, 93, 94, 95, 96, 97]. These currents have a zero scalar square; they are on the light cone because, with (1.104) and (1.105):

$$\begin{aligned}0 &= R^1\overline{R}^1 = \overline{R}^1 R^1 = \widetilde{R}^1\widehat{R}^1 = \widehat{R}^1\widetilde{R}^1,\\ 0 &= L^1\overline{L}^1 = \overline{L}^1 L^1 = \widetilde{L}^1\widehat{L}^1 = \widehat{L}^1\widetilde{L}^1,\\ \mathrm{D}_R^1\cdot\mathrm{D}_R^1 &= \mathrm{D}_R^1\widehat{\mathrm{D}}_R^1 = R^1(\widetilde{R}^1\widehat{R}^1)\overline{R}^1 = 0,\\ \mathrm{D}_L^1\cdot\mathrm{D}_L^1 &= \mathrm{D}_L^1\widehat{\mathrm{D}}_L^1 = L^1(\widetilde{L}^1\widehat{L}^1)\overline{L}^1 = 0.\end{aligned} \tag{1.106}$$

Hence in the wave with spin 1/2 some of the quantities always have properties of light, even at small velocity, and also for an electron at rest. For this reason the approximation (1.16) which suppresses the relativistic invariance is nonsense from the point of view of high-energy physics. Moreover a complex number, however small its modulus may be, may be written in trigonometric form with an argument which can be the phase of a wave. Only zero has no argument.

To build his theory of the leptonic magnetic monopole, Lochak understood that the J and the K current were respectively the sum and the difference of the left and right currents. Now we may extend this to the bivectors, since S_1 and S_2 are also combinations of S_R^1 and S_L^1, with (1.104) and (1.105):

$$\mathrm{J} = \mathrm{D}_0 = \mathrm{D}_R^1 + \mathrm{D}_L^1;\ \mathrm{K} = \mathrm{D}_3 = \mathrm{D}_R^1 - \mathrm{D}_L^1, \tag{1.107}$$

$$S_1 + iS_2 = 2S_R^1;\ S_1 - iS_2 = 2S_L^1. \tag{1.108}$$

We derived in [47] the following relation, which will be generalized in the subsequent chapters:

$$\rho^2 = a_1 a_1^* = 2\mathrm{D}_R^1\cdot\mathrm{D}_L^1. \tag{1.109}$$

Because we have

$$\begin{aligned}\mathrm{J}\cdot\mathrm{J} &= (\mathrm{D}_R^1+\mathrm{D}_L^1)\cdot(\mathrm{D}_R^1+\mathrm{D}_L^1)\\ &= \mathrm{D}_R^1\cdot\mathrm{D}_R^1 + 2\mathrm{D}_R^1\cdot\mathrm{D}_L^1 + \mathrm{D}_L^1\cdot\mathrm{D}_L^1 = 2\mathrm{D}_R^1\cdot\mathrm{D}_L^1,\\ \widetilde{R}^1\widehat{L}^1 &= a_1^*\frac{1+\sigma_3}{2};\ \widetilde{L}^1\widehat{R}^1 = a_1^*\frac{1-\sigma_3}{2},\end{aligned} \tag{1.110}$$

and also

$$\begin{aligned}2\mathrm{D}_R^1\cdot\mathrm{D}_L^1 &= \mathrm{D}_R^1\widehat{\mathrm{D}}_L^1 + \mathrm{D}_L^1\widehat{\mathrm{D}}_R^1 = R^1\widetilde{R}^1\widehat{L}^1\overline{L}^1 + L^1\widetilde{L}^1\widehat{R}^1\overline{R}^1\\ &= R^1 a_1^*\frac{1+\sigma_3}{2}\overline{L}^1 + L^1 a_1^*\frac{1-\sigma_3}{2}\overline{R}^1\\ &= a_1^*(R^1\overline{L}^1 + L^1\overline{R}^1) = a_1^* a_1 = \rho^2.\end{aligned} \tag{1.111}$$

Besides tensor densities coming from one of the two spinors, we have 16 densities that come from the two spinors, the right one and the left one.

This $16 = 2^4$ was the (wrong) maximum number of tensor densities allowed by the old Dirac formalism with complex matrices. Here comes the well known $2^4 = 1+4+6+4+1$ of Pascal's triangle. The 1+1 of the extremities gives a_1, the 4+4 gives the vectors D_1 and D_2, and the 6 is the number of components of S_3:

$$a_1 = S_0 = R^1\overline{L}^1 + L^1\overline{R}^1;\ S_3 = R^1\overline{L}^1 - L^1\overline{R}^1,$$
$$\mathrm{D}_1 + i\mathrm{D}_2 = 2R^1\sigma_1\widetilde{L}_1;\ \mathrm{D}_1 - i\mathrm{D}_2 = 2L^1\sigma_1\widetilde{R}_1. \tag{1.112}$$

The use of Cl_3 is absolutely necessary because the construction of the tensor densities may be generalized only from Cl_3, and we will do this in the next chapter. [21] We will see the importance of the $(\mathrm{D}_0,\ \mathrm{D}_1,\ \mathrm{D}_2,\ \mathrm{D}_3)$ orthogonal basis in Chapter 4. Many other densities may also be useful in the Dirac theory, for instance the densities with first derivatives used in the study of the energy–momentum.

1.3.3 Relativistic transformation of the densities

We already explained how the D_μ vectors transform: they are contravariant vectors ($D' = MDM^\dagger$). Moreover, these formulas of transformation are automatically induced by the transformation of the ϕ wave. Next, in the similitude R induced by any fixed dilator M, the four S_μ quantities become

$$S'_\mu = \phi'\sigma_\mu\overline{\phi}' = M\phi\sigma_\mu\overline{M\phi} = M\phi\sigma_\mu\overline{\phi}\ \overline{M} = MS_\mu\overline{M}. \tag{1.113}$$

Since physics characterizes the tensors by their transformation under a change of reference frame, we see no necessity in distinguishing the different D_μ vectors which transform similarly whether or not they are invariant in the electric gauge. The same situation happens for the different S_μ quantities. For instance we get:

$$\rho' e^{i\beta'} = S'_0 = MS_0\overline{M} = M\rho e^{i\beta}\overline{M} = \rho e^{i\beta}M\overline{M} = \rho e^{i\beta} r e^{i\theta},$$
$$\rho' = r\rho\ ;\quad \beta' = \beta + \theta \mod 2\pi. \tag{1.114}$$

If we restrict the similitudes to only be Lorentz transformations then ρ is invariant, not β. Even in the case where $\det(M) = 1$ we may have $\beta' = \beta + \pi$.

Numerous relations exist between the 36 tensor densities that are dependent on the only eight real parameters of the ϕ wave (see A.4.7). The number 36 is also the result of restrictions for any other possibility: products like R^1R^1 or $R^1\widehat{R}^1$ cannot transform relativistically, because the multiplication by $M^\dagger$ on the right side is not available. And several products cancel, for instance $R^1L^{1\dagger}$.

21. Otherwise R. Boudet and D. Hestenes made too much use of the mobile orthonormal basis $(\mathrm{e}_0,\ \mathrm{e}_1,\ \mathrm{e}_2,\ \mathrm{e}_3)$ such as $\mathrm{D}_\mu = \rho\mathrm{e}_\mu$ [5] [79]. Consequently they have not seen the similarity between the four D_μ and the four S_μ.

The equalities in (1.113) are **entirely new in the physics of tensors**, completely different from the relations for the transformation of antisymmetric tensors of rank 2, which should give: $S'^{\rho\sigma} = R^{\rho}_{\mu} R^{\sigma}_{\nu} S^{\mu\nu}$. Since R^{ν}_{μ} is quadratic in M and multiplies each space-time length by r, the presence of two R factors implies a multiplication by r^2, while (1.113) is quadratic in M and thus multiplies the lengths only by r. Moreover, the J and K currents are perfectly similar since they are simply the sum and difference of the left D^1_L and right D^1_R currents. But the old formalism of 4×4 complex matrices considers J as a space-time vector and K as a pseudovector in space-time, which is wholly inconsistent: D^1_R and D^1_L have of course the same geometric status, they are both contravariant vectors. J and K are also contravariant vectors, necessarily. We may say the same for S_0 et S_3. Therefore the left and the new framework, the old based on complex 4×4 matrices, and the new using Cl_3, cannot be equivalent for space-time geometry. Only the Cl_3 algebra is fully satisfying. This is a sufficient reason to only use the framework of the Clifford algebra Cl_3.

1.4 The invariant form of the Dirac equation

The form invariance of the wave equation of the electron uses the differential operator $\nabla = \overline{M}\nabla'\widehat{M}$. Since $\phi' = M\phi$ implies $\overline{\phi}\,' = \overline{\phi}\ \overline{M}$, the factor $\overline{M}$ on the left side indicates a possible multiplication of the wave equation on the left side by $\overline{\phi}$. When and where $\rho \neq 0$ (and only in this case), $\phi = \phi(\mathrm{x})$ is invertible. Hence by multiplying on the left side by $\overline{\phi}$ the Dirac equation is equivalent to

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi. \tag{1.115}$$

We consider this equation as the true Dirac equation and we now explain why this form is **the invariant form of the Dirac equation**: In a Lorentz similitude R defined by a dilator M in Cl_3 satisfying (1.36), we get (1.62) and (1.64), which imply that if we conserve the gauge invariance we have:

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} = \overline{\phi}(\overline{M}\nabla'\widehat{M}\widehat{\phi})\sigma_{21} = \overline{\phi}\,'(\nabla'\widehat{\phi}')\sigma_{21}, \tag{1.116}$$

$$\overline{\phi}qA\widehat{\phi} = \overline{\phi}\ \overline{M}q'A'\widehat{M}\widehat{\phi} = \overline{\phi}\,'q'A'\widehat{\phi}'. \tag{1.117}$$

The two left terms of (1.115) are then form-invariant, and the mass term is also invariant if we have:

$$m\overline{\phi}\phi = m'\,\overline{\phi}\,'\phi' = m'\,\overline{\phi}\ \overline{M}M\phi = re^{i\theta}m'\,\overline{\phi}\phi, \tag{1.118}$$

which requires:

$$m = re^{i\theta}m'; \quad |m| = r|m'|. \tag{1.119}$$

Of course if we restrict M to $\det(M) = 1$ we have $m = m'$. But we must take caution: For the extended invariance the proper mass is no longer invariant. This is an important change in our habits: it is well known that the quantum wave is necessarily invariant under the Poincaré group formed by all transformations of the complete Lorentz group, plus space-time translations. But the Dirac equation is form-invariant only under these translations and transformations of the restricted Lorentz group, which do not change the orientation of space nor the orientation of time. The similitude group also does not change the orientation of space and the orientation of time. Thus this group does not contain the totality of the Poincaré group, and theorems based on properties of the full Poincaré group cannot apply here. But of course the proper mass remains invariant so long as the transformation belongs to the Poincaré group ($r = 1$). Yet no longer is the proper mass invariant when the transformation does not belong to this group ($r \neq 1$). The mass term reads:

$$m\overline{\phi}\phi = m\Omega_1 + im\Omega_2, \tag{1.120}$$

and is hence the sum of a scalar and a pseudoscalar. The second term of the invariant Dirac equation (1.115) shows another peculiarity: it is a space-time vector that we have calculated in (B.38):

$$\overline{\phi}A\widehat{\phi} = A_\nu \mathrm{D}^\nu_\mu \sigma^\mu = V_\mu \sigma^\mu;\ V_\mu = A_\nu \mathrm{D}^\nu_\mu = A \cdot \mathrm{D}_\mu. \tag{1.121}$$

This also gives:

$$\overline{\phi}\sigma^\nu\widehat{\phi} = \mathrm{D}^\nu_\mu \sigma^\mu. \tag{1.122}$$

Only the first term of (1.115) is a general term in Cl_3, but we can also obtain some properties with

$$\overline{\phi}(\nabla\widehat{\phi}) = \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}] + \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}], \tag{1.123}$$

$$\begin{aligned}\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}] &= \frac{1}{2}\partial_\nu(\overline{\phi}\sigma^\nu\widehat{\phi}) = \frac{1}{2}\partial_\nu(\mathrm{D}^\nu_\mu\sigma^\mu) = \frac{1}{2}(\partial_\nu \mathrm{D}^\nu_\mu)\sigma^\mu \\ &= \frac{1}{2}(\nabla\cdot \mathrm{D}_\mu)\sigma^\mu = v = v_\mu\sigma^\mu;\ 2v_\mu = \nabla\cdot D_\mu, \end{aligned} \tag{1.124}$$

$$\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}] = iw = iw_\mu\sigma^\mu, \tag{1.125}$$

where v and w are two space-time vectors since $v^\dagger = v$ and $(iw)^\dagger = -iw$. This gives:

$$\begin{aligned}&\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} = (v + iw)\sigma_{21} \\ &= (v_0 + v_1\sigma^1 + v_2\sigma^2 + v_3\sigma^3 + iw_0 + w_1 i\sigma^1 + w_2 i\sigma^2 + w_3 i\sigma^3)(i\sigma^3) \\ &= -w_3 + v_2\sigma^1 - v_1\sigma^2 - w^0\sigma^3 + i(v^3 + w_2\sigma^1 - w_1\sigma^2 + v_0\sigma^3).\end{aligned} \tag{1.126}$$

Hence the decomposition of the invariant form (1.115) of the Dirac equation in the basis $(1, \sigma^1, \sigma^2, \sigma^3, i, i\sigma^1, i\sigma^2, i\sigma^3)$ of Cl_3 yields this system of eight real equations:

$$0 = -w_3 + qA \cdot \mathrm{D}_0 + m\Omega_1, \tag{1.127}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_2 + qA \cdot \mathrm{D}_1, \tag{1.128}$$

$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_1 + qA \cdot \mathrm{D}_2, \tag{1.129}$$

$$0 = w_0 + qA \cdot \mathrm{D}_3, \tag{1.130}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_3 + m\Omega_2, \tag{1.131}$$

$$0 = -w_2, \tag{1.132}$$

$$0 = w_1, \tag{1.133}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_0. \tag{1.134}$$

The first equation is exactly the equation of the Lagrangian density $\mathcal{L} = 0$ because of the following (the detailed calculation is in B.1.4):

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\Big[\big(\overline{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi\big) + \big(\overline{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi\big)^\dagger\Big] + m\overline{\psi}\psi \\ &= -w_3 + qA \cdot \mathrm{D}_0 + m\Omega_1.\end{aligned} \tag{1.135}$$

We know that by varying the Lagrangian density $\mathcal{L}$ we obtain the Dirac wave equation. Moreover the fact that the Dirac equation is homogeneous implies that $\mathcal{L} = 0$ when the wave equation is satisfied. Here we have exactly the reciprocal situation; the equation $\mathcal{L} = 0$ is one of the eight real equations equivalent to the wave equation, and **the Lagrangian formalism is an automatic consequence of the wave equation**.

It is known that varying the Lagrangian density $\mathcal{L}$, this gives the Dirac equation itself. Moreover the homogeneous character of the wave equation [22] implies that $\mathcal{L} = 0$ when the wave equation is satisfied. Here we have exactly the reciprocal logical proposition: the equation $\mathcal{L} = 0$ is one of the eight numerical equations, and the Lagrangian formalism is thus an automatic consequence of the wave equation.

22. The wave equations of quantum mechanics are linear. Particularly they are additive, which means that if ϕ_1 and ϕ_2 are two solutions of the wave equation then $\phi_1 + \phi_2$ is also a solution. And they are homogeneous, which means that if ϕ is a solution of the equation and if z is any fixed complex number, then $z\phi$ is also a solution of the equation. The word *homogeneous* is used with its mathematical meaning, it has nothing to do with the concept of physical dimension. Additivity and homogeneity together form the linearity of the wave equation.

But attention please! the homogeneity of the Dirac equation written in the space-time algebra only says that $a\phi$ is solution for a real a, not with a complex a. This is another reason to separate the first Dirac equation, Hamiltonian and completely belonging to quantum mechanics with complex values, and the relativistic Cliffordian Dirac equation, equivalent to the equation with value in Cl_3. The problem comes from the generator of the electric gauge, $\sigma_1\sigma_2$ which does not commute with anything, while i commutes with anything.

Any law of movement, in classical mechanics and in electromagnetism, may be obtained from a Lagrangian formalism. We now know that this results from the Lagrangian form and the universality of quantum mechanics. But where does it come from that quantum mechanics has a Lagrangian form? Here we see that this is completely automatic because the Lagrangian density is the scalar part of the wave equation, and because this Lagrangian density yields anew the whole wave equation. We will detail in Chapter 2 how the single equation of the scalar part gives once again the seven other real equations, not from a physical principle above the differential laws but simply as a consequence of the algebraic structure due to the geometry of space-time. Moreover the four real equations containing the symmetric part v of $\overline{\phi}(\nabla\widehat{\phi})$ are with the D_μ of (1.95) (see A.4.2):

$$0 = \nabla \cdot \mathrm{D}_0, \tag{1.136}$$

$$0 = \nabla \cdot \mathrm{D}_3 + 2m\Omega_2, \tag{1.137}$$

$$0 = \nabla \cdot \mathrm{D}_1 - 2qA \cdot \mathrm{D}_2, \tag{1.138}$$

$$0 = \nabla \cdot \mathrm{D}_2 + 2qA \cdot \mathrm{D}_1. \tag{1.139}$$

The equation (1.136) which is known as the law of conservation of the probability current, is now exactly one of the eight real equations equivalent to the Dirac equation. Next (1.137) is known as Uhlenbeck-Laporte relation. The real equations (1.138) and (1.139) show that the space-time vectors D_1 and D_2 are not gauge-invariant; the gauge transformation operates a rotation in the plane of D_1 D_2 that Boudet named plane of the spin [5].

1.4.1 Charge conjugation

Many years after the discovery of the electron, the positron in turn was discovered. The only difference between electron and positron is the charge sign: negative for the electron, positive for the positron. From the Dirac wave of the particle (1.2) (where the wave of the electron is denoted as ψ_e and the wave of the positron is denoted as ψ_p), quantum mechanics derives the wave equation of the antiparticle as follows. The complex conjugation is used on the Dirac equation:

$$0 = [\gamma^{\mu *}(\partial_\mu - iqA_\mu) - im]\psi_e^*. \tag{1.140}$$

Since (1.4) gives $\gamma_2\gamma^{\mu *} = -\gamma^\mu\gamma_2$, $\mu = 0, 1, 2, 3$, by multiplying (1.140) by $i\gamma_2$ on the left side, we get:

$$0 = -[\gamma^\mu(\partial_\mu - iqA_\mu) + im]i\gamma_2\psi_e^*. \tag{1.141}$$

Then up to an arbitrary phase, quantum mechanics supposes [23] :

$$\psi_p = i\gamma_2\psi_e^*, \tag{1.142}$$

23. Quantum mechanics uses γ_2 because it is the only Dirac matrix with imaginary terms while the three other γ_μ matrices are real, given (1.4). Moreover the relation (1.142) is, by (1.7), independent from the choice of the γ_μ matrices.

which gives

$$0 = [\gamma^\mu(\partial_\mu - iqA_\mu) + im]\psi_p. \tag{1.143}$$

This equation is exactly the same as the equation of the electron up to the change of the sign of the electric charge. We automatically obtain the equality between the mass of the particle and the mass of the antiparticle. Using the decomposition of ψ with left and right waves, and assigning e indices for the electron and p indices for the positron, the link (1.142) between the electron wave and the positron wave reads:

$$\begin{pmatrix}\xi_{1p}\\ \xi_{2p}\\ \eta_{1p}\\ \eta_{2p}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\xi^*_{1e}\\ \xi^*_{2e}\\ \eta^*_{1e}\\ \eta^*_{2e}\end{pmatrix}. \tag{1.144}$$

That gives:

$$\xi_{1p} = \eta^*_{2e}, \quad \xi_{2p} = -\eta^*_{1e}; \quad \eta_{1p} = -\xi^*_{2e}; \quad \eta_{2p} = \xi^*_{1e}. \tag{1.145}$$

Now the same calculation in space algebra, and always with e indices for the electron and p indices for the positron, uses:

$$\widehat{\phi}_e = \sqrt{2}\begin{pmatrix}\eta_{1e} & -\xi^*_{2e}\\ \eta_{2e} & \xi^*_{1e}\end{pmatrix}; \quad \widehat{\phi}_p = \sqrt{2}\begin{pmatrix}\eta_{1p} & -\xi^*_{2p}\\ \eta_{2p} & \xi^*_{1p}\end{pmatrix}. \tag{1.146}$$

Then (1.142), which is equivalent to (1.145), is also equivalent to

$$\widehat{\phi}_p = \widehat{\phi}_e\sigma_1; \quad \phi_p = -\phi_e\sigma_1. \tag{1.147}$$

Once again we must recall that the charge conjugation described here is completely equivalent to the charge conjugation described by all textbooks of quantum physics. Only the style of writing is changed.

1.5 Improved invariant equation

We now come to our main departure from the Dirac theory which underlies all the relativistic components of the Standard Model. This change is also the main difference from Hestenes' work. He always used the linear Dirac equation, and only changed the mathematical framework and the presentation of this equation. **Here we simplify the wave equation itself**. It can be seen from equations in (1.91) that the invariant form of the Dirac equation (1.115) is given by

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\rho\cos(\beta), \tag{1.148}$$

The improvement that we introduced in [12] to that equation was the deletion of $\cos(\beta)$ and now we add the replacement of the scalar mass term m

by a matrix term $\mathbf{m}$ [43] [44]:

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + \mathbf{m}\rho;\ \mathbf{m} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix} \tag{1.149}$$

where $\mathbf{l}$ is the left mass term and $\mathbf{r}$ is the right mass term. The existence of two possibly different masses will be corroborated by their consequences. This improved equation is equivalent to the system of real equations:

$$0 = -w_3 + qA \cdot \mathrm{D}_0 + m_a\rho;\ m_a := \frac{\mathbf{l}+\mathbf{r}}{2} \tag{1.150}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_2 + qA \cdot \mathrm{D}_1, \tag{1.151}$$

$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_1 + qA \cdot \mathrm{D}_2, \tag{1.152}$$

$$0 = w_0 + qA \cdot \mathrm{D}_3 + d\rho;\ d := \frac{\mathbf{l}-\mathbf{r}}{2} \tag{1.153}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_3, \tag{1.154}$$

$$0 = -w_2, \tag{1.155}$$

$$0 = w_1, \tag{1.156}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_0. \tag{1.157}$$

This is actually a simplification since three of the eight equations equivalent to the Dirac equation are simplified: In the first line the invariant $m\Omega_1 = m\rho\cos(\beta)$ is simply replaced by $m_a\rho$ (where m_a is the arithmetic mean). The fourth equation (1.153) now has a $d\rho$ term and becomes very similar to the first equation. This new and apparently slight modification to the Dirac theory does not change five of the eight equations. Nevertheless this slight simplification improves many things: the last change, in (1.154), means the existence of a second conservative current, the K $=$ D_3 current. Since J and K currents are now both conservative, their sum and difference are also conservative [24]: the D^1_R and D^1_L chiral currents are conservative. This will be generalized in the next chapters for the electroweak domain. Moreover the eight equations come two by two. This is a mere consequence of the left–right structure of the wave.

24. The conservation of the left and right currents was obtained as early as 1983 by Lochak in his theory of the leptonic magnetic monopole [90]–[97], the theory from which comes our mass term in a particular case, where the Dirac equation is the linear approximation of our equation. Our equation is nevertheless another, distinct wave equation, because we conserved the electric gauge term of the wave equation of the electron. This gauge term is different from the gauge term of Lochak's monopole. In his theory of the monopole, the invariance of the electric gauge is only global (weaker) and it is the chiral gauge that is local (stronger). On the contrary, for our improved equation the electric gauge is local and the chiral gauge is only global. Since Noether's theorem requires only global invariance, our equation, like Lochak's, has the same conserved currents.

For a comparison between our new improved equation and the Dirac equation it is enough to multiply (1.149) by $\overline{\phi}^{-1}$ on the left side, which gives:

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + \rho\overline{\phi}^{-1}\mathbf{m}. \tag{1.158}$$

We implicitly suppose, when multiplying on the left side by $\overline{\phi}^{-1}$, that $\phi(\mathrm{x})$ is invertible, which means $\rho \neq 0$. Then (see A.3.8) a unique element M exists in Cl_3, defined mod$2i\pi$, satisfying:

$$\phi = e^M;\ M = a + ib + N;\ N = z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3. \tag{1.159}$$

We let:

$$\rho = e^{2a};\ \beta = 2b;\ M_\phi := e^N. \tag{1.160}$$

That gives:

$$\phi = \sqrt{\rho}e^{i\frac{\beta}{2}}M_\phi = e^{a+ib+N};\quad a = \frac{\ln(\rho)}{2};\quad b = \frac{\beta}{2}. \tag{1.161}$$

M_ϕ is an element of $SL(2,\mathbb{C})$, because we have:

$$\det(M_\phi) = \det(e^N) = e^{\mathrm{tr}(N)} = e^0 = 1 = M_\phi\overline{M}_\phi;\ \overline{M}_\phi = M_\phi^{-1}. \tag{1.162}$$

The existence of this M_ϕ element of $SL(2,\mathbb{C})$ was obtained by G. Lochak as early as 1956 [98] and was obtained independently by Hestenes ten years later [77]. This M_ϕ was also the starting point of the work of R. Boudet [5][6]. None of these physicists knew the difference between the field of dilators ϕ (similar to M in 1.1.2) and the field of the induced similitudes (like R). They thus called this M_ϕ Lorentz transformation. That forbids them to understand that **the ϕ wave is a field of dilators**. We then get:

$$\overline{\phi} = e^{\overline{M}} = e^{a+ib}e^{-N};\ \overline{\phi}^{-1} = e^{-(a+ib)}e^N;\ \rho\overline{\phi}^{-1} = e^{-i\beta}\phi. \tag{1.163}$$

Then when we compare with the former Dirac equation, the improved equation appears not with a term less, but with an additional term $e^{-i\beta}$:

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + e^{-i\beta}\phi\mathbf{m}. \tag{1.164}$$

The usual Dirac equation (1.2) is thus the linear approximation of our improved equation (1.164) when the Yvon-Takabayasi β angle is null or negligible and when the difference d between left and right mass terms is null. In the succeeding chapters, only the improved equation will present an ability to generalization. The properties of the improved equation are often simpler than those of the usual Dirac equation, and are closer to physical reality. This is what we explain here.

1.5.1 Uncrossed form of the wave equation

We incorporate in (1.164) the left and right waves defined in (1.70):

$$\begin{aligned}0 &= \nabla(\widehat{L}^1+\widehat{R}^1)(-i\sigma_3)+qA(\widehat{L}^1+\widehat{R}^1)+e^{-i\beta}(R^1+L^1)\mathbf{m} \\ &= (-i\nabla\widehat{L}^1+qA\widehat{L}^1+\mathbf{l}e^{-i\beta}R^1)+(i\nabla\widehat{R}^1+qA\widehat{R}^1+\mathbf{r}e^{-i\beta}L^1).\end{aligned} \tag{1.165}$$

In this last line, the left bracketed quantity is a matrix with two zeros in its second column, while the second bracketed quantity is a matrix with two zeros in its first column. This equation is hence equivalent to the system:

$$\begin{aligned}0 &= -i\nabla\widehat{L}^1+qA\widehat{L}^1+\mathbf{l}e^{-i\beta}R^1, \\ 0 &= i\nabla\widehat{R}^1+qA\widehat{R}^1+\mathbf{r}e^{-i\beta}L^1.\end{aligned} \tag{1.166}$$

We consider :

$$\underline{J} := \mathrm{J}+\frac{d}{m_a}\mathrm{K};\quad d=\frac{\mathbf{l}-\mathbf{r}}{2};\quad m_a=\frac{\mathbf{l}+\mathbf{r}}{2}, \tag{1.167}$$

$$\mathrm{J}=\phi\phi^\dagger=\mathrm{D}_L^1+\mathrm{D}_R^1;\quad \mathrm{K}=\phi\sigma_3\phi^\dagger=\mathrm{D}_L^1-\mathrm{D}_R^1;\quad \underline{J}=\frac{m}{\mathbf{l}}\mathrm{D}_L^1+\frac{m}{\mathbf{r}}\mathrm{D}_R^1$$

$$\rho e^{-i\beta}=\widehat{\phi}\phi^\dagger=\widehat{L}^1\widetilde{R}^1+\widehat{R}^1\widetilde{L}^1. \tag{1.168}$$

We now consider the vector v [25] such that:

$$\mathrm{v}=\frac{\underline{J}}{||\underline{J}||}=\frac{m_g}{m\rho}\underline{J}=\frac{1}{\rho}\Big(\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}\mathrm{D}_L^1+\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{D}_R^1\Big). \tag{1.169}$$

We then get:

$$\begin{aligned}\mathrm{v}\widehat{L}^1 &= \frac{1}{\rho}\Big(\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}\mathrm{D}_L^1+\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\mathrm{D}_R^1\Big)\widehat{L}^1=\frac{1}{\rho}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}R^1\widetilde{R}^1\widehat{L}^1+0, \\ \widetilde{R}^1\widehat{L}^1 &= 2\begin{pmatrix}\xi_1^* & \xi_2^* \\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta_1 & 0 \\ \eta_2 & 0\end{pmatrix}=\begin{pmatrix}a_1^* & 0 \\ 0 & 0\end{pmatrix};\ a_1=\rho e^{i\beta}, \\ R^1\widetilde{R}^1\widehat{L}^1 &= R^1a_1^*\frac{1+\sigma_3}{2}=a_1^*R^1;\quad \mathrm{v}\widehat{L}^1=e^{-i\beta}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}R^1=\frac{\mathbf{l}}{m_g}e^{-i\beta}R^1.\end{aligned} \tag{1.170}$$

Similarly, we have:

$$\begin{aligned}\mathrm{v}\widehat{R}^1 &= \frac{1}{\rho}\Big(\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}L^1\widetilde{L}^1+\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}R^1\widetilde{R}^1\Big)\widehat{R}^1=\frac{1}{\rho}\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}L^1\widetilde{L}^1\widehat{R}^1 \\ &= \frac{1}{\rho}\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}L^1\overline{\widetilde{R}^1\widehat{L}^1}=\frac{a_1^*}{\rho}\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}L^1=e^{-i\beta}\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}L^1, \\ \hat{\mathrm{v}}R^1 &= e^{i\beta}\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}\widehat{L}^1.\end{aligned} \tag{1.171}$$

25. The notation for a vector in space-time is in Roman typeset when using Cl_3 and in bold letters when using $Cl_{1,3}$.

Thus the system (1.166) is equivalent to the (seemingly) uncrossed system:

$$\begin{aligned} 0 &= -i(\nabla + iqA + im_g \mathrm{v})\widehat{L}^1, \\ 0 &= -i(\widehat{\nabla} + iq\widehat{A} + im_g \widehat{\mathrm{v}})R^1. \end{aligned} \tag{1.172}$$

It is to be noted that this system is not completely uncrossed because v is dependent on both the left and right parts of the wave. This system is equivalent to the equation:

$$0 = \nabla\widehat{\phi}\sigma_{21} + (qA + m_g \mathrm{v})\widehat{\phi}. \tag{1.173}$$

If we write this term using the Yvon-Takabayasi β angle and using the right spinor ξ^1 and the left spinor η^1, (1.166) reads:

$$\begin{aligned} 0 &= -i(\nabla + iqA)\eta^1 + \mathbf{l}e^{-i\beta}\xi^1, \\ 0 &= -i(\widehat{\nabla} + iq\widehat{A})\xi^1 + \mathbf{r}e^{i\beta}\eta^1. \end{aligned} \tag{1.174}$$

We can use either form since they are equivalent. As to the Dirac equation itself, the (1.11) system, if we replace m par m_g and with (1.170) and (1.171) identities, becomes:

$$\begin{aligned} 0 &= (-i\nabla + qA + m_g e^{i\beta}\mathrm{v})\eta^1, \\ 0 &= (-i\widehat{\nabla} + q\widehat{A} + m_g e^{-i\beta}\widehat{\mathrm{v}})\xi^1. \end{aligned} \tag{1.175}$$

Since v just as β depend on ξ and η, we must remark that the uncrossing of the Dirac equation is only seeming. We also may consider the Dirac equation as non linear, and even less linear than the improved equation, since identities (1.170) and (1.171) are sufficient to linearise the wave equation. We may notice that only one mass term appears, which is the geometric mean of the two masses.

1.5.2 Gauge invariance

Since the differential term and the gauge term do not change when we shift from the usual Dirac equation to the improved equation, and since the mass term is gauge-invariant, the improved equation is also invariant under the electric gauge. This gauge is expressed in the Cl_3 algebra as:

$$\phi \mapsto \phi' = \phi e^{ia\sigma_3} \ ; \quad A \mapsto A' = A - \frac{1}{q}\nabla a. \tag{1.176}$$

With the usual Dirac equation, the conservative current linked to the electric gauge invariance by Noether's theorem is the $\mathrm{J} = \mathrm{D}_0$ probability current. The first difference introduced by our improved Dirac equation is the status of this conservation law, which is now one of the eight real equations equivalent to the wave equation in invariant form. The second difference is the existence, among these eight equations, of another conservation law (1.154) for the K = D_3 current. This current comes from Lochak's theory of the magnetic monopole [90]–[97], at the origin of our improved wave equation. This second conservation law is linked to the global gauge invariance (chiral gauge):

$$\phi \mapsto \phi' = e^{ia}\phi \ ; \quad \overline{\phi} \mapsto \overline{\phi}' = e^{ia}\overline{\phi} \ ; \quad \partial_\mu a = 0, \tag{1.177}$$

which gives:

$$\rho e^{i\beta} = \phi\overline{\phi} \mapsto \rho' e^{i\beta'} = \phi'\overline{\phi}' = e^{2ia}\phi\overline{\phi} = \rho e^{i(\beta+2a)},$$
$$\rho \mapsto \rho' = \rho \ ; \quad \beta \mapsto \beta' = \beta + 2a. \tag{1.178}$$

We name this invariance the "chiral gauge" since the generator of the gauge group is the i which orients space [26]. We will encounter this chiral gauge again in the study of weak interactions. It is also the gauge of Lochak's magnetic monopole. Since the chiral gauge multiplies ϕ by e^{ia}, therefore $\widehat{\phi}$ is multiplied by e^{-ia}, the ξ spinor that is the left column of ϕ is multiplied by e^{ia}, and η which is the left column of $\widehat{\phi}$ is multiplied by e^{-ia} [92]. Lochak thus remarked that the difference between the electric gauge and the chiral gauge is as following: with both gauge transformations the left and right spinors turn with the same angle. They turn in the same direction for the electric gauge, and in contrary direction for the chiral gauge. The improved equation has lost the linearity of the usual Dirac equation because ρ depends on ϕ, and because the determinant which defines ρ and β is not linear in ϕ. The sum $\phi_1 + \phi_2$ of two solutions of (1.164) is not necessarily a solution of (1.164), but may be a solution. Now, since the equation is homogeneous and invariant on the chiral gauge, if ϕ is a solution and if z is any complex number then $z\phi$ is also a solution of (1.164). This property, common to the Schrödinger, Klein-Gordon and Pauli equations, and also for the first Hamiltonian version of the Dirac equation, is not true for the relativistic Dirac equation (hence these two versions cannot be equivalent) with the i which is a 3-vector in Cl_3. As we will see with the Pauli principle, this completely obscures the nonlinearity of the invariant wave equation, and then induces a false necessity for linearity in relativistic quantum physics. When an electronic wave interferes everything happens as if only one electron is at play. Moreover, the calculation of interferences with Young's slits, for instance, are not made with a single relativistic electron (the slits should be much narrower). Thus we are in the case where the approximation by the Pauli equation used by Gondran [100] is perfectly legitimate.

Indeed the Dirac equation in space algebra contains both ϕ and $\widehat{\phi}$. And if we multiply by i we must not forget that $\widehat{i} = -i$. The multiplication by the i commutative of $\mathbb{C}$ is different from the multiplication by the i commutative in Cl_3. Consequently the isomorphism existing between Cl_3 and $M_2(\mathbb{C})$ is

26. In Cl_3, for any orthonormal (u, v, w) basis, this basis is direct if and only if $uvw = i$, and is inverse if and only if $uvw = -i$ (see A.3.1).

only an isomorphism of algebras on the real field, not on the complex field. In fact, the multiplication by i in Cl_3, a pseudoscalar term, does not match the multiplication by i into $M_2(\mathbb{C})$. In Cl_3 instead, this multiplication by i matches the multiplication on the right side by $i\sigma_3 = \sigma_1\sigma_2$, a term which is a 2-vector (an oriented area), not a 3-vector (an oriented volume). Since Cl_3 is isomorphic to $Cl_{1,3}^+$ this $i\sigma_3$ becomes in space-time algebra the $\gamma_1\gamma_2$ 2-vector used by Hestenes [78]. With Cl_3 we must not forget that $\widehat{i} = -i$. This explains why we get three conjugations, when only one exists with the usual formalism of quantum mechanics.

The restricted isomorphism between the old formalism and relativistic Cliffordian formalism, as we previously explained, is also the reason for the discordance between the earlier form of the Dirac equation, using α_k and β matrices, and the relativistic Dirac equation. [27] The Hamiltonian formalism that we get with these matrices acts on a wave equation that is not relativistic, and moreover that is indeed equivalent to the Dirac equation expressed in $M_4(\mathbb{C})$ (with the unique i of quantum mechanics), but it cannot be equivalent to the Dirac equation expressed in Cl_3 or $Cl_{1,3}$, where the unique i is replaced by a 2-vector. This equation can indeed be multiplied by the i unique of quantum mechanics, which is also the i of the iI_4 matrix of $M_4(\mathbb{C})$. But also this iI_4 does not belong to the matrix formulation of the space-time algebra $Cl_{1,3}$. The unique i of the Dirac theory, commuting with everything, is proof of the fact that the theory is expressed not in $Cl_{1,3}$ or $Cl_{3,1}$ but in $M_4(\mathbb{C})$ which is isomorphic to the Clifford algebras $Cl_{2,3}$, or $Cl_{4,1}$ or $Cl_{0,5}$. Hence it is the algebra of a space-time with a supplementary dimension. This dimension is a supplementary dimension of time with $Cl_{2,3}$, or of space with $Cl_{4,1}$, or without ability to distinguish time from space with $Cl_{0,5}$. So it is very tempting to use this supplementary dimension to unify gravitation and electromagnetism, following Kaluza and many other physicists, among them Einstein and de Broglie. But these many attempts never prove their utility.

1.5.3 Plane wave

We take again (with the same reservations) the calculation made in 1.3.3 for the usual Dirac equation. Our improved equation is now reduced, for $A = 0$, to

$$\nabla\widehat{\phi} + e^{-i\beta}\phi\mathbf{m}\sigma_{12} = 0. \tag{1.179}$$

We consider the same plane wave with the φ phase satisfying

$$\phi = \phi_0 e^{\varphi\sigma_{21}} \ ; \quad \varphi = m_g \mathrm{v}_\mu \mathrm{x}^\mu \ ; \quad \mathrm{v} = \sigma^\mu \mathrm{v}_\mu, \tag{1.180}$$

27. The γ_5 matrix does not belong to the $Cl_{1,3}$ Clifford algebra generated by γ_0, γ_1, γ_2, γ_3.

where v is a fixed reduced velocity ($\mathrm{v}\widehat{\mathrm{v}} = 1$) and ϕ_0 is also a constant factor, and hence we get

$$\nabla\widehat{\phi} = \sigma^\mu \partial_\mu(\widehat{\phi}_0 e^{\varphi\sigma_{21}}) = -m_g \mathrm{v}\widehat{\phi}\sigma_{12}. \tag{1.181}$$

Then (1.164) is equivalent to

$$0 = (-m_g \mathrm{v}\widehat{\phi} + e^{-i\beta}\phi\mathbf{m})\sigma_{12};\ \phi\mathbf{m} = m_g e^{i\beta}\mathrm{v}\widehat{\phi}, \tag{1.182}$$

$$\widehat{\mathrm{v}}\phi\mathbf{m} = m_g e^{i\beta}\widehat{\mathrm{v}}\mathrm{v}\widehat{\phi} = m_g e^{i\beta}\widehat{\phi}, \tag{1.183}$$

Conjugating we get

$$\mathrm{v}\widehat{\phi}\widehat{\mathbf{m}} = m_g e^{-i\beta}\phi, \tag{1.184}$$

which implies

$$\phi = e^{i\beta}\mathrm{v}\widehat{\phi}\frac{\widehat{\mathbf{m}}}{m_g};\ \widehat{\phi} = e^{-i\beta}\widehat{\mathrm{v}}\phi\frac{\mathbf{m}}{m_g}. \tag{1.185}$$

We then have:

$$\phi = e^{i\beta}\mathrm{v}\Big(e^{-i\beta}\widehat{\mathrm{v}}\phi\frac{\mathbf{m}}{m_g}\Big)\frac{\widehat{\mathbf{m}}}{m_g} = \phi\frac{\mathbf{m}\widehat{\mathbf{m}}}{m_g^2}. \tag{1.186}$$

Therefore if ϕ_0 is invertible we must take:

$$m_g^2 = \mathbf{m}\widehat{\mathbf{m}} = (m_a + d\sigma_3)(m_a - d\sigma_3) = m_a^2 - d^2 = \mathbf{lr},$$
$$m_g = \sqrt{\mathbf{lr}}. \tag{1.187}$$

Thus the mass term m_g is the geometric mean of the left and right mass terms ($m_g < m_a$ if $\mathbf{l} \neq \mathbf{r}$). Multiplying (1.183) by $\phi^\dagger$ on the right side we obtain:

$$\widehat{\mathrm{v}}\phi\mathbf{m}\phi^\dagger = m_g e^{i\beta}\widehat{\phi}\phi^\dagger, \tag{1.188}$$

$$\widehat{\mathrm{v}}(\mathbf{l}\mathrm{D}_R^1 + \mathbf{r}\mathrm{D}_L^1) = m_g e^{i\beta}\rho e^{-i\beta}, \tag{1.189}$$

$$\widehat{\mathrm{J}}(\mathbf{l}\mathrm{D}_R^1 + \mathbf{r}\mathrm{D}_L^1) = m_g\rho^2 = \sqrt{\mathbf{lr}}\widehat{\mathrm{J}}\mathrm{J}. \tag{1.190}$$

Dividing by $\widehat{\mathrm{J}}$ this implies:

$$\mathbf{l}\mathrm{D}_R^1 + \mathbf{r}\mathrm{D}_L^1 = \sqrt{\mathbf{lr}}(\mathrm{D}_R^1 + \mathrm{D}_L^1), \tag{1.191}$$

$$\frac{\mathrm{J}}{\sqrt{\mathbf{lr}}} = \frac{\mathrm{D}_R^1}{\mathbf{r}} + \frac{\mathrm{D}_L^1}{\mathbf{l}}, \tag{1.192}$$

$$\mathbf{l}\mathrm{D}_R^{10} + \mathbf{r}\mathrm{D}_L^{10} = \sqrt{\mathbf{lr}}(\mathrm{D}_R^{10} + \mathrm{D}_L^{10}) = \sqrt{\mathbf{lr}}\rho\mathrm{v}^0. \tag{1.193}$$

Since $\mathrm{D}_R^{10} = |\xi_1^1|^2 + |\xi_2^1|^2 > 0$, $\mathrm{D}_L^{10} = |\eta_1^1|^2 + |\eta_2^1|^2 > 0$, $\sqrt{\mathbf{lr}} > 0$ and $\rho > 0$ we obtain:

$$\mathbf{l} > 0;\ \mathbf{r} > 0;\ \mathrm{v}^0 > 0. \tag{1.194}$$

We thus have, as in the case where $\mathbf{l} = \mathbf{r}$:

$$1 = \mathrm{v}\widehat{\mathrm{v}} = (\mathrm{v}^0 - \vec{\mathrm{v}})(\mathrm{v}^0 + \vec{\mathrm{v}}) = (\mathrm{v}^0)^2 - \vec{\mathrm{v}}^2, \tag{1.195}$$

$$\mathrm{v}^0 = \sqrt{1 + \vec{\mathrm{v}}^2}. \tag{1.196}$$

Thus we solve here in the simplest manner the old problem of unphysical negative energy: the plane wave of the electron may only have positive energy and positive proper mass (we will see later the question of charge conjugation). **The improved wave equation is thus much better than the linear Dirac equation**: the non-existence of negative energies, never observed in particle physics, does not need second quantization to find an explanation.

1.5.4 Extended invariance

We start from the invariant form (1.149). The similitude R induced by the dilator M with ratio $r = |\det(M)|$ satisfies:

$$\begin{aligned} &\mathrm{x}' = R(\mathrm{x}) = M\mathrm{x}M^\dagger \ , \quad \det(M) = re^{i\theta} \ , \quad \phi' = M\phi, \\ &\nabla = \overline{M}\nabla'\widehat{M} \ ; \quad qA = \overline{M}q'A'\widehat{M}. \end{aligned} \tag{1.197}$$

We also have:

$$\begin{aligned} &\rho' e^{i\beta'} = \det(\phi') = \det(M\phi) = \det(M)\det(\phi) = re^{i\theta}\rho e^{i\beta} = r\rho e^{i(\beta+\theta)}, \\ &\quad \rho' = r\rho; \ \beta' = \beta + \theta \mod 2\pi. \end{aligned} \tag{1.198}$$

And we obtain:

$$\begin{aligned} 0 &= \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + \mathbf{m}\rho = \overline{\phi}\ \overline{M}\nabla'\widehat{M}\widehat{\phi}\sigma_{21} + \overline{\phi}\ \overline{M}q'A'\widehat{M}\widehat{\phi} + \mathbf{m}\rho \\ &= \overline{\phi}'(\nabla'\widehat{\phi}')\sigma_{21} + \overline{\phi}'q'A'\widehat{\phi}' + \mathbf{m}\rho. \end{aligned} \tag{1.199}$$

The improved equation is form-invariant under Cl_3^*, which is the multiplicative group of the invertible elements of Cl_3, if and only if:

$$\mathbf{m}\rho = \mathbf{m}'\rho'; \ \mathbf{m}\rho = \mathbf{m}'r\rho. \tag{1.200}$$

We then obtain the form invariance of the wave equation under Cl_3^*, a group isomorphic to $GL(2, \mathbb{C})$, if and only if:

$$\mathbf{m} = \mathbf{m}'r; \ \mathbf{l} = \mathbf{l}'r; \ \mathbf{r} = \mathbf{r}'r; \ m_a = m'_a r; \ m_g = m'_g r; \ d = d'r. \tag{1.201}$$

These equalities are simpler than the $m = m're^{i\theta}$ that the usual Dirac equation gives – and this is a powerful argument for our improved equation.

What is the meaning of these equalities for physics? If the true invariance group of the electromagnetic laws is not only the Lorentz group, not even its covering group, but the more binding Cl_3^* group, similar things must happen as when Galilean physics was replaced by relativistic physics, which put together mass and momentum, or electric field and magnetic field. The proper mass m_0 and the density $\rho = ||\mathrm{J}||$ are both invariant under Lorentz transformations. Under the similitude induced by any dilator M, we see a similar grouping: we find that $\mathbf{m}$ and ρ are no longer separately invariant, and only their product $\mathbf{m}\rho$ remains invariant:

$$\mathbf{m}\rho = \mathbf{m}' r\rho = \mathbf{m}'\rho',$$
$$d\rho = d' r\rho = d'\rho'. \tag{1.202}$$

Hence only the product of a reduced mass and a ratio of similitude is fully invariant. And the reduced mass $m = m_0 c/\hbar$ is proportional to the inverse of length in space-time, which means a frequency. Let us consider an analogy: the fact that gravitational acceleration is proportional to the acceleration due to inertia results in a constant ratio between gravitational mass and inertial mass (and so this ratio may be put equal to one). This is historically the starting point of Einstein's gravitation. Now since, when the scale parameter r changes arbitrarily, the ratio between ρ and $1/m = \hbar/m_0 c$ is constant. This needs the existence of a constant, which is the Planck constant. We may then say that **the existence of the Planck constant is a consequence of the invariance under the Cl_3^* group**, which is a more binding group than the local invariance group of either special or general relativity. We will see in the next chapter how the quantization of the action is linked to this invariance under Cl_3^*. From this point of view we may also say this: the existence of the Planck constant has not been fully understood. The consideration of a greater invariance group will allow us to see things differently, and will later allow us in chapter 2 to understand **why the kinetic momentum is quantized with the value** $\hbar/2$.

1.5.5 Normalization of the wave

We start from the improved wave equation with the system in (1.174), and we use:

$$\mathrm{J} = \mathrm{D}_L^1 + \mathrm{D}_R^1 = \rho\mathrm{v};\ \mathrm{J}\widehat{\mathrm{J}} = \rho^2;\ \mathrm{D}_L^{1\mu} = \eta^{1\dagger}\sigma^\mu\eta^1;\ \mathrm{D}_R^{1\mu} = \xi^{1\dagger}\widehat{\sigma}^\mu\xi^1. \tag{1.203}$$

We may express the Lagrangian density of the improved wave equation as:

$$\mathcal{L} = \frac{m_a}{k\mathbf{l}}\mathcal{L}_L + \frac{m_a}{k\mathbf{r}}\mathcal{L}_R;\ \mathcal{L}_L = \Re[\eta^{1\dagger}(-i\nabla + qA + m_g\mathrm{v})\eta^1], \tag{1.204}$$
$$\mathcal{L}_R = \Re[\xi^{1\dagger}(-i\widehat{\nabla} + q\widehat{A} + m_g\widehat{\mathrm{v}})\xi^1],$$

where k is a constant which is further explained. As for the Dirac equation, the Lagrangian density becomes:

$$\mathcal{L} = \frac{m_a}{k\mathbf{l}}\mathcal{L}_L + \frac{m_a}{k\mathbf{r}}\mathcal{L}_R;\ \mathcal{L}_L = \Re[\eta^{1\dagger}(-i\nabla + qA + m_g e^{i\beta}\mathrm{v})\eta^1], \tag{1.205}$$
$$\mathcal{L}_R = \Re[\xi^{1\dagger}(-i\widehat{\nabla} + q\widehat{A} + m_g e^{-i\beta}\widehat{\mathrm{v}})\xi^1],$$

because we have, with (1.170) and (1.171), next with (1.92) :

$$\begin{aligned}&\frac{m_a}{\mathbf{l}}\eta^{1\dagger}m_g\mathrm{v}\eta^1+\frac{m_a}{\mathbf{r}}\xi^{1\dagger}m_g\widehat{\mathrm{v}}\xi^1=m_ae^{-i\beta}\eta^{1\dagger}\xi^1+m_ae^{i\beta}\xi^{1\dagger}\eta^1\\&=\frac{m_a}{2}e^{-i\beta}\rho e^{i\beta}+\frac{m_a}{2}e^{i\beta}\rho e^{-i\beta}=m_a\rho.\end{aligned}\tag{1.206}$$

Similarly, for the Dirac equation, we have:

$$\begin{aligned}&\frac{m_a}{\mathbf{l}}\eta^{1\dagger}m_ge^{i\beta}\mathrm{v}\eta^1+\frac{m_a}{\mathbf{r}}\xi^{1\dagger}m_ge^{-i\beta}\widehat{\mathrm{v}}\xi^1=m_a\eta^{1\dagger}\xi^1+m_a\xi^{1\dagger}\eta^1\\&=\frac{m_a}{2}\rho e^{i\beta}+\frac{m_a}{2}\rho e^{-i\beta}=m_a\rho\cos\beta=m_a\overline{\psi}\psi.\end{aligned}\tag{1.207}$$

With the covariant derivatives

$$d_\mu:=-i\partial_\mu+qA_\mu+m_g\mathrm{v}_\mu,\tag{1.208}$$

for the improved equation, and with:

$$d_\mu:=-i\partial_\mu+qA_\mu+m_g\cos\beta\mathrm{v}_\mu,,\tag{1.209}$$

for the Dirac equation, we can read the Lagrangian density as:

$$\mathcal{L}=\Re\Big[-i\Big(\frac{m_a}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu d_\mu\eta^1+\frac{m_a}{k\mathbf{l}}\xi^{1\dagger}\widehat{\sigma}^\mu d_\mu\xi^1\Big)\Big].\tag{1.210}$$

The invariance of the Lagrangian density under the translations in space-time, like in the linear Dirac theory, implies the existence of a conserved tensor density of energy-impulse, This tensor is known as Tétrode's tensor. Since the wave equation is homogeneous, the Lagrangian density is exactly null for any solution of the wave equation, and the Tétrode's tensor reads:

$$\begin{aligned}T_\nu^\mu&=\Re\Big[-i\Big(\frac{m_a}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu d_\nu\eta^1+\frac{m_a}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu d_\nu\xi^1\Big)\Big]-\delta_\nu^\mu\mathcal{L}\\&=\Re\Big[-i\Big(\frac{m_a}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu d_\nu\eta^1+\frac{m_a}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu d_\nu\xi^1\Big)\Big].\end{aligned}\tag{1.211}$$

For a wave with an energy E satisfying

$$-id_0\eta^1=\frac{E}{\hbar c}\eta^1;\ -id_0\xi^1=\frac{E}{\hbar c}\xi^1,\tag{1.212}$$

we get

$$\begin{aligned}T_0^0&=\Re\Big[-i\Big(\frac{m_a}{k\mathbf{l}}\eta^{1\dagger}d_0\eta^1+\frac{m_a}{k\mathbf{r}}\xi^{1\dagger}d_0\xi^1\Big)\Big]\\&=\frac{E}{\hbar c}\Big(\frac{m_a}{k\mathbf{l}}\eta^{1\dagger}\eta^1+\frac{m_a}{k\mathbf{r}}\xi^{1\dagger}\xi^1\Big)=E\frac{\mathbf{J}^0}{\hbar c},\\\mathbf{J}:&=\frac{m_a}{k\mathbf{l}}\mathrm{D}_L^1+\frac{m_a}{k\mathbf{r}}\mathrm{D}_R^1=\frac{1}{k}\underline{J}.\end{aligned}\tag{1.213}$$

The condition for normalization of the wave function:

$$\iiint dv \frac{\mathbf{J}^0}{\hbar c} = 1, \tag{1.214}$$

is hence equivalent to:

$$E = \iiint dv\, T_0^0. \tag{1.215}$$

The left term of this sum is the total energy E of the electron, which de Broglie conceived of as a very small clock with frequency $E = h\nu$, while the right term is the sum of the local energy density of the electron. We will see that this local density is linked to inertia through the Lorentz force. Hence it is not because we must have a probability that the wave must be normalized. The physical wave is normalized, always, because the inertial mass-energy acted on by all forces (electromagnetic, weak, ...) is equal to the absolute value of the gravitational mass-energy. So this energy has a determined value, not an arbitrary one. The normalization of the electron wave that is a law in quantum mechanics is hence only the consequence of the equivalence between gravitational mass and inertial mass, [28] the principle at the basis of general relativity. The existence of a probability density, for any electron wave and in any possible case, is not a principle on which any physical theory must be built: It is simply the necessary equality between inertial mass and gravitational mass. And this is the same whether for the usual Dirac equation, or for the improved equation which the usual Dirac equation linearly approximates [21].

Since T^μ_ν must have the dimension of an energy density, ML^2T^{-2}/L^3, and since $\mathbf{J}$ has the dimension $\hbar c/L^3 = M/T^2$, we see that $k\mathbf{J}$ which has the dimension of D_L and D_R, has also the dimension: $\dim(k)M/T^2$. And the dimension of D_L and D_R is the dimension of $\phi\phi^\dagger$. Then ϕ is without physical dimension if and only if k has dimension T^2/M, that means that $k\hbar c$ has dimension L^3. The only length which has an intrinsic physical meaning being the Planck length, it is possible that:

$$k\hbar c = l_P^3;\ l_P = \sqrt{\frac{G\hbar}{c^3}}. \tag{1.216}$$

After normalization, the numerical value of each current does not depend on the choice of the k value.

Normalization obviously applies to the solutions for the hydrogen atom

28. This normalization is so important that it was included among the postulates imposed on any quantum wave. In fact normalization is allowed by the wave equations but is not deduced from them. It is the cause of great difficulties, like the collapse of the ψ or Schrödinger's dead–living cat. The issue of normalization also precipitated the setback of de Broglie's pilot–wave and afterward Bohm's. Consequently de Broglie elaborated his theory of the double solution.

that we study in Appendix C [29]. Moreover, since the $|\psi|^2$ of the Schrödinger wave is a particular case of the approximation of the Dirac wave by a part of its components, the need for normalizing the wave function of an electron – which is part of the principles of nonrelativistic quantum theory – follows as in the relativistic case from the equivalence principle. This is very important for the unification of all interactions, because until now the existence of probabilities in quantum mechanics was thought of as a metaphysical principle governing any present and future theory, while in fact this is only the consequence of the equality between gravitational and inertial mass. At the same time we understand better why Bohr was able to rebut all of Einstein's arguments against Born's probabilistic interpretation: The existence of a probability density comes from gravitation.

The probabilities that Einstein was thinking of derive from thermodynamics, in which case there is not only one particle, but myriads of particles moving in all directions. Furthermore we used the expression "probability density" and we carefully avoided the expression "probability of presence." The first expression makes sense because the theory of probabilities, like integral calculus, was developed from the same mathematics, measure theory. The second expression cannot make sense for the electron because any experimental verification of the probability of presence of the electron-particle, for instance a probability of 0.1 in a domain D of space, supposes that we can attain the convergence of statistical frequency at 0.1. And though it is possible to obtain the statistics from the myriads of photons moving on a single light wave, it is absolutely impossible to obtain statistics from the single electron that can occupy an electron wave. It is possible to obtain statistics from electrons only if we have a great number of them and each one necessarily has its own particular wave. We can say that the probability of the domain D is 0.1 if the sum over D of the probability density is 0.1, and this is all that may be said about the wave of an electron. For a system of electrons it is necessary to use the Pauli principle that we will study later. Now we can see why any scientific discussion about probabilities in quantum physics needs very careful phrasing. A general theoretical discussion about probabilities has little meaning: properties of electrons, each one being alone on its wave, are radically different from properties of photons which can move on the same wave. For instance the violation of Bell's inequalities was experimentally observed only for photons. For electrons this remains unproved. Entanglement needs at least two waves.

29. We must recall that the density $\mathbf{J}^0$ is not equal to the relativistic invariant ρ. Instead it is the time component of a contravariant space-time vector. We also recall that T_0^0 is a component of a nonsymmetric tensor. It is well known that the integration of the spin 1/2 into relativistic gravitation is only possible with a nonzero torsion [95] (see also Chapter 4).

Scalar product

The current which is normalized is

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{1}{l_P^3}\Big(\mathrm{D}_0^0 + \frac{d}{m_a}D_3^0\Big) = \frac{m}{l_P^3}\Big(\frac{\eta^\dagger\eta}{\mathbf{l}} + \frac{\xi^\dagger\xi}{\mathbf{r}}\Big). \tag{1.217}$$

The scalar product hence satisfies:

$$\langle\phi|\phi\rangle := \iiint \mathrm{dv}\frac{\mathbf{J}^0}{\hbar c} = \iiint \mathrm{dv}\frac{m}{l_P^3}\Big(\frac{\eta^\dagger\eta}{\mathbf{l}} + \frac{\xi^\dagger\xi}{\mathbf{r}}\Big), \tag{1.218}$$

$$\langle\phi|\phi'\rangle := \iiint \mathrm{dv}\frac{m}{l_P^3}\Big(\frac{\eta^\dagger\eta'}{\mathbf{l}} + \frac{\xi^\dagger\xi'}{\mathbf{r}}\Big), \tag{1.219}$$

This scalar product is identical to the Hermitian scalar product of quantum mechanics.

1.5.6 Charge conjugation

We again begin with $\phi_p = -\phi_e\sigma_1$, the link between the wave of the particle and the wave of the antiparticle in relativistic quantum mechanics. The improved wave equation (1.164) reads for the particle:

$$\nabla\widehat{\phi}_e\sigma_{21} + qA\widehat{\phi}_e + e^{-i\beta_e}\phi_e\mathbf{m} = 0. \tag{1.220}$$

We also have

$$\rho_e e^{i\beta_e} = \phi_e\overline{\phi}_e. \tag{1.221}$$

This then gives:

$$\rho_e e^{i\beta_e} = \phi_e\overline{\phi}_e = \phi_p(-\sigma_1)(\widehat{\phi}_p\sigma_1)^\dagger = -\phi_p\overline{\phi}_p = -\rho_p e^{i\beta_p}. \tag{1.222}$$

Therefore (1.220) takes the form:

$$\nabla\widehat{\phi}_p\sigma_1\sigma_{21} + qA\widehat{\phi}_p\sigma_1 + (-e^{-i\beta_p})(-\phi_p\sigma_1\mathbf{m}) = 0. \tag{1.223}$$

Multiplying on the right side by σ_1, this is equivalent to

$$\begin{aligned} 0 &= -\nabla\widehat{\phi}_p\sigma_{21} + qA\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}, \\ 0 &= \nabla\widehat{\phi}_p\sigma_{21} - qA\widehat{\phi}_p - e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}. \end{aligned} \tag{1.224}$$

Next, multiplying on the left side by $\overline{\phi}_p$, we get the invariant wave equation of the positron:

$$\begin{aligned} 0 &= -\overline{\phi}_p\nabla\widehat{\phi}_p\sigma_{21} + q\overline{\phi}_pA\widehat{\phi}_p + \widehat{\mathbf{m}}\rho_p, \\ 0 &= \overline{\phi}_p\nabla\widehat{\phi}_p\sigma_{21} - q\overline{\phi}_pA\widehat{\phi}_p - \widehat{\mathbf{m}}\rho_p \end{aligned} \tag{1.225}$$

The first equation means that the lone differential term of the wave equation changes sign. This was the reason for Feynman [73] to interpret charge conjugation as parity–time (PT) symmetry. The second equation means that the charge term is seen changing sign, and that the mass term is changed in a way that the arithmetic average $m = (\mathbf{r} + \mathbf{l})/2$ changes sign while the difference $d = (\mathbf{l} - \mathbf{r})/2$ remains unchanged, so we can say that the role of left and right parts are exchanged. Next we are able to recast the previous equations as:

$$0 = \nabla\widehat{\phi}_p\sigma_{12} + qA\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}. \tag{1.226}$$

Plane waves

The improved wave equation is reduced, if $A = 0$, to:

$$0 = -\nabla\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}\sigma_{12}. \tag{1.227}$$

We consider a solution such as:

$$\phi_p := \phi_0 e^{\varphi_p\sigma_{21}};\ \varphi_p := m_g \mathrm{v}_{p\mu}\mathrm{x}^\mu;\ \mathrm{v}_p := -\mathrm{v}. \tag{1.228}$$

We obtain the same results as in 1.5.3 :

$$m_g = \sqrt{\mathbf{lr}};\ \mathrm{v}^0 = \sqrt{1 + \vec{\mathrm{v}}^2}. \tag{1.229}$$

We thus have:

$$\mathrm{v}_p^0 = -\sqrt{1 + \vec{\mathrm{v}}_p^2} \tag{1.230}$$

Hence we again obtain plane wave solutions with a negative time coefficient, necessary for Fourier transformation and for very small wave packets, but with a positive mass-energy, in accordance with experiment. Created with the same energy, the electron and positron are moving with opposite velocity vectors, which is also in full accordance with experiment.

Numeric equations

Multiplying on the left side by $\overline{\phi}_p$ we get the invariant wave equation:

$$0 = \overline{\phi}_p\nabla\widehat{\phi}_p\sigma_{12} + q\overline{\phi}_p A\widehat{\phi}_p + \widehat{\mathbf{m}}\rho_p;\ \widehat{\mathbf{m}} = \begin{pmatrix} \mathbf{r} & 0 \\ 0 & \mathbf{l} \end{pmatrix}. \tag{1.231}$$

Nonrelativistic quantum mechanics, using a single i, could not truly understand charge conjugation, which simply changes the sign of the $\sigma_{21} = \sigma_2\sigma_1$ term into $\sigma_{12} = \sigma_1\sigma_2$ and the sign of the difference d. The first change of sign is thus only a change of direction in the series of σ_k, which is also a change of space orientation, the left wave becoming right and conversely, and changing also the sign of d. Instead of the system of the eight equations of the particle, we now have the same system, albeit one where the components of all ∂_μ and d change sign:

$$0 = w_3 + qA \cdot \mathrm{D}_0 + m_a\rho, \tag{1.232}$$
$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_2 + qA \cdot \mathrm{D}_1, \tag{1.233}$$
$$0 = +\frac{1}{2}\nabla \cdot \mathrm{D}_1 + qA \cdot \mathrm{D}_2, \tag{1.234}$$
$$0 = -w_0 + qA \cdot \mathrm{D}_3 - d\rho, \tag{1.235}$$
$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_3, \tag{1.236}$$
$$0 = w_2, \tag{1.237}$$
$$0 = -w_1, \tag{1.238}$$
$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_0. \tag{1.239}$$

Charge conjugation changes the sign of the charge and the sign of chiral masses, because we cannot change the arrow of time nor the orientation of space [62]. Actually, only the differential terms of the wave equations and d change sign. The electric gauge invariance is now obtained as:

$$\phi_p \mapsto \phi'_p = \phi_p e^{ia\sigma_3}, \tag{1.240}$$
$$A \mapsto A' = A - \frac{1}{q}(-\nabla a) = A - \frac{1}{-q}\nabla a.$$

Thus the positron seems to have a charge opposite to that of the electron. But in fact it is not q but $\partial_\mu a$ that changes sign. And so only ∂_μ, v_μ, w_μ and d change sign. With the covariant derivatives

$$\overline{d}_\mu = i\partial_\mu + qA_\mu + m_g \mathrm{v}_\mu, \tag{1.241}$$

we can express the Lagrangian density as:

$$\overline{\mathcal{L}} = \Re\Big(\frac{m}{k\mathbf{r}}\eta_p^{1\dagger}\widehat{\sigma}^\mu \overline{d}_\mu \eta_p^1 + \frac{m}{k\mathbf{r}}\xi_p^{1\dagger}\overline{d}_\mu \sigma^\mu \xi_p^1\Big). \tag{1.242}$$

The normalization of the wave always given a stationary state, is thus equivalent to:

$$\iiint dv T_0^0 = -E. \tag{1.243}$$

The mass-energy m_a/c^2, positive, of the positron is exactly the opposite of the negative energy-coefficient of the stationary wave. **The improved wave equation thus resolves the problem of the energy sign in a way that is much easier to understand than second quantization**: we indeed have the negative coefficients $-|E|$ necessary to obtain the Fourier transformation, but the true energy density is the T_0^0 component of the energy–momentum that remains positive. Since the wave equation of the antiparticle is obtained from that of the particle simply by changing ∂_μ into $-\partial_\mu$ and d into $-d$, which also results from the PT transformation, the CPT theorem of quantum field theory is trivially satisfied. Therefore, charge conjugation is the purely quantum and purely relativistic phenomenon of a wave which in a sense sees space-time upside down. This was the point of view of Feynman [73]. And since the usual Dirac equation is the linear approximation of our improved equation, we derive the Dirac equation of the positron from the improved equation of the positron by changing the mass term: we must account for the fact that $\beta_p = \beta_e + \pi$ and that $\Omega_{1e} = -\Omega_{1p}$. The linear approximation of the improved wave equation of the positron satisfies $m = \mathbf{l} = \mathbf{r} = m_a$, which implies:

$$0 = -\overline{\phi}_p \nabla \widehat{\phi}_p \sigma_{21} + q\overline{\phi}_p A\widehat{\phi}_p - m\Omega_{1p}, \tag{1.244}$$
$$0 = -\nabla \widehat{\phi}_p \sigma_{21} + qA\widehat{\phi}_p - m\phi_p,$$
$$0 = \nabla \widehat{\phi}_p \sigma_{21} + (-q)A\widehat{\phi}_p + m\phi_p. \tag{1.245}$$

This is precisely the Dirac equation of the positron, with the charge appearing with a changed sign. We have, for the sign of E and of T_0^0, the same results as with the improved equation: E is negative while T_0^0 is positive.

1.6 The hydrogen atom

Early quantum mechanics obtained the quantization of the energy levels by solving the Schrödinger equation in the case of the hydrogen atom, an electron "revolving" around a proton. Obtaining the quantization was a brilliant result. Moreover the Hamiltonian formulation of quantum mechanics allowed physicists to extend the understanding to the electron systems, so important to chemistry. But the other results were not as satisfying. For instance the energy levels were not very precise. And the total number of quantum states for the principal quantum number $\mathbf{n}$ was thought to be $\mathbf{n}^2$ when the actual number should be $2\mathbf{n}^2$ states.

The detailed calculation using our improved equation is presented in Appendix C. This resolution is quite different from the one used in the early years of quantum mechanics. At that time the theory of proper values and proper vectors in Hermitian spaces was developed mainly for application to the angular momentum operators. Bohr understood the Mendeleev periodic table by counting all possible values of the angular momentum of atomic electron-particles. That gave Bohr the expected energy levels $k/\mathbf{n}^2$, but not the expected number of states. Next Sommerfeld looked into relativistic dynamics of particles to obtain more states, as well as the fine structure of atomic spectral lines. Those models of electrons as particles could not explain the half-odd numbers that nevertheless were necessary to understand the emission and absorption spectra of atoms. Quantum mechanics replaced this counting with the calculation of solutions of the electron wave equation, which were proper vectors of operators with the same algebraic properties as classical angular momentum. Using the Schrödinger wave equation, only Bohr's model was reproduced. The relativistic Klein-Gordon equation was able to obtain the second quantum number introduced by Sommerfeld, but not with the correct values, because integer numbers (0, 1, 2, ...), which are the only possible values with angular momentum operators, must be replaced by half-integer values (1/2, 3/2, 5/2, ...) to account for spectroscopic lines. This profound divergence between theory and experiments on light led to the hypothesis of the spin of the electron, a set of operators having the same algebraic properties as angular momentum operators, with only two proper values, $+\frac{1}{2}$ and $-\frac{1}{2}$. Next Pauli got a wave equation for an electron with spin.

That equation uses the 2×2 complex matrices of (1.4) which are the generators of the Cl_3 algebra. This algebra contains the Lie algebra of the $SU(2)$ group, and also of the $SO(3)$ group. The representations of the $SU(2)$ group are characterized by a number which has 0, 1/2, 1, 3/2, 2, 5/2, 3, ... values. So the half-odd numbers are possible, but that do not explain the absence of the 0, 1, 2, 3, ... values, which are precisely those linked to the representations of $SU(2)$ which are also representations of $SO(3)$. Consequently, and because the Pauli equation is nonrelativistic, this wave equation cannot be the true wave equation of the electron [56].

Therefore Dirac looked for and quickly found a better equation, which immediately was improved to respect the invariance rules of restricted relativity. Only the Dirac equation was able to get the true quantum numbers and the true energy levels in the case of the Hydrogen atom. The set of adequate solutions was obtained by C.G. Darwin [11], using operators issued from the Pauli equation. That resolution, despite its excellent precision, showed two defects: it did not explain why it was necessary that the electron states must be proper vectors of operators constructed from the Pauli wave equation. It also did not explain why these solutions were the only ones possible, notably when was wrongly refused the existence of states with both a $\kappa < 0$ number and constant radial polynomials. Finally and above all, de Broglie [87] thought that the linearity of the Dirac equation did not allowed us to explain the spatial limitation of wave trains. The linear equation could only be the linear approximation of the true wave equation.

As for us, we use a completely different method obtained by H. Krüger [86], a marvelous classical one from the mathematical point of view, separating the variables in spherical coordinates (r, θ, φ). In the present edition, we completely solved the improved equation, accounting for left and right mass terms. This resolution computes in an exact manner (not only approximating) all possible solutions with separated variables, in the case of the Coulombian potential [26]. We show that these solutions are the only ones possible, from our hypotheses: the wave is a function of space-time with value in the Cl_3^* Lie group, solution of the improved wave equation, such that its energy is the sum on the whole space of the density energy of the electron wave. The wave must thus be normalized. The principal quantum number $\mathbf{n}$ remains a sum: $\mathbf{n} = |\kappa| + n$ [106]; κ is introduced in the separation between the variables $\mathrm{x}^0 = ct$ and φ on one side, from the variables θ and radial r on the other side. The study of functions of θ links the necessary normalization of the wave to conditions governing $|\kappa|$ and n constants: $|\kappa|$ must be a nonzero integer number and n is the degree of the polynomial functions included in the series expansion of radial functions. The mere necessity of normalizing the solution of the wave equation allows us the existence of a probability density, hence gives the $|\kappa|$ and n integers, and thus the $\mathbf{n}$ number. The last quantum number, here called λ (to avoid a possible confusion with the mass), is obtained from the sole condition that the wave must be a well-defined function in space-time, with unique value in Cl_3. There is absolutely no need for operators of angular momentum, since we obtain all quantum numbers without any need of the l number in non relativistic quantum mechanics. And if the λ numbers are integers, not half-odd numbers, that only results of the use of a moving frame when the calculation works in spherical coordinates. This moving frame is generated by a $S = e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2}$ rotator, with half-angles that are thus purely geometrical. The number of turns $(\lambda \pm \frac{1}{2})\frac{\varphi}{2\pi}$ being necessarily an integer number, for each part of the wave, λ may be only the half of an odd relative integer.

Let us see how, without any integer angular momentum l, the various states are obtained. We consider for instance the case $\mathbf{n} = 5$. The degree n of the radial polynomials, integer number, is $\mathbf{n} - |\kappa|$ and the smallest value of $|\kappa|$ is 1. The n degree may hence be only 4, 3, 2, 1 or 0. And the number λ is restricted by the $|\lambda| < |\kappa|$ condition. The calculation of the radial functions contains two arbitrary a and b constants which cannot be both non zero : Should the contrary occur, a value of the r variable exists in the equatorial plane where the wave does not take value in Cl_3^*. It is thus the necessity to take value in Cl_3^* which is the true reason of the success of the wave equation, giving the true number of solutions and the uniqueness of these solutions.

1. If $n = 4$ then $|\kappa| = 1$, thus we get 4 states: $\lambda = -1/2$ and $a_0 = 0$, $\lambda = -1/2$ and $b_0 = 0$, $\lambda = 1/2$ and $a_0 = 0$, $\lambda = 1/2$ and $b_0 = 0$.

2. If $n = 3$ then $|\kappa| = 2$, thus we get 8 states: $\lambda = -3/2$ and $a_0 = 0$, $\lambda = -3/2$ and $b_0 = 0$, $\lambda = -1/2$ and $a_0 = 0$, $\lambda = -1/2$ and $b_0 = 0$, $\lambda = 1/2$ and $a_0 = 0$, $\lambda = 1/2$ and $b_0 = 0$, $\lambda = 3/2$ and $a_0 = 0$, $\lambda = 3/2$ and $b_0 = 0$.

3. If $n = 2$ then $|\kappa| = 3$, thus we get 12 stats: $\lambda = -5/2$ and $a_0 = 0$, $\lambda = -5/2$ and $b_0 = 0$, $\lambda = -3/2$ and $a_0 = 0$, $\lambda = -3/2$ and $b_0 = 0$, $\lambda = -1/2$ and $a_0 = 0$, $\lambda = -1/2$ and $b_0 = 0$, $\lambda = 1/2$ and $a_0 = 0$, $\lambda = 1/2$ and $b_0 = 0$, $\lambda = 3/2$ and $a_0 = 0$, $\lambda = 3/2$ and $b_0 = 0$, $\lambda = 5/2$ and $a_0 = 0$,

$\lambda = 5/2$ and $b_0 = 0$.

4. If $n = 1$ then $|\kappa| = 4$, thus we get 16 states: $\lambda = -7/2$ and $a_0 = 0$, $\lambda = -7/2$ and $b_0 = 0$, $\lambda = -5/2$ and $a_0 = 0$, $\lambda = -5/2$ and $b_0 = 0$, $\lambda = -3/2$ and $a_0 = 0$, $\lambda = -3/2$ and $b_0 = 0$, $\lambda = -1/2$ and $a_0 = 0$, $\lambda = -1/2$ and $b_0 = 0$, $\lambda = 1/2$ and $a_0 = 0$, $\lambda = 1/2$ and $b_0 = 0$, $\lambda = 3/2$ and $a_0 = 0$, $\lambda = 3/2$ and $b_0 = 0$, $\lambda = 5/2$ and $a_0 = 0$, $\lambda = 5/2$ and $b_0 = 0$. $\lambda = 7/2$ and $a_0 = 0$, $\lambda = 7/2$ and $b_0 = 0$.

5. If $n = 0$ then $|\kappa| = 5$, but in this case the radial polynomial are reduced to constants, and in this case neither a_0 nor b_0 can cancel. The solutions depend on a single constant, its modulus is fixed by the normalization of the wave. We hence have here only 10 states: $\lambda = -9/2$, $-7/2$, $-5/2$, $-3/2$, $-1/2$, $1/2$, $3/2$, $5/2$, $7/2$, $9/2$.

That finally gives $4 + 8 + 12 + 16 + 10 = 50 = 2 \times 5^2$ (and more genrally $2\mathbf{n}^2$) states, two by two orthogonal (what de Broglie yet explained in a detailed manner in 1934 [56]). The argument of signs given by Darwin to suppress half of the solutions with $n = 0$ was not the correct explanation, this was known from the separation of variables [13]. We put at the end of the Chapter C the detailed calculation of all solutions with a principal quantum number $\mathbf{n} = 1$ and $\mathbf{n} = 2$. Among the 50 states with total quantum number $\mathbf{n} = 5$, let us consider the state characterized by $n = 2$, $|\kappa| = 3$, $\lambda = 3/2$ and $b_0 = 0$. The value of the wave satisfies:

$$\phi = \Omega X e^{(\frac{3}{2}\varphi - E\mathrm{x}^0)i_3};\ \Omega = \frac{1}{r\sqrt{\sin\theta}} e^{-\frac{\varphi}{2}i_3} e^{-\frac{\theta}{2}i_2}, \tag{1.246}$$

$$X = \begin{pmatrix} AU & -\bar{B}V \\ CV & \bar{D}U \end{pmatrix};\ U = U(\theta);\ V = V(\theta),$$

$$A = A(r);\ B = B(r);\ C = C(r);\ D = D(r), \tag{1.247}$$

where U and V are functions with real value, A, B, C and D are functions with complex value. Each of these functions are calculated in Chapter C. We then get:

$$\phi = \sqrt{2} \begin{pmatrix} \eta_1 & \bar{\xi}_2 \\ \eta_2 & \bar{\xi}_1 \end{pmatrix};\ \underline{c} := \cos\frac{\theta}{2};\ \ \underline{s} := \sin\frac{\theta}{2}, \tag{1.248}$$

$$\eta_1 = \frac{A\underline{c}U - C\underline{s}V}{r\sqrt{2\sin\theta}} e^{i(\varphi - E\mathrm{x}^0)};\ \ \bar{\xi}_1 = \frac{\bar{D}\underline{c}U - \bar{B}\underline{s}V}{r\sqrt{2\sin\theta}} e^{-i(\varphi - E\mathrm{x}^0)},$$

$$\eta_2 = \frac{A\underline{s}U + C\underline{c}V}{r\sqrt{2\sin\theta}} e^{i(2\varphi - E\mathrm{x}^0)};\ -\bar{\xi}_2 = -\frac{\bar{D}\underline{s}U + \bar{B}\underline{c}V}{r\sqrt{2\sin\theta}} e^{-i(2\varphi - E\mathrm{x}^0)}, \tag{1.249}$$

Clearly it appears that the phase-wave which was the starting idea of L. de Broglie is, for the electron, a wave with **twice two phases**: a part of the wave spins $\frac{-1}{2}+\frac{\lambda}{2}=1$ turn when another part spins $\frac{1}{2}+\frac{\lambda}{2}=2$ turns. One part spins in a direction while the other part spins in the contrary direction. Similarly, for a state with spin $7/2$, we get two components spining 3 turns, in contrary directions, while the two other components spin exactly 4 turns, in contrary direction. That gives indeed a mean of $7/2$ turns, but none rotation has a half-turn. So with the spin $7/2$ there are four phases spining respectivelevy 3, -3, 4 and -4 turns. When the sign of the magnetic quantum number is changed which means for the spin $-7/2$, all numbers change sign: -3, 3, -4 and 4 turns. To say that the spin of the electron can only take two values, $+1/2$ or $-1/2$, is certainly false, since λ may be $7/2$ or $-9/2$. But there are truly only two possibilities: the quantum number λ may be only positive or negative!

Pauli's explanation of the $2\mathbf{n}^2$ states from the $\mathbf{n}^2$ coming from the Schrödinger equation, plus the two values of the spin (up – down) is now only a children's story. Luckily for Pauli and chemistry, and unfortunately for the understanding of physics, this children's story is still popular. It is even sufficient, due to the parting of all electron states into two sets exactly symmetric, according to the sign of λ (see C.1.2).

The study of the solutions of the improved wave equation hence shows that a family of solutions exists (that is absolutely not trivial for a nonlinear wave equation). These solutions are very close to the solutions of the linear Dirac equation previously obtained [13], such as the Yvon-Takabayasi angle is everywhere defined and small. More, if ϕ_1 and ϕ_2 are two solutions in that family, then $e^{ia}\phi_1$ and $e^{ib}\phi_2$, a and b constant, are also solutions of the improved equation, since it is both homogeneous and globally invariant under the chiral gauge (multiplication by i). But the sum $\phi_1+\phi_2$ has no reason to be a solution of the wave equation, since it is non additive. The solutions labeled by the quantum numbers $\mathbf{n}$, $|\kappa|$, λ, n and by the two possible cancellation, either a or b, thus give the only possible solutions for the stationary states of the hydrogen atom, and of the other atom with an alone electron. That explain simply why an electron in a hydrogen atom usually lie in one of those labeled states, never in a linear combination of such states. That is experimentally well satisfied, resulting into well defined spectral lines. And it is a reality that the theory based on Hilbert spaces and linear operators on these Hilbert spaces never explained.

Moreover, in the case of the linear Dirac equation, the set of all solutions for a particular state defined by a precise set of quantum numbers $\mathbf{n},\kappa$,n, λ, forms if $n>0$ a 2-dimensional Hilbert space. Experimentally we know that these states are however non degenerate. Now, if we suppose that the wave has value in the Lie group Cl_3^*, for each set $\mathbf{n},\kappa$,n, λ only one function may be associated. The improved equation thus seems nearer physical reality than its linear approximation, which is the Dirac equation. The improved equation is then, as far as we know, the only nonlinear wave equation such that the quantized energy levels exist, without degeneracy, with exactly the true energy levels, with exactly all properties needed to explain the different electron states.

1.6.1 Lamb effect

It is well known that the Dirac equation was perfect for the electron, yet nevertheless two properties were not obtained: the anomalous magnetic moment and the Lamb shift. The Lamb shift is about a small difference in the electron energy observed between levels that the Dirac equation predicts as strictly equal, like the $\mathbf{2s}_{1/2}$ and $\mathbf{2p}_{1/2}$ energy levels. The shift is maximal for the $\mathbf{1s}_{1/2}$ states, a shift between the energy level calculated from the Dirac equation and the level calculated from quantum fields theory [76]. The Dirac equation did not allow a difference between the energy levels of the four $\mathbf{2s}_{1/2}$ states, which were moreover presented as two $\mathbf{2s}_{1/2}$ states and two $\mathbf{2p}_{1/2}$ states. The detailed calculation in Chapter C allows us to question the calculation made from quantum fields theory. It accounts for several effects, adding and subtracting the contributions: void polarization, size of the proton, and also the behavior at the origin of the different radial functions. But when we effectively compute the normalization of the wave (a calculation that the Darwin's resolution is not able to end), none difference exists for the behavior at the origin between the different states. Since the Lamb effect has the same order of magnitude that the hyperfine structure of the hydrogen spectrum, it will be necessary to exactly calculate the electronic states by accounting for the proton wave, and by accounting for the relative movement of electron and proton to hope a correct calculation of the Lamb effect. This movement being a rotation, it will be also necessary to integrate to the calculation the contribution coming from relativistic gravitation.

However, with the improved equation, a difference appears between the four states when b_0 or a_0 cancels. In the first case the normalization gives:

$$|a_0| = \sqrt{\frac{l_P^3 m_a (2\Lambda m)^{2s+1}}{2\pi \mathbf{l} S \Gamma(2s+1)}}, \tag{1.250}$$

while in the second case the normalization gives:

$$|b_0| = \sqrt{\frac{l_P^3 m_a (2\Lambda m)^{2s+1}}{2\pi \mathbf{r} S \Gamma(2s+1)}}, \tag{1.251}$$

It is then possible that all or a part of the shift can come from the difference between $\mathbf{l}$ and $\mathbf{r}$.

1.6.2 About probability

The wave equation of the electron is a wave equation for a single object: the Pauli exclusion principle forbids to place more than one electron on a single electron state. When the physical theory needs to calculate the probability of emission or absorption of a photon, as a function of time, most often not an alone electron is considered, but a great number of electrons, each with its own wave. So there is *a priori* no link between those probabilities functions of time and the density of probability $\mathbf{J}^0/\hbar c$ (function of space). If a link exists between these probabilities of a different nature, that link must be proved from a theoretical standpoint, and experimentally validated by appropriate statistics. Nothing, in the previous paragraphs, necessitates the existence of an essential probabilism, different from the probabilities coming from chaotic phenomenons. Nevertheless a quantity always exists, that is the ratio between the local density of energy divided by the total energy, which is a positive real number such as the total sum is always 1. Hence this ratio may always be called probability, in the sense of the mathematical measure theory.

1.7 Three generations

When, in the 1930s, physicists understood that other particles exist besides electrons, photons and protons, one of the first newly discovered particles was the muon (1936), a discovery completely unexpected. The muon look just like an electron: same charge, same spin, same properties in electromagnetic and weak interactions, same lack of strong interactions. But the proper mass of the muon is much greater than that of the electron: $105,6583755(23)\text{MeV}/c^2$. Later physicists understood that fundamental particles could be put into three similar "generations". And the particle corresponding to the electron in the third generation, the tau (or tauon) was discovered only forty years later. It is still heavier: $1776,86(12)\text{MeV}/c^2$. Muons and tau are unstable particles, with a half-life $2.1969811(22)\times 10^{-6}s$ for the muon, and only $2.903(5)\times 10^{-13}s$ for the tau, thus much more difficult to study.

It is thus necessary to explain why these three generations, why such differences on the proper masses, why only the electron is stable. Since all three have very similar properties, it is supposed that they follow a same kind of wave equation, where only the mass term changes. An empiric formula was found by Yoshio Koide in 1981 [85] :

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}, \tag{1.252}$$

where the mass m_e of the electron $(0,51099895069(16)\text{MeV/c}^2)$ and the mass of the muon $(105,6583755(23)\text{MeV}/c^2)$ are known with a much higher precision than the tau mass. With the experimental precise values mentioned above, we get not exactly $\frac{2}{3}$. But the last experimental results on the tau proper mass are upper the first published values, and they are perfectly compatible with the exact value $\frac{2}{3}$. So we shall consider exact the

Koide formula.

Koide himself, several physicists working on the subject, among them O. Rousselle (private communication), sought to justify this equality from a symmetry breaking by the Higgs boson. Rousselle's relation reads:

$$m_k = \frac{2}{\sqrt{6}} y_F \nu \Big(\frac{1}{\sqrt{2}} + \sin(\theta_k) \Big)^2; \quad \theta_k = \theta_F + \frac{2\pi(3-k)}{3}, \ k = 1, 2, 3.$$
$$\frac{\sqrt{m_k}}{\sqrt{K}} = \frac{1}{\sqrt{2}} + \sin(\theta_k); \quad K = \frac{2 y_F \nu}{\sqrt{6}}, \tag{1.253}$$

with as approximate values:

$$\theta_e = -\frac{\pi}{4} + \epsilon = \theta_F - \frac{2\pi}{3} \approx -42.73^\circ > -45^\circ,$$
$$\theta_\mu = \theta_F + \frac{2\pi}{3} \approx 197.27^\circ; \quad \theta_\tau = \theta_F \approx 77.27^\circ. \tag{1.254}$$

The $\frac{2\pi}{3}$ angles are those that draw the numbers 1, $j = e^{i\frac{2\pi}{3}}$ and j^2 (the three cubic roots of 1) in the complex plane. That induces a comparison to the similar drawing formed by the roots of the Lie algebra of $SU(3)$, drawing at the origin of the quark hypothesis. We hence study:

$$a_n^k := c_n^k + i s_n^k = e^{i(k\theta_e - \frac{2n\pi}{3})}; \quad n = 0, 1, 2; \quad \theta_e = -\frac{\pi}{4} + \epsilon, \tag{1.255}$$
$$\sqrt{m_e} = \sqrt{K}(\delta + s_0^1); \ \sqrt{m_\mu} = \sqrt{K}(\delta + s_1^1); \ \sqrt{m_\tau} = \sqrt{K}(\delta + s_2^1), \tag{1.256}$$

where δ and ϵ are real number that we can calculate rather precisely the value, by using the mass value of the electron and the muon. With

$$r_j := \delta + s_j^1, \ j = 0, 1, 2. \tag{1.257}$$
$$a_0 = c_0^1 + i s_0^1 = e^{i\theta_e} = e^{i(\epsilon - \frac{\pi}{4})}, \tag{1.258}$$
$$a_1 = c_1^1 + i s_1^1 = e^{i\theta_\mu} = e^{i(\theta_e - \frac{2\pi}{3})} = j^2 a_0, \tag{1.259}$$
$$a_2 = c_2^1 + i s_2^1 = e^{i\theta_\tau} = e^{i(\theta_e + \frac{2\pi}{3})} = j a_0. \tag{1.260}$$

We then have:

$$r_0 = \frac{\sqrt{m_e}}{\sqrt{K}} = \delta + \sin(\theta_e), \tag{1.261}$$
$$r_1 = \frac{\sqrt{m_\mu}}{\sqrt{K}} = \delta + \sin(\theta_\mu) = \delta - \frac{1}{2}\sin(\theta_e) - \frac{\sqrt{3}}{2}\cos(\theta_e), \tag{1.262}$$
$$r_2 = \frac{\sqrt{m_\tau}}{\sqrt{K}} = \delta + \sin(\theta_\tau) = \delta - \frac{1}{2}\sin(\theta_e) + \frac{\sqrt{3}}{2}\cos(\theta_e), \tag{1.263}$$
$$\cos\theta_e = \frac{r_2 - r_1}{\sqrt{3}}. \tag{1.264}$$

And since $1 + j + j^2 = 0$ and $j^3 = 1$:

$$\sin(\theta_e) + \sin(\theta_\mu) + \sin(\theta_\tau) = 0, \tag{1.265}$$

$$c_3^k = c_0^k; \quad s_3^k = s_0^k; \quad c_0^k + c_1^k + c_2^k = 0; \quad s_0^k + s_1^k + s_2^k = 0, \tag{1.266}$$

$$3\delta = r_0 + r_1 + r_2 = \frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{K}} \tag{1.267}$$

That gives:

$$r_2 - r_1 = \sqrt{3}\cos\theta_e; \quad r_1 + r_2 = 2\delta - \sin\theta_e \tag{1.268}$$

$$= \frac{2}{3}(r_0 + r_1 + r_2) - \sin\theta_e; \quad \sin\theta_e = \frac{2}{3}r_0 - \frac{1}{3}(r_1 + r_2),$$

$$\tan\theta_e = \frac{\frac{2}{3}r_0 - \frac{1}{3}(r_1 + r_2)}{\frac{r_2 - r_1}{\sqrt{3}}} = \sqrt{3}\frac{\frac{2}{3}\sqrt{m_e} - \frac{1}{3}(\sqrt{m_\mu} + \sqrt{m_\tau})}{\sqrt{m_\tau} - \sqrt{m_\mu}} \tag{1.269}$$

$$\theta_e = \text{Arctan}\Big[\frac{2\sqrt{m_e} - \sqrt{m_\mu} - \sqrt{m_\tau}}{\sqrt{3}(\sqrt{m_\tau} - \sqrt{m_\mu})}\Big]. \tag{1.270}$$

From the identity $1 = \cos^2\theta_e + \sin^2\theta_e$ we then get:

$$1 = \Big(\frac{r_2 - r_1}{\sqrt{3}}\Big)^2 + \Big(\frac{2}{3}r_0 - \frac{1}{3}(r_1 + r_2)\Big)^2$$

$$= \frac{4}{9}[r_0^2 + r_1^2 + r_2^2 - (r_0 r_1 + r_1 r_2 + r_2 r_0)], \tag{1.271}$$

$$\frac{9}{4} = r_0^2 + r_1^2 + r_2^2 - (r_0 r_1 + r_1 r_2 + r_2 r_0)$$

$$= (r_0 + r_1 + r_2)^2 - 3(r_0 r_1 + r_1 r_2 + r_2 r_0)$$

$$= 9\delta^2 - 3(r_0 r_1 + r_1 r_2 + r_2 r_0), \tag{1.272}$$

$$r_0 r_1 + r_1 r_2 + r_2 r_0 = 3\delta^2 - \frac{3}{4}, \tag{1.273}$$

$$9\delta^2 = (r_0 + r_1 + r_2)^2 = r_0^2 + r_1^2 + r_2^2 + 2(r_0 r_1 + r_1 r_2 + r_2 r_0)$$

$$= r_0^2 + r_1^2 + r_2^2 + 6\delta^2 - \frac{3}{2}, \tag{1.274}$$

We hence have:

$$r_0^2 + r_1^2 + r_2^2 = 3\Big(\delta^2 + \frac{1}{2}\Big), \tag{1.275}$$

$$\frac{r_0^2 + r_1^2 + r_2^2}{(r_0 + r_1 + r_2)^2} = 3\frac{\delta^2 + \frac{1}{2}}{9\delta^2} = \frac{1 + \frac{1}{2\delta^2}}{3}, \tag{1.276}$$

$$\frac{r_0^2 + r_1^2 + r_2^2}{(r_0 + r_1 + r_2)^2} = \frac{2}{3} \Leftrightarrow 1 + \frac{1}{2\delta^2} = 2 \Leftrightarrow \delta = \frac{1}{\sqrt{2}}. \tag{1.277}$$

To put exactly $\frac{2}{3}$ in the Koide's formula is thus equivalent to put the value $\frac{1}{\sqrt{2}}$ in the formula (1.253). We then deduce the values of m_τ and of the ϵ

angle:

$$m_\tau \approx 1776,969028(39)\mathrm{MeV}/c^2; \quad \epsilon \approx 2,23761847(51)^\circ \tag{1.278}$$

The value obtained for the tau mass is perfectly in adequacy with the last measures of this proper mass. The approximate value of the θ_e angle is also very precise. As for the value of the Koide's ratio, so near $\frac{2}{3}$ that this value is considered exact, the value of ϵ satisfies:

$$\theta_e = -\frac{\pi}{4} + \epsilon = -\frac{\pi}{6} - 0,22222204711 \approx -\frac{\pi}{6} - \frac{2}{9}. \tag{1.279}$$

Now look at what comes when the fractional numbers $\frac{2}{3}$ and $-\frac{2}{9}$ are exact values.[30] We then have:

$$\theta_e = -\frac{\pi}{4} + \epsilon = \frac{\pi}{2} - \frac{2}{9} - \frac{2\pi}{3}, \tag{1.280}$$

$$\epsilon = \frac{\pi}{4} + \frac{\pi}{2} - \frac{2}{9} - \frac{2\pi}{3} = \frac{\pi}{12} - \frac{2}{9}; \quad \theta_e = -\frac{\pi}{6} - \frac{2}{9}. \tag{1.281}$$

Thus we get:

$$\sqrt{m_k} = \sqrt{K}\Big[\frac{1}{\sqrt{2}} + \sin\Big(-\frac{\pi}{6} - \frac{2}{9} - k\frac{2\pi}{3}\Big)\Big], \; k = 0, 1, 2, \tag{1.282}$$

$$m_e = m_0; \; m_\mu = m_1; \; m_\tau = m_2, \tag{1.283}$$

$$r_0 r_1 + r_1 r_2 + r_2 r_0 = 3\delta^2 - \frac{3}{4} = \frac{3}{4}, \tag{1.284}$$

$$\frac{m_e + m_\mu + m_\tau}{K} = r_1^2 + r_2^2 + r_3^2 = 3; \quad K = \frac{m_e + m_\mu + m_\tau}{3}, \tag{1.285}$$

$$m_\tau = 3K - m_e - m_\mu; \quad 3\frac{\sqrt{K}}{\sqrt{2}} = \sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau} \tag{1.286}$$

Hence we obtain:

$$\sin\theta_e = \sin\Big(-\frac{\pi}{6} - \frac{2}{9}\Big); \quad r_0 = \frac{\sqrt{m_e}}{\sqrt{K}} = \frac{1}{\sqrt{2}} + \sin\theta_e, \tag{1.287}$$

$$\sqrt{K} = \frac{\sqrt{m_e}}{\frac{1}{\sqrt{2}} - \sin\left(\frac{2}{9} + \frac{\pi}{6}\right)} \approx 25,05419727 \tag{1.288}$$

$$\sqrt{m_\mu} = \sqrt{K}\Big[\frac{1}{\sqrt{2}} + \frac{1}{2}\sin\Big(\frac{2}{9} + \frac{\pi}{6}\Big) - \frac{\sqrt{3}}{2}\cos\Big(\frac{2}{9} + \frac{\pi}{6}\Big)\Big], \tag{1.289}$$

$$m_\mu \approx 105,6594144827 \tag{1.290}$$

$$\sqrt{m_\tau} = \sqrt{K}\Big[\frac{1}{\sqrt{2}} + \frac{1}{2}\sin\Big(\frac{2}{9} + \frac{\pi}{6}\Big) + \frac{\sqrt{3}}{2}\cos\Big(\frac{2}{9} + \frac{\pi}{6}\Big)\Big]. \tag{1.291}$$

$$m_\tau \approx 1776,984971 \tag{1.292}$$

30. That has then as first advantage the reduction by two units instead one of the number of free parameters of the standard model.

The value obtained for the proper mass of the muon happens to be completely outside the confidence interval accepted today. But this value is completely in accordance with what was published a half-century ago (105.659 MeV/c^2). Is the muon mass known with an experimental accuracy directly coming from new experiments, or is the accuracy coming from an improvement of the theory interpreting these experiments ? If the experimental accuracy is truly better than fifty years ago, it will be necessary to consider the 2/9 value as a first approximation. If the accuracy on the proper mass comes only from a better calculation of this value, It will be interessant to know why the 2/3 and 2/9 ratios give, with an extraordinary precision, these mass ratios.

1.8 The invariant – dimension (invdim)

The *invariant – dimension* of a physical quantity is defined as the power of $\sqrt{r}$ where $r = |\det(M)|$ in the transformation of that quantity by the similitude R generated by any dilator M in Cl_3^*. It is abbreviated into the acronym **invdim**.

1. Since ϕ becomes $\phi' = M\phi$ and $M = \sqrt{r}e^{i\theta}P$, where P belongs to $SL(2,\mathbb{C})$, the invdim of ϕ is 1.
The invdim of $\widehat{\phi}$, $\widetilde{\phi} = \phi^\dagger$ and $\overline{\phi}$ is also 1. The invdim of ϕ^{-1} is -1.

2. Any contravariant vector such as x or J = D$_0$ = $\phi\phi^\dagger$, which transforms into J$' = M$J$M^\dagger$, has an invdim equal to 2.

3. Any covariant vector as $\nabla = \overline{M}\nabla'\widehat{M}$ has an invdim equal to -2.

4. Since we have $m = m'r$ and $\rho' = r\rho$, we can say that ρ has an invdim 2 and m has an invdim -2.

5. Since time and space change in the same manner, any velocity has a null invdim.

6. Since an acceleration is the derivative of a velocity its invdim is -1.

7. The electromagnetic potential A, in the second-order equation, is linked to the J current in a scalar product. This vector must be, like J, a contravariant vector. We may also use the fact that the electromagnetic potential is linked to its sources, which are the particles having an electric charge (or other charges: magnetic, strong and so on). Hence A must have an invdim +2 and must satisfy

$$A' = MAM^\dagger. \tag{1.293}$$

8. So that the gauge invariance may be compatible with relativistic invariance, qA must transform like a covariant vector while A is contravariant.

We thus have:

$$qA = \overline{M}q'A'\widehat{M} = \overline{M}q'MAM^{\dagger}\widehat{M} = q're^{i\theta}Are^{-i\theta} = r^2q'A,$$
$$q = q'r^2;\ q' = qr^{-2}. \tag{1.294}$$

The invdim of q is thus -4. We may remark that m and q do not have the same invdim. This is an important difference between mass and charge which have the same status in relativistic invariance but not in the extended invariance under Cl_3^*. That is why a synthesis between gravitation and gauge theories was so hard until now.

9. Next, we have:

$$q = \frac{e}{\hbar c}\ ;\ qe = \frac{e^2}{\hbar c} = \alpha = q'e'\ ;\ \ qe = q'r^2e = q'e'. \tag{1.295}$$
$$e' = r^2e. \tag{1.296}$$

An electric (or magnetic) charge thus has an invdim +4, which is also the invdim of a surface.

10. We thus have:

$$\alpha = \frac{e^2}{\hbar c} = \frac{e'^2}{\hbar' c} = \frac{r^4e^2}{\hbar' c}\ ;\ \ e^2\hbar' c = \hbar c r^4 e^2. \tag{1.297}$$
$$\hbar' = r^4\hbar. \tag{1.298}$$

The "Planck constant" is thus a variable when the ratio of similitude is not reduced to 1 and the invdim of the action is 8 – this is the invdim of a space-time volume (this is convenient for relativistic thermodynamics). We also remark that it is not consistent to give to J^0 a status of probability density, but it is consistent for $J^0/\hbar c$ which has the expected invdim -6.

11. For a proper mass m_0 we have

$$\frac{m_0c}{\hbar} = m = rm' = r\frac{m_0'c}{\hbar'} = r\frac{m_0'c}{r^4\hbar} = \frac{m_0'c}{r^3\hbar}. \tag{1.299}$$

And this gives

$$m_0' = r^3m_0. \tag{1.300}$$

A proper mass thus has an invdim +6; this is the invdim of a volume. Both a charge or a proper mass are no longer invariant, and this requires a change in our habits. Among the bad habits needing a quick change is taking $\hbar = 1$. This is nonsense here since $\hbar$ is variable with r, as shown in (1.298). All these variations do not contradict relativistic invariance in the restricted sense, which is the particular case where $r = 1$: the concept of invdim is not pertinent in this case [31] .

31. The variation of r changing m_0 and $\hbar$ has no consequence on measurements of

12. Pressure, with dimension $\mathrm{ML}^{-1}\mathrm{T}^{-2}$ thus has a null invdim.

13. We now consider the classical part $F = \nabla\widehat{A}$ of the electromagnetic field. We have:

$$F = \overline{M}\nabla'\widehat{M}\widehat{A},$$
$$MFM^{-1} = M\overline{M}\nabla'\widehat{M}\widehat{A}M^{-1} = re^{i\theta}\nabla'\widehat{M}\widehat{A}M^{-1} \tag{1.301}$$
$$= \nabla'\widehat{M}\widehat{A}re^{i\theta}M^{-1} = \nabla'\widehat{M}\widehat{A}\overline{M}MM^{-1} = \nabla'\widehat{A}' = F'. \tag{1.302}$$

Since M brings $\sqrt{r}$ and since M^{-1} brings a $1/\sqrt{r}$ factor, the electromagnetic field has an invdim 0 (and this is necessarily the same for the other gauge fields). All these results are consistent with the laws of mechanics and of electromagnetism: mass, energy and momentum have the same invdim +6. A mechanical or electromagnetic force has an invdim 4: this is consistent with the force exerted on a charge since the invdim of a charge is 4 and the invdim of a field is 0.

The fact that the invdim of gauge fields is null, and the fact that they transform following the $F' = MFM^{-1}$ law is very important, as this implies that an F_1F_2 product of two such fields again satisfies the same rule:

$$F_1'F_2' = MF_1M^{-1}MF_2M^{-1} = MF_1F_2M^{-1}. \tag{1.303}$$

This is why products of photon fields may be added together and may follow Bose-Einstein statistics (actually found in the thesis of L. de Broglie [55]). For the fields of second quantization which act as operators on the wave itself, this also allows us the definition of creation and annihilation operators.

1.9 Invariant space-time

When we presented this double space-time in the book with the same name [21], we implicitly worked with ρ on an equal footing with r. This is natural since $\rho' = r\rho$. More generally there is no difference of structure between a dilator M defining the similitude R, and $\phi(\mathrm{x})$, which are both complex 2×2 matrices, which means two elements of the Cl_3 algebra. More precisely ϕ is a function of space-time with value in Cl_3. Therefore ϕ, like

mass, which are always measurements of the ratio between two masses. When physics went from classical mechanics into relativistic mechanics, where masses are no longer invariant, there was no need to change the mass unit: any measurement of mass is obtained at zero velocity in the laboratory. It is the same here, because any proper mass and any action varies with the same ratio (r^3 for a proper mass, r^4 for an action), in the laboratory at the time when the measurement is made. The variation of $\hbar$, which remains relativistically invariant, is thus perfectly compatible with the replacement of the standard mass by a standard action, more accurate and more stable than the previous International Prototype Kilogram (IPK).

M, allows us to define a similitude D_{x}, with ratio $\rho = \rho(\mathrm{x})$, by:

$$D_{\mathrm{x}} : X \mapsto \mathrm{x} = \phi X \phi^{\dagger}. \tag{1.304}$$

And the components D^{ν}_{μ} of the four vectors D_{μ} are the terms of the matrix of this similitude D_{x} because we have:

$$\mathrm{x} = \mathrm{x}^{\mu}\sigma_{\mu} = \phi X^{\nu}\sigma_{\nu}\phi^{\dagger} = X^{\nu}\phi\sigma_{\nu}\phi^{\dagger} = X^{\nu}\mathrm{D}_{\nu} = X^{\nu}\mathrm{D}^{\mu}_{\nu}\sigma_{\mu}\ ;\quad \mathrm{x}^{\mu} = \mathrm{D}^{\mu}_{\nu}X^{\nu}. \tag{1.305}$$

There is no difference between the $M'M$ product giving the composition $R' \circ R$ of the similitudes, and the product $M\phi$ which gives the transformation of the wave under a similitude, and which also induces a composition of the similitudes $\mathrm{D}'_{\mathrm{x}'} = R \circ \mathrm{D}_{\mathrm{x}}$, since:

$$\mathrm{x}' = M\mathrm{x}M^{\dagger} = M\phi X\phi^{\dagger}M^{\dagger} = (M\phi)X(M\phi)^{\dagger} = \phi' X \phi'^{\dagger} \tag{1.306}$$

This implies that the X introduced in (1.304) does not change when seen by the observer at x or by the observer at x$'$. It is independent of the observer. We may also remark that the invdim of X is null, because the invdim of x is 2 while the invdim of ϕ and of $\phi^{\dagger}$ are 1. We may then name the set of X the **invariant space-time**.

1.10 Energy–momentum, Lorentz force

The calculation of the Lorentz force may be done in space-time algebra; this needs the use of the method of calculation explained in [83] and [89]. Since the wave of the electron only has value in the even subalgebra and since this even subalgebra is isomorphic to the Cl_3 algebra, we are able to calculate the Lorentz force easier using Cl_3. First question: what energy–momentum density may be attached to the Dirac wave? Quantum field theory derives this density from the Lagrangian density, and the invariance of the Lagrangian density under space-time translation allows us to define a tensor density of energy–momentum from Noether's theorem. But the Lagrangian density is also a problem: several textbooks, old [2][112] or new like Wikipedia, give this Lagrangian density as:

$$\mathcal{L} = i\overline{\psi}\partial\psi - m\overline{\psi}\psi. \tag{1.307}$$

Besides the error in sign, because they wrongly indicate a negative energy density, these authors are apparently unaware of the complex character of this Lagrangian density. Some other authors are more precise [102] and give the Lagrangian density as:

$$\mathcal{L} = \Re(-i\overline{\psi}\partial\psi) + m\overline{\psi}\psi. \tag{1.308}$$

that is better, even if the term of electromagnetic interaction is missing, hence that is about an electron without its electric charge! We must also recall that passing from this form to the improved equation replaces $\overline{\psi}\psi = \rho\cos(\beta)$ with ρ and needs to account for the two proper masses. It is interesting to see why the absence of rigor in this part of the theory has no impact on the studies that came after and hence remains unnoticed. We have obtained this Lagrangian density in (1.135) as the real part of the wave (because the real field is conventionally included in any real Clifford algebras). Since we work in the Pauli algebra, this is the real part (in the complex field) of the trace of the matrix. This trace also has an imaginary part which gives the conservation of the probability current ($\partial_\mu \mathrm{J}^\mu = 0$). This explains why the Lagrangian density of the seemingly complex form (1.308) is nevertheless correct, the imaginary part being automatically null. Moreover the formula is shorter, hence more convenient for an introduction. The physical reason explaining why the two formulas for $\mathcal{L}$ give the same result is the invariance under the electric gauge, associated with the conservation of the probability current by Noether's theorem. We may then begin all calculations from (1.308), using complex variables to calculate densities which will nevertheless all be real as a result of the electric gauge invariance.

We saw in 1.5.5 that the Lagrangian density is sum of a right part and a left part. Therefore the energy–momentum tensor T linked to that Lagrangian density is also the sum of a left part depending only on the left wave η and of a right part depending only on the right wave ξ (but with together, in each part of the sum, the vector v and the angle β which depend on both ξ and η waves). We will again see in Chapter 2 this important partition between right and left waves, a partition which is invariant under Cl_3^*, thus relativistic invariant. The notion that these left and right waves are the fundamental fields was obtained and used by G. Lochak [90]–[96]. That was his starting point for his theory of the magnetic monopole. This partition is actually important for all tensor densities that we obtain from the spinors. The existence of an energy–momentum tensor T_R for the right waves and another tensor T_L for the left waves implies the existence of **two tensors of energy–momentum**, the tensor $T = T_R + T_L$ and the tensor $V = T_L - T_R$ noted by O. Costa de Beauregard [53] as not yet classically interpreted. This tensor V is simply the difference between the right and left tensors. [32] . For the Dirac equation we have:

$$T^\mu_\nu = \frac{m_a}{k\mathbf{l}} T^\mu_{L\nu} + \frac{m_a}{k\mathbf{r}} T^\mu_{R\nu};\ V^\mu_\nu = \frac{m_a}{k\mathbf{l}} T^\mu_{L\nu} - \frac{m_a}{k\mathbf{r}} T^\mu_{R\nu}, \tag{1.309}$$

32. To derive the dynamics of the electron Hestenes started from a different tensor [79] containing only the differential terms. Yet he added the electromagnetic part, his tensor is thus identical to ours. Hestenes' calculation is complicated by failing to distinguish the right and left parts of the wave. It is only with a convenient choice of the γ_μ matrices that we can easily see the tensors as sums of a left and a right part. But Hestenes, since he uses only the space-time algebra $Cl_{1,3}$, strict sub-algebra of the algebra $M_4(\mathbb{C})$ of Dirac matrices, never uses the facilities of matrix calculus.

$$\begin{aligned} T^\mu_{L\nu} &= \Re\Big[\eta^{1\dagger}\sigma^\mu\Big(-i\partial_\nu + qA_\nu + m_g e^{i\beta}\mathrm{v}_\nu\Big)\eta^1\Big] = \Re\big(\eta^{1\dagger}\sigma^\mu d_\nu \eta^1\big), \\ d^L_\nu &= -i\partial_\nu + qA_\nu + m_g\cos\beta\mathrm{v}_\nu, \qquad (1.310) \\ T^\mu_{R\nu} &= \Re\Big[\xi^{1\dagger}\widehat{\sigma}^\mu\Big(-i\partial_\nu + qA_\nu + m_g e^{-i\beta}\mathrm{v}_\nu\Big)\xi^1\Big] = \Re(\xi^{1\dagger}\widehat{\sigma}^\mu d^R_\nu \xi^1). \end{aligned}$$

With the improved equation we always have (1.309). The T_L and T_R tensors simplify with:

$$T^\mu_{L\nu} = \Re\Big[-i\eta^{1\dagger}\sigma^\mu\Big(\partial_\nu + iqA_\nu + im_g\mathrm{v}_\nu\Big)\eta^1\Big], \qquad (1.311)$$

$$T^\mu_{R\nu} = \Re\Big[-i\xi^{1\dagger}\widehat{\sigma}^\mu\Big(\partial_\nu + iqA_\nu + im_g\mathrm{v}_\nu\Big)\xi^1\Big]. \qquad (1.312)$$

In space-time algebra, the energy–momentum tensor $T(\mathrm{u}) = T(\mathrm{u}, \mathrm{x})$ is interpreted by Hestenes [79] as the flux of energy–momentum through a hypersurface with normal vector u at the space-time point x. It is a vectorial function of a vectorial variable [33]:

$$T(\mathrm{u}) = T(\mathrm{u}_\mu\sigma^\mu) = \mathrm{u}_\mu T(\sigma^\mu). \qquad (1.313)$$

Hence this tensor is completely defined by the four vectors:

$$T^\mu = T(\sigma^\mu), \qquad (1.314)$$

which satisfy:

$$T^\mu = T^\mu_\nu\sigma^\nu \ ; \ \ T^\mu_\nu = T^\mu\cdot\sigma_\nu. \qquad (1.315)$$

We then get with the improved equation:

$$\begin{aligned} \partial_\mu T^\mu &= (\partial_\mu T^\mu_\nu)\sigma^\nu, \qquad (1.316) \\ \partial_\mu T^\mu_{L\nu} &= \frac{i}{2}\partial_\mu[-\eta^{1\dagger}\sigma^\mu\partial_\nu\eta^1 + (\partial_\nu\eta^{1\dagger})\sigma^\mu\eta^1] + \partial_\mu[(qA_\nu + m_g\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L], \\ \partial_\mu T^\mu_{R\nu} &= \frac{i}{2}\partial_\mu[-\xi^{1\dagger}\widehat{\sigma}^\mu\partial_\nu\xi^1 + (\partial_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1] + \partial_\mu[(qA_\nu + m_g\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R], \end{aligned}$$

where partial derivatives commute, and the **J** current is conservative. We then have:

$$\begin{aligned} &\partial_\mu[(qA_\nu + m_g\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L] = (q\partial_\mu A_\nu + m_g\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L + (qA_\nu + m_g\mathrm{v}_\nu)\partial_\mu\mathrm{D}^{1\mu}_L \\ &= (q\partial_\mu A_\nu + m_g\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L \qquad (1.317) \\ &\partial_\mu[(qA_\nu + m_g\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R] = (q\partial_\mu A_\nu + m_g\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R + (qA_\nu + m_g\mathrm{v}_\nu)\partial_\mu\mathrm{D}^{1\mu}_R \\ &= (q\partial_\mu A_\nu + m_g\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R \qquad (1.318) \end{aligned}$$

33. For GR this is an important point that comes from quantum physics: vectors are the only useful tensors.

That gives:

$$\partial_\mu \mathrm{T}^\mu_{L\nu} = \frac{i}{2}[-(\nabla\eta^1)^\dagger\partial_\nu\eta^1 - \eta^{1\dagger}\partial_\nu(\nabla\eta^1) + \partial_\nu(\nabla\eta^1)^\dagger\eta^1 + \partial_\nu\eta^{1\dagger}\nabla\eta^1] + (q\partial_\mu A_\nu + m_g\partial_\mu \mathrm{v}_\nu)\mathrm{D}^{1\mu}_L, \tag{1.319}$$

$$\partial_\mu \mathrm{T}^\mu_{R\nu} = \frac{i}{2}[-(\widehat{\nabla}\xi^1)^\dagger\partial_\nu\xi^1 - \eta^{1\dagger}\partial_\nu(\widehat{\nabla}\xi^1) + \partial_\nu(\widehat{\nabla}\xi^1)^\dagger\xi^1 + \partial_\nu\xi^{1\dagger}\widehat{\nabla}\xi^1] + (q\partial_\mu A_\nu + m_g\partial_\mu \mathrm{v}_\nu)\mathrm{D}^{1\mu}_R. \tag{1.320}$$

Then we use the wave equations of η^1 and ξ^1 which are equivalent to the system:

$$\nabla\eta^1 = -i(qA + m_g\mathrm{v})\eta^1;\ (\nabla\eta^1)^\dagger = i\eta^{1\dagger}(qA + m_g\mathrm{v}),$$
$$\widehat{\nabla}\xi^1 = -i(q\widehat{A} + m_g\widehat{\mathrm{v}})\xi^1;\ (\widehat{\nabla}\xi^1)^\dagger = i\xi^{1\dagger}(q\widehat{A} + m_g\widehat{\mathrm{v}}). \tag{1.321}$$

We then get:

$$\partial_\mu T^\mu_{L\nu} = [q(\partial_\mu A_\nu - \partial_\nu A_\mu) + m_g(\partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu)]\mathrm{D}^{1\mu}_L,$$
$$\partial_\mu T^\mu_{R\nu} = [q(\partial_\mu A_\nu - \partial_\nu A_\mu) + m_g(\partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu)]\mathrm{D}^{1\mu}_R, \tag{1.322}$$
$$k\partial_\mu T^\mu_\nu = q(\partial_\mu A_\nu - \partial_\nu A_\mu)\Big(\frac{m_a}{\mathbf{l}}\mathrm{D}^{1\mu}_L + \frac{m_a}{\mathbf{r}}\mathrm{D}^{1\mu}_R\Big) + \frac{m_a m_g}{m}(\partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu)\underline{J}^\mu.$$

The electromagnetic field F and the gravitational field G come here with:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu;\ G_{\mu\nu} := \partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu. \tag{1.323}$$

We then get:

$$\frac{m}{m_a}\partial_\mu \mathrm{T}^\mu_\nu = qF_{\mu\nu}\mathbf{J}^\mu + \frac{m_g}{k}G_{\mu\nu}\mathrm{J}^\mu, \tag{1.324}$$

$$\mathbf{J} = \frac{m}{k\mathbf{l}}\mathrm{D}^1_L + \frac{m}{k\mathbf{r}}\mathrm{D}^1_R. \tag{1.325}$$

If $m \approx \mathbf{l} \approx \mathbf{r}$ and with the total field:

$$\mathbf{F} := F + \frac{m}{q}G, \tag{1.326}$$

we have:

$$\partial_\mu T^\mu \approx (F_{\mu\nu} + \frac{m}{q}G_{\mu\nu})q\mathrm{J}^\mu\sigma^\nu = \mathbf{F}_{\mu\nu}q\mathrm{J}^\mu\sigma^\nu. \tag{1.327}$$

We then get the Lorentz force for a space-time vector j (charge density – electric current density) on the relativistic form:

$$\mathrm{j} = q\mathrm{J} = \frac{q}{k}(\mathrm{D}^1_L + \mathrm{D}^1_R);\ \partial_\mu T^\mu = \mathbf{F}_{\mu\nu}\mathrm{j}^\mu\sigma^\nu. \tag{1.328}$$

Hence with:

$$\mathbf{F} = \vec{E} + i\vec{H}$$
$$\mathrm{j} = q\mathrm{J} = \rho_e + \vec{\mathrm{j}};\ \mathrm{f} = \mathrm{f}_0 + \vec{\mathrm{f}}, \tag{1.329}$$

where $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field, ρ_e is the electric charge density, $\vec{\mathrm{j}}$ is the density of electric current $\vec{\mathrm{f}}$ is the density of force, (1.328) is equivalent to:

$$\vec{\mathrm{f}} = \rho_e \vec{E} + \vec{\mathrm{j}} \times \vec{H};\ \mathrm{f}_0 = \vec{E} \cdot \vec{\mathrm{j}}. \tag{1.330}$$

This is obviously very important to unify the laws of physics: except the Lorentz force that we simply obtained as a consequence of the Dirac equation, linear or improved, only the gravitational field is able to yield the gravitational force as a consequence of field equations. We can even say that relativistic quantum mechanics outperforms general relativity, since this result is obtained with a quantized charge, finite, when the gravitational force is obtained as a limit case for an infinitesimal mass. This means that the Standard Model, in the fully relativistic manner used here, is at least as suitable as general relativity to obtain the motion of field sources from the evolution law of the field.

1.11 Electromagnetic field

The laws of electromagnetism, when we account for the existence of magnetic charges (magnetic monopoles) are reducible to two simple laws (a detailed calculation is in A.3.10):

$$F = \nabla \widehat{A + iB};\ \nabla \widehat{F} = \mathrm{j} - \mathrm{ik}, \tag{1.331}$$

where F is the electromagnetic field (bivector in space-time), A is the electromagnetic potential and j the density of electric current, iB the pseudo-vector magnetic-weak potential and k the density of magnetic current. A, B, j et k are space-time vectors, for instance j bring together a space vector $\vec{\mathrm{j}}$ and a time component which is the density of electric charge. When accounting for magnetic charges, the two laws of electromagnetism change only by the addition of iB to A and of $-ik$ to j. Since $\overline{F} = -F$ and $\widehat{F} = -F^\dagger$, we have at the second order:

$$\Box \widehat{A + iB} = (\widehat{\nabla} \nabla) \widehat{A + iB} = \widehat{\nabla}(\nabla \widehat{A + iB}) = \widehat{\nabla} F = \widehat{\mathrm{j}} + i\widehat{\mathrm{k}} \tag{1.332}$$

$$F = F_e + F_m;\ F_e := \nabla \widehat{A};\ F_m := \nabla(\widehat{iB});\ \Box A = \mathrm{j};\ \Box B = -\mathrm{k}. \tag{1.333}$$

Since J is the sum of the chiral currents, we study the left and right fields such as:

$$F_L := \nabla \widehat{i\mathrm{D}}^1_L;\ F_R := \nabla \widehat{i\mathrm{D}}^1_R. \tag{1.334}$$

The left field F_L satisfies :

$$\begin{aligned} F_L = \vec{E}_L + i\vec{H}_L &= (-i)(\partial_0 - \vec{\partial})(\mathrm{D}^{10}_L - \vec{\mathrm{D}}^1_L) \\ &= -i\partial_\mu \mathrm{D}^{1\mu}_L + i(\partial_0 \vec{\mathrm{D}}^1_L + \vec{\partial} \mathrm{D}^{10}_L) + \vec{\partial} \times \vec{\mathrm{D}}^1_L. \end{aligned} \tag{1.335}$$

We thus get:

$$0 = \partial_\mu \mathrm{D}_L^{1\mu} \tag{1.336}$$

$$\vec{H}_L = \partial_0 \vec{\mathrm{D}}_L^1 + \vec{\partial}\mathrm{D}_L^{10};\ \vec{E}_L = \vec{\partial} \times \vec{\mathrm{D}}_L^1. \tag{1.337}$$

The right field F_R similarly satisfies:

$$0 = \partial_\mu \mathrm{D}_R^{1\mu} \tag{1.338}$$

$$\vec{H}_R = \partial_0 \vec{\mathrm{D}}_R^1 + \vec{\partial}\mathrm{D}_R^{10};\ \vec{E}_R = \vec{\partial} \times \vec{\mathrm{D}}_R^1. \tag{1.339}$$

We read the covariant derivatives of (1.208), for the improved wave equation, as follows:

$$d_\mu = -i\partial_\mu + s_\mu;\ s_\mu := qA_\mu + m_g \mathrm{v}_\mu. \tag{1.340}$$

The wave equations of η^1 and ξ^1 are expressed in the form:

$$\sigma^\mu \partial_\mu \eta^1 = -i\sigma^\mu s_\mu \eta^1, \tag{1.341}$$

$$\widehat{\sigma}^\mu \partial_\mu \xi^1 = -i\widehat{\sigma}^\mu s_\mu \xi^1. \tag{1.342}$$

The tensor densities of energy–momentum left and right read with (1.204), for the Dirac equation as for the improved equation:

$$\begin{aligned} T^\mu_{L\nu} &= \frac{1}{2}\Big[\eta^{1\dagger}\sigma^\mu(-i\partial_\nu\eta^1 + s_\nu\eta^1) + (i\partial_\nu\eta^{1\dagger} + s_\nu\eta^{1\dagger})\sigma^\mu\eta^1\Big] \\ &= \frac{i}{2}\Big[-\eta^{1\dagger}\sigma^\mu\partial_\nu\eta^1 + (\partial_\nu\eta^{1\dagger})\sigma^\mu\eta^1\Big] + s_\nu\eta^{1\dagger}\sigma^\mu\eta^1, \end{aligned} \tag{1.343}$$

$$\begin{aligned} T^\mu_{R\nu} &= \frac{1}{2}\Big[\xi^{1\dagger}\widehat{\sigma}^\mu(-i\partial_\nu\xi^1 + s_\nu\xi^1) + (i\partial_\nu\xi^{1\dagger} + s_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1\Big] \\ &= \frac{i}{2}\Big[-\xi^{1\dagger}\widehat{\sigma}^\mu\partial_\nu\xi^1 + (\partial_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1\Big] + s_\nu\xi^{1\dagger}\widehat{\sigma}^\mu\xi^1. \end{aligned} \tag{1.344}$$

The improved equation implies for the left wave:

$$\partial_0\eta^1 + \sigma^1\partial_1\eta^1 = \sigma_2\partial_2\eta^1 + \sigma_3\partial_3\eta^1 - i(s_0 + s_1\sigma^1 + s_2\sigma^2 + s_3\sigma^3)\eta^1, \tag{1.345}$$

Multiplying on the left side by $\eta^{1\dagger}\sigma^1$ we get:

$$\begin{aligned} &\eta^{1\dagger}\sigma^1\partial_0\eta^1 + \eta^{1\dagger}\partial_1\eta^1 \\ &= i\eta^{1\dagger}\sigma^3\partial_2\eta^1 - i\eta^{1\dagger}\sigma^2\partial_3\eta^1 - i\eta^{1\dagger}(s_0\sigma^1 + s_1 - i\sigma^3 s_2 + i\sigma^2 s_3)\eta^1. \end{aligned} \tag{1.346}$$

Taking the adjoint, next adding, we get:

$$\begin{aligned} &\partial_0(\eta^{1\dagger}\sigma^1\eta^1) + \partial_1(\eta^{1\dagger}\sigma^0\eta^1) \\ &= i\eta^{1\dagger}\sigma^3\partial_2\eta^1 - i(\partial_2\eta^{1\dagger})\sigma^3\eta^1 - 2s_2\eta^{1\dagger}\sigma^3\eta^1 + 2s_3\eta^{1\dagger}\sigma^2\eta^1 \\ &- i\eta^{1\dagger}\sigma^2\partial_3\eta^1 + i(\partial_3\eta^{1\dagger})\sigma^2\eta^1. \end{aligned} \tag{1.347}$$

It must be noted that the present calculation is completely dependent on the dimension three of space. It is linked to the existence of the cross product and to the mixed product, through the dimension $1+3+3+1$ of the Cl_3 algebra, which gives $\sigma_1\sigma_2\sigma_3 = i$, etc. These equalities are nearly hidden when only the Clifford algebra of space-time is used. We also get:

$$\begin{aligned}\partial_0 \mathrm{D}^{11}_L + \partial_1 \mathrm{D}^{10}_L &= -2T^3_{L2} + 2T^2_{L3},\\ H^1_L &= 2(T^2_{L3} - T^3_{L2}).\end{aligned} \tag{1.348}$$

By circular permutation of the indices we habe:

$$\begin{aligned}H^1_L &= 2(T^2_{L3} - T^3_{L2}),\\ H^2_L &= 2(T^3_{L1} - T^1_{L3}),\\ H^3_L &= 2(T^1_{L2} - T^2_{L1}),\end{aligned} \tag{1.349}$$

Starting now of (1.346) and subtracting we get:

$$\begin{aligned}&\eta^{1\dagger}\sigma^1\partial_0\eta^1 - (\partial_0\eta^{1\dagger})\sigma^1\eta^1 + 2is_0\eta^{1\dagger}\sigma^1\eta^1\\ &+ \eta^{1\dagger}\sigma^0\partial_1\eta^1 - (\partial_1\eta^{1\dagger})\sigma^0\eta^1 + 2is_1\eta^{1\dagger}\sigma^0\eta^1\\ &= i\partial_2(\eta^{1\dagger}\sigma^3\eta^1) - i\partial_3(\eta^{1\dagger}\sigma^2\eta^1).\end{aligned} \tag{1.350}$$

Dividing by i and permuting the indices, we get:

$$\begin{aligned}E^1_L &= 2(T^1_{L0} + T^0_{L1}),\\ E^2_L &= 2(T^2_{L0} + T^0_{L2}),\\ E^3_L &= 2(T^1_{L0} + T^0_{L1}).\end{aligned} \tag{1.351}$$

The strong link obtained here between the electromagnetic field and the energy–momentum tensor of the electron is thus proper to the space of dimension three. It is a sufficient reason to follow Baylis [2] and to prefer Cl_3 to $Cl_{1,3}$, the space-time algebra previously used by many physicists, like Hestenes, Boudet, Lasenby. Neither of them got the relations (1.351), because the space-time algebra cannot be the best tool to separate right and left waves. The Dirac equation gives for the right waves:

$$\partial_0\xi^1 + \sigma_1\partial_1\xi^1 = \sigma^2\partial_2\xi^1 + \sigma^3\partial_3\xi^1 - i(s_0 + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3)\xi^1. \tag{1.352}$$

Multiplying on the left side by $\xi^{1\dagger}\sigma_1$ we get:

$$\begin{aligned}&\xi^{1\dagger}\widehat{\sigma}^1\partial_0\xi^1 + \xi^{1\dagger}\widehat{\sigma}^0\partial_1\xi^1 + i\xi^{1\dagger}\widehat{\sigma}^3\partial_2\xi^1 - i\xi^{1\dagger}\widehat{\sigma}^2\partial_3\xi^1\\ &\quad = -i(s_0\mathrm{D}^{11}_R + s_1\mathrm{D}^{10}_R + is_2\mathrm{D}^{13}_R - is_3\mathrm{D}^{12}_R).\end{aligned} \tag{1.353}$$

Next we use the adjoint, we add and that gives:

$$\partial_0 \mathrm{D}^{11}_R + \partial_1 \mathrm{D}^{10}_R = 2T^3_{R2} - 2T^2_{R3}, \tag{1.354}$$

$$\begin{aligned}H^1_R &= 2(T^2_{R3} - T^3_{R2}),\\ H^2_R &= 2(T^3_{R1} - T^1_{R3}),\\ H^3_R &= 2(T^1_{R2} - T^2_{R1}).\end{aligned} \tag{1.355}$$

While subtracting, we get:

$$\begin{aligned} E_R^1 &= 2(T_{R0}^1 + T_{R1}^0), \\ E_R^2 &= 2(T_{R0}^2 + T_{R2}^0), \\ E_R^3 &= 2(T_{R0}^3 + T_{R3}^0). \end{aligned} \tag{1.356}$$

Now we consider the complete electromagnetic field, with a left and a right part:

$$F = \vec{E} + i\vec{H} = \frac{m_a}{k\mathbf{l}} F_L + \frac{m_a}{k\mathbf{r}} F_R, \tag{1.357}$$

$$\vec{E} = \frac{m_a}{k\mathbf{l}} \vec{E}_L + \frac{m_a}{k\mathbf{r}} \vec{E}_R;\ \vec{H} = \frac{m_a}{k\mathbf{l}} \vec{H}_L + \frac{m_a}{k\mathbf{r}} \vec{H}_R, \tag{1.358}$$

that gives:

$$\begin{aligned} H^1 &= \frac{m_a}{k\mathbf{l}} H_L^1 + \frac{m_a}{k\mathbf{r}} H_R^1 \\ &= \frac{2m_a}{k\mathbf{l}} (T_{L3}^2 - T_{L2}^3) + \frac{2m_a}{k\mathbf{r}} (T_{R3}^2 - T_{R2}^3) \\ &= 2\Big(\frac{m_a}{k\mathbf{l}} T_{L3}^2 + \frac{m_a}{k\mathbf{r}} T_{R3}^2\Big) - 2\Big(\frac{m_a}{k\mathbf{l}} T_{L2}^3 + \frac{m_a}{k\mathbf{r}} T_{R2}^3\Big) \\ &= 2(T_3^2 - T_2^3). \end{aligned} \tag{1.359}$$

And similarly:

$$\begin{aligned} E^1 &= \frac{m_a}{k\mathbf{l}} E_L^1 - \frac{m_a}{k\mathbf{r}} E_R^1 \\ &= \frac{2m_a}{k\mathbf{l}} (T_{L0}^1 + T_{L1}^0) + \frac{2m_a}{k\mathbf{r}} (T_{R0}^1 + T_{R1}^0) \\ &= 2\Big(\frac{m_a}{k\mathbf{l}} T_{L0}^1 + \frac{m_a}{k\mathbf{r}} T_{R0}^1\Big) + 2\Big(\frac{m_a}{k\mathbf{l}} T_{L1}^0 + \frac{m_a}{k\mathbf{r}} T_{R1}^0\Big) \\ &= 2(T_0^1 + T_1^0). \end{aligned} \tag{1.360}$$

Hence if we suppose:

$$B = \frac{m_a}{m}\mathbf{J} = \frac{m_a}{k\mathbf{l}} \mathrm{D}_L^1 + \frac{m_a}{k\mathbf{r}} \mathrm{D}_R^1;\ F = \vec{E} + i\vec{H} = \nabla \widehat{iB}, \tag{1.361}$$

then we get:

$$\begin{aligned} H^1 &= 2(T_3^2 - T_2^3);\ E^1 = 2(T_0^1 + T_1^0), \\ H^2 &= 2(T_1^3 - T_3^1);\ E^2 = 2(T_0^2 + T_2^0), \\ H^3 &= 2(T_2^1 - T_1^2);\ E^3 = 2(T_0^3 + T_3^0). \end{aligned} \tag{1.362}$$

The components of the electromagnetic field are thus sum and difference of the components of the energy–momentum tensor of the electron wave, named Tétrode's tensor, T. This electromagnetic field is exactly a bivector field ($F = \vec{E} + i\vec{H}$), without scalar nor pseudo-scalar part: ($F = a + \vec{E} + i\vec{H} + ib$ with $a = 0$ and $b = 0$). That comes from the fact that F is the gradient of a pseudo-vector in space-time (thus without pseudo-scalar part), and with a null divergence in space-time thus without scalar part. And this also comes from the conservation of the left and right currents D_L and D_R. We saw in 1.8 that the F field has a null invdim, since it is transformed under the similitude induced by a dilator M in Cl_3^* as: $F \mapsto F' = MFM^{-1}$. The potential A is not only a mathematical tool for the calculation of the electromagnetic field, it has a kind of physical reality; this was claimed by O. Costa de Beauregard [54], following L. de Broglie [57] [58]. Still more important, the potential A is not exterior to the wave, but totally dependent on the wave, which is necessary in any true theory of fields. The potential A present in the wave equation is considered as "exterior" because the potential iB is a pseudo-vector, not a vector in space-time. We have:

$$\Box(A + iB) = q(\mathrm{j} - i\mathrm{k}); \quad \Box A = q\mathrm{j}; \quad \Box B = -q\mathrm{k}. \tag{1.363}$$

And consequently the A potential is non proportional to $\mathbf{J}$ but depends on this current through the Dalembertian:

$$A = \Box^{-1}(e\mathbf{J}), \tag{1.364}$$

which induces the calculation via the advanced or retarded Green functions.

The electromagnetic field has none proper Lagrangian density nor associated energy-momentum tensor, because **the electromagnetic field is itself that energy-momentum**. In light, that is directly linked to the existence of energy-momentum quanta, as Einstein [67] understood, quanta today called photons.

1.12 The Pauli exclusion principle

The calculation of the energy levels in the case of the ion He^+ was considered as a big success of Bohr's model, where the electron is a charged point particle moving in the electric field created by the double charge of the nucleus. Later the quantum wave of the electron was discovered, then the calculation became much more difficult, especially since it was necessary to justify why two electrons in a neutral helium atom cannot occupy the same state. This impossibility was moreover generalized by Pauli : each list of quantum numbers characterizing each state is necessarily different for each electron in an electron cloud (exclusion principle), in any atom. This principle has been accounted for supposing that the wave of a system of two electrons was the anti-symmetric product $\psi = \psi_1\psi_2 - \psi_2\psi_1$ of the wave of each electron [59]. Even in the non relativistic case that induces difficulties, since the ψ wave becomes a function of seven variables, the three spatial coordinates of each particle, plus the time. And for an atom with n

electrons the wave becomes a function of $3n+1$ variables. Even if the use of a configuration space is usual in Hamiltonian mechanics, that is inevitably strange compared to the gravitation field, which does not change its nature when a thing is broken into two separated parts.

And the difficulties grow considerably up with the spin, because with the spin 1/2 the wave of each electron has value in $\mathbb{C}^2$ with the Pauli equation, and in $\mathbb{C}^4$ with the Dirac equation. Since $\mathbb{C}^4$ has none internal multiplication, a tensor product is used to antisymmetrise the wave. That falls back on the difficulty of the change of nature for the wave with the number of particles, which gets away any unitary field including gravitation.

Non relativistic quantum mechanics uses a linear wave equation, that accounts for the addition and subtraction of waves and hence allows a simple explanation of interference phenomenons. Therefore it is the multiplication that is used to justify the exclusion principle. For a two electrons system, the supposition is:

$$\psi_{12} = \psi_1\psi_2 - \psi_2\psi_1 = -\psi_{21}; \quad \psi_{11} = 0. \tag{1.365}$$

But we now have a non linear wave equation, for our improved equation the sum of two wave functions that are solutions of the equation has no reason to be, allowing possible exception, a solution of the wave equation. And the first version of the exclusion principle simply says: **for electrons in atoms and for each list of quantum numbers, only one electron may be present**. If an electron is an electron wave and nothing more, the exclusion is automatically realized if the two following conditions are satisfied:

1- For each list of quantum numbers, a single solution exists for the wave equation. We explain in Chapter C why, with our improved equation, this rule is always satisfied.

2- The sum of a wave function with itself is impossible. And that is also the situation if the wave is normalized and if the addition of the wave also adds the currents, because that gives:

$$\begin{gathered}\langle\phi_1+\phi_2|\phi_1+\phi_2\rangle = \langle\phi_1|\phi_1\rangle + \langle\phi_2|\phi_2\rangle = 1+1 = 2,\\ \langle\phi+\phi|\phi+\phi\rangle = 4\langle\phi|\phi\rangle = 4 \neq 2,\end{gathered} \tag{1.366}$$

And this normalization comes from the identity between the mass-energy of the wave frequency and the sum on the whole space of the energy density of the wave. The impossibility to get two electrons in the same state can hence follow from the fact that the left and right currents of the two electrons add together (addition of the electromagnetic potentials) and from the addition of the energy densities, time component of the Tétrode's tensor (addition of the electromagnetic fields). Moreover mechanical laws include the addition of angular momentum, this is traduced in relativistic quantum mechanics by adding the magnetic quantum numbers λ. Since these quantum numbers may be negative, this addition of the angular momenta may be traduced by a subtraction of the $|\lambda|$ numbers. The general form of the solutions of our improved equation, in spherical coordinaters, is (see Chapter C):

$$\phi = \Omega X e^{(\lambda\varphi - E\mathrm{x}^0)i_3}; \quad X = \begin{pmatrix} AU & -\bar{C}V \\ BV & \bar{D}U \end{pmatrix}; \; \Omega = \frac{e^{(-\varphi/2)i_3}e^{(-\theta/2)i_2}}{r\sqrt{\sin\theta}},$$

$$U' - \frac{\lambda}{\sin\theta}U = -\kappa V; \quad V' + \frac{\lambda}{\sin\theta}V = \kappa U. \tag{1.367}$$

where κ is a nonzero integer number, and where λ may be any half-integer value from $-\kappa + \frac{1}{2}$ until $\kappa - \frac{1}{2}$. A, B, C and D functions are functions of the radial variable r with complex values, solutions of the radial system:

$$\begin{aligned}
&\left(E + \frac{\alpha}{r}\right)D - iD' - i\frac{\kappa}{r}B + \mathbf{l}e^{ir_1}A = 0,\\
&\left(E + \frac{\alpha}{r}\right)B + iB' + i\frac{\kappa}{r}D + \mathbf{l}e^{ir_1}C = 0,\\
&\left(E + \frac{\alpha}{r}\right)A + iA' + i\frac{\kappa}{r}C - \mathbf{r}e^{-ir_1}D = 0,\\
&\left(E + \frac{\alpha}{r}\right)C - iC' - i\frac{\kappa}{r}A - \mathbf{r}e^{-ir_1}B = 0.
\end{aligned} \tag{1.368}$$

where r_1 is any real number. Since λ may be positive or negative, and since to subtract is equivalent to add the opposite, it is possible to consider only the addition of the quantum number λ. For two electron waves, when the electrons add their magnetic quantum numbers λ_1 and λ_2, their wave cannot have as magnetic quantum number $\lambda_1 + \lambda_2$, which cannot be a half-odd number, but an integer number. That should make the wave function ϕ undefined. The wave of the system is hence necessarily the sum of a wave with a magnetic quantum number$\lambda_1 + \lambda_2 + \frac{1}{2}$ and a wave with a magnetic quantum number $\lambda_1 + \lambda_2 - \frac{1}{2}$. For the κ numbers, which are positive, κ becomes $\kappa_1 + \kappa_2 - 1$, that adds the $j_1 = \kappa_1 - 1/2$, $j_2 = \kappa_2 - 1/2$, or $\kappa_1 - \kappa_2$ that subtracts j_2 from j_1 which is possible only if $\kappa_1 > \kappa_2$.

$$\phi = \phi_1 + \phi_2 = \phi_2 + \phi_1, \tag{1.369}$$

$$\phi_1 = \Omega X_1 e^{[(\lambda_1+\lambda_2+\frac{1}{2})\varphi - (E_1+E_2)\mathrm{x}^0]}; \quad X_1 = \begin{pmatrix} A_1U_1 & -\bar{C}_1V_1 \\ B_1V_1 & \bar{D}_1U_1 \end{pmatrix}, \tag{1.370}$$

$$U_1' - \frac{\lambda_1 + \lambda_2 + \frac{1}{2}}{\sin\theta}U_1 = -(\kappa_1 + \kappa_2 - 1)V_1, \tag{1.371}$$

$$V_1' + \frac{\lambda_1 + \lambda_2 + \frac{1}{2}}{\sin\theta}V_1 = (\kappa_1 + \kappa_2 - 1)U_1,$$

$$\phi_2 = \Omega X_2 e^{[(\lambda_1+\lambda_2-\frac{1}{2})\varphi - (E_1+E_2)\mathrm{x}^0]}; \quad X_2 = \begin{pmatrix} A_2U_2 & -\bar{C}_2V_2 \\ B_2V_2 & \bar{D}_2U_2 \end{pmatrix}, \tag{1.372}$$

$$U_2' - \frac{\lambda_1 + \lambda_2 - \frac{1}{2}}{\sin\theta}U_2 = -(\kappa_1 + \kappa_2 - 1)V_2, \tag{1.373}$$

$$V_2' + \frac{(\lambda_1 + \lambda_2 - \frac{1}{2}}{\sin\theta}V_2 = (\kappa_1 + \kappa_2 - 1)U_2.$$

That is the commutativity of the sum which gives the indistinguishability of the two electrons: it is impossible to know which is which. Notice also that the participation to a system of electrons do not suppress the existence of each electron, with its own list of quantum numbers. That simply adds to these individual waves the wave (or waves if there are more than two electrons) of systems adding phases and quantum numbers. Accounting for the fact that the mass (approximately) and the charge (exactly) are doubled, the radial functions are no more those of alone electrons, and satisfy the radial system:

$$\begin{aligned}
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)D_1 - iD_1' - i\frac{\kappa_1+\kappa_2-1}{r}B_1 + \mathbf{l}e^{ir_1}A_1 = 0,\\
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)B_1 + iB_1' + i\frac{\kappa_1+\kappa_2-1}{r}D_1 + \mathbf{l}e^{ir_1}C_1 = 0,\\
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)A_1 + iA_1' + i\frac{\kappa_1+\kappa_2-1}{r}C_1 - \mathbf{r}e^{-ir_1}D_1 = 0,\\
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)C_1 - iC_1' - i\frac{\kappa_1+\kappa_2-1}{r}A_1 - \mathbf{r}e^{-ir_1}B_1 = 0.
\end{aligned} \tag{1.374}$$

where Z is the number of protons contributing to the electric potential, and r_1 is any real number,

$$\begin{aligned}
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)D_2 - iD_2' - i\frac{\kappa_1+\kappa_2-1}{r}B_2 + \mathbf{l}e^{ir_2}A_2 = 0,\\
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)B_2 + iB_2' + i\frac{\kappa_1+\kappa_2-1}{r}D_2 + \mathbf{l}e^{ir_2}C_2 = 0,\\
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)A_2 + iA_2' + i\frac{\kappa_1+\kappa_2-1}{r}C_2 - \mathbf{r}e^{-ir_2}D_2 = 0,\\
&\Big(E_1+E_2+\frac{Z\alpha}{r}\Big)C_2 - iC_2' - i\frac{\kappa_1+\kappa_2-1}{r}A_2 - \mathbf{r}e^{-ir_2}B_2 = 0,
\end{aligned} \tag{1.375}$$

where r_2 is any real number. We remark that, in the particular case where $\kappa_1 = \kappa_2 = 1$ and where the possible magnetic quantum numbers $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$ are occupied, those quantum numbers do not change, since $\kappa_1+\kappa_2-1 = 1$, $\lambda_1+\lambda_2+\frac{1}{2} = \frac{1}{2}$ and $\lambda_1+\lambda_2-\frac{1}{2} = -\frac{1}{2}$.

In the case of a system with three electron with ϕ_1, ϕ_2 and ϕ_3 waves, what prededes applies to each of the three pairs constituted of the 1 and 2 electron, the 2 and 3 electrons, 1 and 3 electrons, plus a wave formed by the three electrons. And so on for a n electrons system.

1.13 Iterative form of the improved equation

We calculate $\widehat{\mathrm{J}}\phi$:

$$\widehat{\mathrm{J}}\phi = \widehat{\phi\widetilde{\phi}}\phi = \widehat{\phi}\rho e^{i\beta}. \tag{1.376}$$

Using the P conjugation we then get

$$\mathrm{J}\widehat{\phi} = \rho e^{-i\beta}\phi;\ e^{-i\beta}\phi = \frac{\mathrm{J}}{\rho}\widehat{\phi}. \tag{1.377}$$

With (1.109) and (1.110), we define the unit vector:

$$\mathrm{v} = \mathrm{v}^\mu \sigma_\mu = \frac{\mathrm{J}}{\rho};\ \mathrm{v}\cdot\mathrm{v} = \mathrm{v}\widehat{\mathrm{v}} = \frac{\mathrm{J}\widehat{\mathrm{J}}}{\rho^2} = \frac{\rho^2}{\rho^2} = 1;\ \ \widehat{\mathrm{v}} = \mathrm{v}^{-1}. \tag{1.378}$$

Similarly we obtain the mass term:

$$\mathbf{m}\widehat{\mathbf{m}} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix} \begin{pmatrix} \mathbf{r} & 0 \\ 0 & \mathbf{l} \end{pmatrix} = \begin{pmatrix} \mathbf{lr} & 0 \\ 0 & \mathbf{rl} \end{pmatrix} = m_g^2 \tag{1.379}$$

This allows us to express the improved wave equation as follows[47][49]:

$$\widehat{\phi} = -\widehat{\mathrm{v}}(\nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi})\frac{\widehat{\mathbf{m}}}{m_g^2}. \tag{1.380}$$

Conjugating, this gives the recursive functional equation:

$$\phi = f(\phi); f(\phi) = -\mathrm{v}(\widehat{\nabla}\phi\sigma_{21} + q\widehat{A}\phi)\frac{\mathbf{m}}{m_g^2}. \tag{1.381}$$

We then get by iterating:

$$\phi = f(f(\phi));\ \phi = f(f(f(\phi))),\dots \tag{1.382}$$

To get only multiplication by the left side, we use the uncrossed form (1.172):

$$\begin{aligned} &i\nabla\eta^1 = qA\eta^1 + \mathbf{l}\mathrm{v}\eta^1;\ i\nabla\eta^1 = p_L\eta^1;\ p_L := qA + \mathbf{l}\mathrm{v}, \\ &i\widehat{\nabla}\xi^1 = q\widehat{A}\xi^1 + \mathbf{r}\widehat{\mathrm{v}}\xi^1;\ i\widehat{\nabla}\xi^1 = \widehat{p}_R\xi^1;\ \widehat{p}_R := q\widehat{A} + \mathbf{r}\widehat{\mathrm{v}}. \end{aligned} \tag{1.383}$$

This gives at the second order:

$$i\widehat{\nabla}\nabla\eta^1 = \widehat{\nabla}(p_L\eta^1);\ i\nabla\widehat{\nabla}\xi^1 = \nabla(\widehat{p}_R\xi^1). \tag{1.384}$$

Iterating the wave equation once, we obtain second derivatives, and we then use the d'Alembert operator:

$$\Box = \widehat{\nabla}\nabla = \nabla\widehat{\nabla} = \partial_{00}^2 - \partial_{11}^2 - \partial_{22}^2 - \partial_{33}^2. \tag{1.385}$$

We indeed get:

$$\begin{aligned} i\Box\eta^1 &= (\widehat{\nabla}p_L)\eta^1 + 2p_L^\mu\partial_\mu\eta^1 - \widehat{p}_L\nabla\eta^1, \\ i\Box\xi^1 &= (\nabla(\widehat{p_R})\xi^1) + 2p_R^\mu\partial_\mu\xi^1 - p_R\widehat{\nabla}\xi^1. \end{aligned} \tag{1.386}$$

We let:

$$F := \nabla\widehat{A};\ F_l := \nabla\widehat{p}_L;\ F_r := \nabla\widehat{p}_R. \tag{1.387}$$

This gives

$$i(\Box - p_L\cdot p_L)\eta^1 = \widehat{F}_l\eta^1 + 2p_L^\mu\partial_\mu\eta^1, \tag{1.388}$$

$$i(\Box - p_R\cdot p_R)\xi^1 = F_r\xi^1 + 2p_R^\mu\partial_\mu\xi^1. \tag{1.389}$$

We indeed get:

$$i\Box\eta^1 = (\widehat{\nabla}p_L)\eta^1 + 2p_L^\mu\partial_\mu\eta^1 - \widehat{p}_L\nabla\eta^1,$$
$$i\Box\xi^1 = (\nabla(\widehat{p_R})\xi^1) + 2p_R^\mu\partial_\mu\xi^1 - p_R\widehat{\nabla}\xi^1. \tag{1.390}$$

We let:

$$F := \nabla\widehat{A};\ F_l := \nabla\widehat{p}_L;\ F_r := \nabla\widehat{p}_R. \tag{1.391}$$

This gives

$$i(\Box - p_L\cdot p_L)\eta^1 = \widehat{F}_l\eta^1 + 2p_L^\mu\partial_\mu\eta^1, \tag{1.392}$$
$$i(\Box - p_R\cdot p_R)\xi^1 = F_r\xi^1 + 2p_R^\mu\partial_\mu\xi^1. \tag{1.393}$$

These equations are more similar on the left side, but this is only because the d'Alembert operator suppresses the difference between $\widehat{\nabla}\nabla$ and $\nabla\widehat{\nabla}$. Two remarks: as in the linear Dirac equation case, these equations do not be Klein-Gordon equations, because we also have $A^\mu\partial_\mu$ term. (This fact is rather well concealed in many books where the second-order wave equation is given for an electron without electromagnetic interaction, and thus without real physical existence.) Next, the electromagnetic field is introduced as two chiral fields F_l and F_r. We have:

$$\nabla\widehat{\mathrm{v}} = \nabla(\frac{\widehat{\mathrm{J}}}{\rho}) = \nabla(\frac{\rho\widehat{\mathrm{J}}}{\rho^2}) = \nabla(\frac{\rho\widehat{\mathrm{J}}}{\mathrm{J}\widehat{\mathrm{J}}}) = \nabla(\rho\mathrm{J}^{-1}) := G. \tag{1.394}$$

These fields satisfy

$$F_l = qF + \mathbf{l}G;\ F_r = qF + \mathbf{r}G. \tag{1.395}$$

Chapter 2

Weak Interactions (Lepton case)

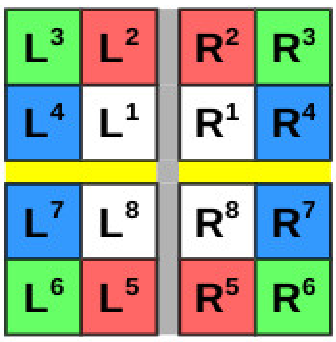
We use the Clifford algebra $Cl_{3,3}$ for the waves of all fermions and antifermions of the first generation. This includes a magnetic monopole that is also the complete neutrino, with both left and right waves. We study the new tensor densities that come from the extension of the electron wave. We transpose to Clifford algebra the covariant derivative of the electroweak gauge group. This covariant derivative is compatible with the mass term of the improved wave equation of the electron. We generalize to the lepton wave the Lagrangian density of the electron as well as its double link with the wave equations. The recursion of the wave equations allows us to obtain the properties of the gauge bosons. The lepton wave equation is form-invariant under the Cl_3^* group. The gauge invariance group is the $U(1) \times SU(2)$ group of the electroweak interactions. We also obtain the value of each charge of particle and antiparticle. The constraints imposed by this gauge group allow us to calculate the gauge potentials and to simplify the wave equation. The particular case of the electron fixes the value of the Weinberg–Salam angle to $30°$. We study the energy–momentum tensor density. We obtain the Lorentz force. We derive the dynamics of the magnetic monopole. We study the kinetic momentum tensor density and we derive the quantization of the kinetic momentum from the form-invariance of the kinetic momentum under Cl_3^*. We present a soliton solution for the self-interacting electron.

2.1 From the electron wave to the complete wave

In the first chapter we ascribed an invdim 1 to the wave of the electron, and we saw that the electromagnetic field had an invdim of 0. Thus some quantities have an invdim while others do not. The origin of the concept of an invdim is relativistic quantum physics. Since QFT replaces the electromagnetic field with a field of creator and annihilator operators, we thus postulate this fermion field as a field of operators:

$$\Psi : \phi \mapsto \phi_e,\ \phi_e = \Psi(\phi); \Psi \in \mathrm{End}(Cl_3);\ \mathrm{x} = \phi_e X \widetilde{\phi}_e. \tag{2.1}$$

where X belongs to the self-adjoint part of Cl_3, and ϕ_e is the wave of the electron. This is the fourth major change that we introduce: **space-time is not a starting point, but the consequence of the fermion field value**. This will be essential to incorporating gravitation into the Theory of Everything (see Chapter 4). The $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu \in Cl_3$ is the general linear element of space-time in special relativity, and the general element of any tangent space-time in general relativity. Time is the fourth component: $\mathrm{x}^0 = ct$. We need the absence of any additional dimension for the space-time manifold. For the first generation of fundamental fermions the SM accounts for 16 fermions: eight particles and their corresponding antiparticles. We just studied the case of the electron and its antiparticle the positron. These objects are not the only ones in the Standard Model. They are only examples of what are called *fermions*. In ordinary matter other fermions exist within the atoms, whose nuclei are made of protons and neutrons that are themselves composed of colored quarks. Besides an electron and its neutrino and their anti-particles, this ordinary matter forms what is called the "first generation." Each generation includes two quarks with three color states each. Thus we get eight waves similar to the electron wave ϕ_e. We label these waves from one to eight. Each one of these eight waves so labeled has a left part and a right part (including the neutrino). Here we study the general case, while in [28] we simplified the study by neglecting the right waves of the quarks. In [50] we summarized the scene in the following picture:



The quarks of the first generation are called the up (u) and down (d) quarks, and the couple $d-u$ is similar to $n-e$ in electroweak interactions, but with differences since the electric charge of the u quark will be $\frac{2}{3}|e|$, while the charge of the d quark will be $-\frac{1}{3}|e|$. For the lepton sector of each generation, the charges of the antiparticles appear opposite to the charges of particles. As we saw in Chapter 1, neither the charges nor the mass actually change. But the wave equations change because all partial derivatives change sign, and the right and left parts of the waves are exchanged. Without this identity between the wave of the particle and the wave of the antiparticle we should count not only 64 parameters but 128, and End(Cl_3) offers only 64 dimensions. The three "color charges" are called r, g, b (red, green, blue). The lepton wave that we study in this chapter is the white one at the center of the diagram. The quark waves are placed at the colored perimeter of the diagram; we will study them in the next chapter. This diagram indicates two symmetries that are both left–right symmetries: we placed the left waves on the left side and the right waves on the right side. We recall that the invariance group acts differently on right R^n and left L^n waves – it is precisely the source of that symmetry. The Cl_3^* invariance group is also the source of the second symmetry between the upper part of the diagram, on which the action is a multiplication on the left side, while the action on the lower part is a multiplication on the right side [1]. This second symmetry exchanges, for instance, the four red cases: those of the upper part containing the waves L^2 and R^2 of the d quark with color r, and those of the lower part containing the waves L^5 and R^5 of the u quark with color r. This double symmetry is well known in the framework of Lie groups and Lie algebras: the $GL(n,\mathbb{C})$ groups have four kinds of representations. In these symmetries quarks and leptons are highly similar.

For each quarter of the diagram we have one white box and three colored boxes; thus the whole wave of the first generation, including leptons and quarks, also comes from a single mathematical object linked to Cl_3 since it takes value in the algebra of all endomorphisms on this linear space: End(Cl_3). It also happens that this ring is the Clifford algebra $Cl_{3,3}$ (we study this algebra in B.2). This algebra is a 64-dimensional linear space on the $\mathbb{R}$ field. Therefore we will use the function Ψ, with value in $Cl_{3,3}$, as quantum wave of second quantization. This algebra contains eight supplementary linear spaces similar to Cl_3. So we will use these eight linear spaces to obtain eight waves linearly similar to the wave of the electron studied in Chapter 1. With (B.94) we have:

$$\Psi = \Psi(\mathrm{x}) = \begin{pmatrix} \Psi_l + i\Psi_b & \Psi_r + \Psi_g \\ \Psi_r - \Psi_g & \Psi_l - i\Psi_b \end{pmatrix}, \tag{2.2}$$

1. This symmetry that inverts the order of all products is called reversion in Clifford algebra (see A.1).

$$\Psi^{3,3} = \begin{pmatrix} \Psi_l + i\Psi_b & \Psi_r + \Psi_g \\ \Psi_r - \Psi_g & \Psi_l - i\Psi_b \end{pmatrix}, \tag{2.3}$$

$$\Psi_l := \mathcal{P}_1 - i\mathcal{I}_1 = \Psi^1 = \alpha + \mathbf{A} + \zeta\mathbf{i} - i(\mathbf{g} + \mathbf{c}\mathbf{i}), \tag{2.4}$$

$$\mathcal{P}_1 = \begin{pmatrix} \phi_e & 0 \\ 0 & \widehat{\phi}_e \end{pmatrix}; \ \mathcal{I}_1 = \begin{pmatrix} 0 & \phi_n \\ \widehat{\phi}_n & 0 \end{pmatrix}; \ \Psi_l = \begin{pmatrix} \phi_e & -i\phi_n^\dagger \\ -i\overline{\phi}_n & \widehat{\phi}_e \end{pmatrix},$$

$$i\Psi_r = \mathcal{P}_2 + i\mathcal{I}_2 = \begin{pmatrix} \phi_{dr} & i\phi_{ur} \\ i\widehat{\phi}_{ur} & \widehat{\phi}_{dr} \end{pmatrix}, \tag{2.5}$$

$$i\Psi_g = \mathcal{P}_3 + i\mathcal{I}_3 = \begin{pmatrix} \phi_{dg} & i\phi_{ug} \\ i\widehat{\phi}_{ug} & \widehat{\phi}_{dg} \end{pmatrix}, \tag{2.6}$$

$$i\Psi_b = \mathcal{P}_4 + i\mathcal{I}_4 = \begin{pmatrix} \phi_{db} & i\phi_{ub} \\ i\widehat{\phi}_{ub} & \widehat{\phi}_{db} \end{pmatrix}. \tag{2.7}$$

The Ψ term is hence composed of two different kinds of terms: Ψ_l which is a single term, and Ψ_r, Ψ_g and Ψ_b, which are three similar terms, all different from Ψ_l. The distinction between a lepton part Ψ_l and a quark part (Ψ_r, Ψ_g, Ψ_b) does not need to be postulated, it directly comes from the mathematical nature of the whole quantum wave.

In this chapter we study the Ψ_l wave, which is a function of space-time in $Cl_{3,1}$. And since i is the 3-vector term of Cl_3, which commutes with any term in Cl_3, when we restrict Ψ_l to its first row containing the 1 and 8 indices we may consider a function with value in $Cl_{3,1}$ as a function in $Cl_3 \times Cl_3$:

$$\Psi_l = (\phi_e \quad -i\phi_n^\dagger) \in Cl_3 \times Cl_3. \tag{2.8}$$

To get the Ψ wave of Chapter 1, with value in $Cl_{1,3}$, we associate to Ψ_l the Ψ^1 wave function linking $Cl_3 \times Cl_3$ to $Cl_{1,3}$ by:

$$\Psi^1 := (\phi_e \quad -i\phi_n^\dagger) = (\phi^1 \quad \phi^{8\dagger}) = \begin{pmatrix} \phi^1 & \phi^{8\dagger} \\ \overline{\phi}^8 & \widehat{\phi}^1 \end{pmatrix}. \tag{2.9}$$

The Ψ_l wave is made of two similar waves, $\phi_e = \phi^1$ which is, in the picture of second quantization, the electron wave. The electron wave thus plays a dual and special role, being included both in Cl_3 and $\text{End}(Cl_3)$ by (2.3) and (2.4). So we may say that the electron is both an example of a fermion and the quintessential fermion. The wave $i\phi_n = \phi^8$ is the wave of the neutrino, and also the wave of Lochak's magnetic monopole when it has both a left part and a right part. That are observations on presumed magnetic monopoles [105] [51] [52] which induced the hypothesis of a different proper mass for each part, left or right, of the wave. We previously placed the waves of antiparticles on the second row of each matrix in (2.4) to (2.7) [49, 50]. With the charge conjugation studied in 1.4.1 the second row is determined by the first row of the matrix, and we can use *ad libitum* the complete matrix element of $Cl_{3,1}$ or the first row allowing us to work with $Cl_3 \times Cl_3$. We shall thus employ the convenience of these algebras previously used in [50].

We also saw this essential property of the electron wave: the wave is double, with a right spinor and a left spinor (see (1.62)). The mathematical origin of this dualism is the existence of two nonequivalent homomorphisms from the $SL(2,\mathbb{C})$ group into the proper Lorentz group [101]. We also know that not only electrons exist: The β radioactivity that emits electrons also emits another particle nowadays called the electron antineutrino. The electron neutrino and antineutrino induce the existence of another pair of spinors, a left one and a right one. In a theory that unifies all interactions, and since gravitation and the geometry of space-time are strongly linked, the origin of this quartet of spinors which together constitute the lepton wave is necessarily geometric. And the whole of electromagnetism, including the electron wave, is form-invariant under the greater geometric group Cl_3^*. This group is isomorphic to the $GL(2,\mathbb{C})$ group which includes $SL(2,\mathbb{C})$ as a subgroup. The $GL(n,\mathbb{C})$ groups are well known to be the simplest Lie groups: their Lie algebra is the matrix algebra $M_n(\mathbb{C})$. Also well known are the four kinds (not only two) of nonequivalent representations. Our hypothesis is: these four kinds of representations are the origin of the existence of the four kinds of spinors forming the lepton wave (and also for the complete wave where the 16 spinors are divided into four parts of four spinors, that we will study in the next chapter dealing with the quarks.

The Standard Model first considered a neutrino reduced to its left wave only, and without proper mass. Modern experiments on neutrinos show that they must have a proper mass, which is very small indeed yet nonetheless still nonzero, and thus a right neutrino wave must also exist. The Standard Model actually has none objection against the existence of this right wave; it is simply considered useless. Yet the fact that the neutrino travels in space with the speed of light, or in any case with a velocity extraordinarily near light speed, justifies a null proper mass. With the Dirac wave equation for the neutrino, there is thus a problem, which is derived from the mass connection between left and right waves. We will see in this chapter how the improved equation, nonlinear, easily solves all these difficulties.

The starting point of this work was Lochak's theory of the leptonic magnetic monopole [90]-[97], where the wave of the monopole is also a function of space-time with value in Cl_3. Whether for the electron–positron pair or for the electron–neutrino pair or for the electron–monopole pair we obtain in each case four waves with a spinor value, with two left spinors and two right spinors. Moreover, as a particular case, we must again end up with only the electron or only the left electron − left neutrino pair. And we saw in the first chapter that the charge conjugation is simply the change of σ_{21} into σ_{12} in the wave equation of the electron. Hence the two pairs of spinors may be able to account for both leptons and antileptons, or to descrbe foth the electron and the magnetic monopole. **This leads us to think that these two objects, the neutrino with proper mass and the magnetic monopole, are one and the same particle**.

An extension of the Dirac equation for the electroweak interactions [114] was studied by Hestenes [81] and by Boudet [5] [6] in the framework of the $Cl_{1,3}$ algebra. The extension of the gauge invariance with the improved Dirac equation necessarily leads to the gauge invariance under $U(1)\times SU(2)$, already obtained in [12]. This comes from the existence of four independent generators with square -1 in the Cl_3 algebra: $i = \sigma_1\sigma_2\sigma_3$, $i\sigma_1$, $i\sigma_2$ and $i\sigma_3$. They are also the generators of the Lie algebra of $U(1) \times SU(2)$. Since the form invariance is always governed by the Cl_3^* group, and since $Cl_3 \times Cl_3$ is a Cl_3 left modulus, we may express everything in Cl_3: all complex 4×4 matrices of the Dirac theory may be calculated by blocks made of 2×2 matrices.

Under the similitude D generated by any dilator M in Cl_3^* we recall that we have: $\mathrm{x} \mapsto \mathrm{x}' = D(\mathrm{x}) = M\mathrm{x}M^\dagger$, $\nabla = \overline{M}\nabla'\widehat{M}$ and $\det(M) = re^{i\theta}$, and also

$$R^1 \mapsto R'^1 = MR^1;\ \widehat{L}^1 \mapsto \widehat{L}'^1 = \widehat{M}\widehat{L}^1.$$

The R^8 and L^8 waves use the two other representations with a reversion:

$$\begin{aligned} R^8 \mapsto R'^{\,8} &= R^8\widetilde{M};\ \widetilde{R}'^{\,8} = M\widetilde{R}^8, \\ \widehat{L}^8 \mapsto \widehat{L}'^{\,8} &= \widehat{L}^8\overline{M};\ \overline{L}'^{\,8} = \widehat{M\overline{L}}^{8}, \end{aligned} \tag{2.10}$$

The duality between a charged lepton and another lepton without electric charge is reproduced for each one of the three generations, through the simple generalization of Section 1.6. For the four spinors we use the following expressions: [2]:

$$\xi^n = \begin{pmatrix} \xi^n_1 \\ \xi^n_2 \end{pmatrix};\ \eta^n = \begin{pmatrix} \eta^n_1 \\ \eta^n_2 \end{pmatrix};\ \widehat{\eta}^n = \begin{pmatrix} -\bar{\eta}^n_2 \\ \bar{\eta}^n_1 \end{pmatrix};\ \widehat{\xi}^n = \begin{pmatrix} -\bar{\xi}^n_2 \\ \bar{\xi}^n_1 \end{pmatrix}, \tag{2.11}$$

$$\phi^1 = R^1 + L^1 = \sqrt{2}\begin{pmatrix}\xi^1 & \widehat{\eta}^1\end{pmatrix};\ R^1 = \sqrt{2}\begin{pmatrix} \xi^1_1 & 0 \\ \xi^1_2 & 0 \end{pmatrix};\ L^1 = \sqrt{2}\begin{pmatrix} 0 & -\bar{\eta}^1_2 \\ 0 & \bar{\eta}^1_1 \end{pmatrix},$$

$$\widehat{\phi}^1 = \widehat{R}^1 + \widehat{L}^1 = \sqrt{2}\begin{pmatrix}\eta^1 & \widehat{\xi}^1\end{pmatrix};\ \widehat{R}^1 = \sqrt{2}\begin{pmatrix} 0 & -\bar{\xi}^1_2 \\ 0 & \bar{\xi}^1_1 \end{pmatrix};\ \widehat{L}^1 = \sqrt{2}\begin{pmatrix} \eta^1_1 & 0 \\ \eta^1_2 & \end{pmatrix}$$

$$\widetilde{\phi}^8 = \widetilde{R}^8 + \widetilde{L}^8 = \sqrt{2}\begin{pmatrix}\xi^8 & \widehat{\eta}^8\end{pmatrix};\ \widetilde{R}^8 = \sqrt{2}\begin{pmatrix} \xi^8_1 & 0 \\ \xi^8_2 & 0 \end{pmatrix};\ \overline{L}^8 = \sqrt{2}\begin{pmatrix} \eta^8_1 & 0 \\ \eta^8_2 & 0 \end{pmatrix}.$$

We saw that the wave equations of the right and left parts of the electron wave satisfy a first-order partial differential equation, with only two other terms: a gauge term and a mass term. The gauge term is from the geometric point of view a covariant vector, and the mass term is the product of the reduced mass by a unitary vector v. This vector is the local reduced velocity of the relativistic fluid. The invdim of the different terms allows us to understand why no other term is possible in a first-order wave equation (see 5.6) thus we can only generalize these equations. We may hence suppose a

2. Here we use the usual notation for the complex conjugate.

similar wave equation for the four parts of the wave:

$$i\nabla\eta^1 = l^1\eta^1, \tag{2.12}$$
$$i\widehat{\nabla}\xi^1 = \widehat{r}^1\xi^1, \tag{2.13}$$
$$i\nabla\eta^8 = l^8\eta^8, \tag{2.14}$$
$$i\widehat{\nabla}\xi^8 = \widehat{r}^8\xi^8. \tag{2.15}$$

In this chapter, the M in Cl_3^* which allows us to get the form invariance of the wave equations is constant. Moreover we have $\nabla = \widetilde{\widehat{\nabla}}$ and the four differential operators are reduced to ∇ and $\widehat{\nabla}$. The l^n and r^n are four covariant space-time vectors, and we will see their connection with the potentials of the electroweak gauge group, as well as with the reduced proper mass generalizing the proper mass of the electron.

Let us first see how, when ϕ^8 is zero, the two equations (2.12) (2.13) may be the wave equations of the electron. Since η^1 is, when multiplied by $\sqrt{2}$, the left column of $\widehat{\phi}^1$ while $\widehat{\xi}^1$ gives the right column, these equations, when multiplied by $\sqrt{2}$, and using the P conjugation on (2.13) read as follows:

$$\begin{aligned} i\nabla\widehat{\phi}_e p_r &= l^1\widehat{\phi}_e p_r; \quad p_r := \frac{1+\sigma_3}{2} \\ -i\nabla\widehat{\phi}_e p_l &= r^1\widehat{\phi}_e p_l; \quad p_l := \frac{1-\sigma_3}{2}. \end{aligned} \tag{2.16}$$

Adding these equations we get:

$$i\nabla\widehat{\phi}_e\sigma_3 = \frac{l^1+r^1}{2}\widehat{\phi}_e + \frac{l^1-r^1}{2}\widehat{\phi}_e\sigma_3. \tag{2.17}$$

If we let:

$$l^1 := q\mathrm{A} + \mathbf{l}\mathrm{v};\ r^1 := q\mathrm{A} + \mathbf{r}\mathrm{v} \tag{2.18}$$

we then obtain the improved wave equation of the electron [3] :

$$\nabla\widehat{\phi}^1\sigma_{12} = q\mathrm{A}\widehat{\phi}^1 + \mathrm{v}\widehat{\phi}^1\mathbf{m}, \tag{2.19}$$

because the v vector is equal to the $\underline{J}/\rho$ in Chapter 1. We also use in Chapter 1 the following vectors, that it will be necessary to generalize:

$$\mathbf{J} := \frac{m}{k\mathbf{l}}\mathrm{D}_L^1 + \frac{m}{k\mathbf{r}}\mathrm{D}_R^1 = \frac{1}{k}\Big(\mathrm{J} + \frac{d}{m_a}\mathrm{K}\Big), \tag{2.20}$$
$$\mathbf{K} := -\frac{m}{k\mathbf{l}}\mathrm{D}_L^1 + \frac{m}{k\mathbf{r}}\mathrm{D}_R^1 = \frac{1}{k}\Big(-\mathrm{J} + \frac{d}{m_a}\mathrm{K}\Big). \tag{2.21}$$

3. It is well known that the energy–momentum of the electron is the sum of a matter term and an electromagnetic term. This duality is again obtained here with the tensor densities.

2.1.1 New tensor densities

In the case of a single electron we used four currents $\mathrm{D}_\mu = \phi\sigma_\mu\phi^\dagger$, particularly the $\mathrm{J} = \mathrm{D}_0 = \phi\phi^\dagger$. This current is the sum of two chiral currents $\mathrm{D}^1_R = R^1\widetilde{R}^1$ and $\mathrm{D}^1_L = L^1\widetilde{L}^1$. Moreover these currents are now similar to two other chiral currents:

$$\mathrm{D}^8_R := \widetilde{R}^8 R^8;\ \mathrm{D}^8_L := \widetilde{L}^8 L^8. \tag{2.22}$$

And these currents, like those of the electron, have a null scalar square in space-time:

$$\begin{aligned}\mathrm{D}^8_R \cdot \mathrm{D}^8_R &= \mathrm{D}^8_R \widehat{\mathrm{D}}^8_R = \widetilde{R}^8(R^8\overline{R}^8)\widehat{R}^8 = 0,\\ \mathrm{D}^8_L \cdot \mathrm{D}^8_L &= \mathrm{D}^8_L \widehat{\mathrm{D}}^8_L = \widetilde{L}^8(L^8\overline{L}^8)\widehat{L}^8 = 0,\end{aligned} \tag{2.23}$$

because the bracketed quantities are both zero. The natural generalization of the probability current of the electron is the lepton current $\mathbf{J}_l$ such that:

$$\xi^1 = \sqrt{\frac{\mathbf{r}^1 l_P^3}{2m}}\begin{pmatrix} a^1_1 + ia^1_2 \\ a^1_3 + ia^1_4\end{pmatrix};\ \eta^1 = \sqrt{\frac{\mathbf{l}^1 l_P^3}{2m}}\begin{pmatrix} a^1_5 + ia^1_6 \\ a^1_7 + ia^1_8\end{pmatrix}, \tag{2.24}$$

$$\xi^8 = \sqrt{\frac{\mathbf{r}^8 l_P^3}{2m}}\begin{pmatrix} a^8_1 + ia^8_2 \\ a^8_3 + ia^8_4\end{pmatrix};\ \eta^8 = \sqrt{\frac{\mathbf{l}^8 l_P^3}{2m}}\begin{pmatrix} a^8_5 + ia^8_6 \\ a^8_7 + ia^8_8\end{pmatrix}, \tag{2.25}$$

$$\mathbf{J}_l = \frac{m}{k}\Big(\frac{\mathrm{D}^1_R}{\mathbf{r}^1} + \frac{\mathrm{D}^1_L}{\mathbf{l}^1} + \frac{\mathrm{D}^8_R}{\mathbf{r}^8} + \frac{\mathrm{D}^8_L}{\mathbf{l}^8}\Big). \tag{2.26}$$

This current generalizes indeed the $\mathbf{J}$ current since we have:

$$\begin{aligned}\mathbf{J}^0_l = {}&\frac{m}{k\mathbf{r}^1}(|\xi^1_1|^2 + |\xi^1_2|^2) + \frac{m}{k\mathbf{l}^1}(|\eta^1_1|^2 + |\eta^1_2|^2)\\ &+ \frac{m}{k\mathbf{r}^8}(|\xi^8_1|^2 + |\xi^8_2|^2) + \frac{m}{k\mathbf{l}^8}(|\eta^8_1|^2 + |\eta^8_2|^2),\end{aligned} \tag{2.27}$$

The norm and the associated scalar product generalizing those of the single electron case becomes:

$$||\Psi_l|| := \iiint dv\Big(\sum_{j=1}^{j=8}(a^1_j)^2 + (a^8_j)^2\Big), \tag{2.28}$$

$$\langle\Psi_l|\Psi'_l\rangle := \iiint dv\Big(\sum_{j=1}^{j=8} a^1_j a^{1\prime}_j + a^8_j a^{8\prime}_j\Big). \tag{2.29}$$

This probability density is the generalization of the probability density of the electron in Chapter 1. The time component J^0_l is now one of $17\times 16/2 = 136$ tensor densities that we may define without derivatives from our four spinors with four real components each (16, thus (16+1)16/2 densities). We are now far ahead of the mere 16 tensor densities coming from the $M_4(\mathbb{C})$ algebra generated by the Dirac matrices, yet presented in most course books [4] as the only possible tensor densities! With (2.23) we have:

$$\begin{aligned}
&\mathbf{J}_l \cdot \mathbf{J}_l = \widehat{\mathbf{J}}_l \mathbf{J}_l \\
&= \frac{m^2}{k^2}\Big(\frac{\widehat{\mathrm{D}}^1_R}{\mathbf{r}^1} + \frac{\widehat{\mathrm{D}}^1_L}{\mathbf{l}^1} + \frac{\widehat{\mathrm{D}}^8_R}{\mathbf{r}^8} + \frac{\widehat{\mathrm{D}}^8_L}{\mathbf{l}^8}\Big)\Big(\frac{\mathrm{D}^1_R}{\mathbf{r}^1} + \frac{\mathrm{D}^1_L}{\mathbf{l}^1} + \frac{\mathrm{D}^8_R}{\mathbf{r}^8} + \frac{\mathrm{D}^8_L}{\mathbf{l}^8}\Big) \\
&= \frac{m^2}{k^2}\Big(\frac{\widehat{\mathrm{D}}^1_R\mathrm{D}^1_R}{\mathbf{r}^1\mathbf{r}^1} + \frac{\widehat{\mathrm{D}}^1_L\mathrm{D}^1_L}{\mathbf{l}^1\mathbf{l}^1} + \frac{\widehat{\mathrm{D}}^8_R\mathrm{D}^8_R}{\mathbf{r}^8\mathbf{r}^8} + \frac{\widehat{\mathrm{D}}^8_L\mathrm{D}^8_L}{\mathbf{l}^8\mathbf{l}^8} \\
&\qquad + \frac{\widehat{\mathrm{D}}^1_R\mathrm{D}^1_L + \widehat{\mathrm{D}}^1_L\mathrm{D}^1_R}{\mathbf{r}^1\mathbf{l}^1} + \frac{\widehat{\mathrm{D}}^1_R\mathrm{D}^8_R + \widehat{\mathrm{D}}^8_R\mathrm{D}^1_R}{\mathbf{r}^1\mathbf{r}^8} + \frac{\widehat{\mathrm{D}}^1_R\mathrm{D}^8_L + \widehat{\mathrm{D}}^8_L\mathrm{D}^1_R}{\mathbf{r}^1\mathbf{l}^8} \\
&\qquad + \frac{\widehat{\mathrm{D}}^1_L\mathrm{D}^8_R + \widehat{\mathrm{D}}^8_R\mathrm{D}^1_L}{\mathbf{l}^1\mathbf{r}^8} + \frac{\widehat{\mathrm{D}}^1_L\mathrm{D}^8_L + \widehat{\mathrm{D}}^8_L\mathrm{D}^1_L}{\mathbf{l}^1\mathbf{l}^8} + \frac{\widehat{\mathrm{D}}^8_R\mathrm{D}^8_L + \widehat{\mathrm{D}}^8_L\mathrm{D}^8_R}{\mathbf{r}^8\mathbf{l}^8}\Big) \\
&= \frac{2m^2}{k^2}\Big(\frac{\mathrm{D}^1_R}{\mathbf{r}^1}\cdot\frac{\mathrm{D}^1_L}{\mathbf{l}^1} + \frac{\mathrm{D}^1_R}{\mathbf{r}^1}\cdot\frac{\mathrm{D}^8_R}{\mathbf{r}^8} + \frac{\mathrm{D}^1_R}{\mathbf{r}^1}\cdot\frac{\mathrm{D}^8_L}{\mathbf{l}^8} \\
&\qquad + \frac{\mathrm{D}^1_L}{\mathbf{l}^1}\cdot\frac{\mathrm{D}^8_R}{\mathbf{r}^8} + \frac{\mathrm{D}^1_L}{\mathbf{l}^1}\cdot\frac{\mathrm{D}^8_L}{\mathbf{l}^8} + \frac{\mathrm{D}^8_R}{\mathbf{r}^8}\cdot\frac{\mathrm{D}^8_L}{\mathbf{l}^8}\Big).
\end{aligned} \tag{2.30}$$

We saw that $2\mathrm{D}^1_R \cdot \mathrm{D}^1_L = a_1 a_1^*$ where $a_1 = \Omega_1 + i\Omega_2$, a sum obtained from the two relativistic invariants Ω_1 and Ω_2 of the electron wave. With four spinors we form $6 = 4\times 3/2$ pairs, each giving a term similar to a_1. Thus we now have 12 invariant densities which give the 6 following complex terms:

$$\begin{aligned}
a_1 &= 2(\xi^1_1\overline{\eta}^1_1 + \xi^1_2\overline{\eta}^1_2) = R^1\overline{L}^1 + L^1\overline{R}^1, \\
a_2 &= 2(\eta^8_1\eta^1_2 - \eta^8_2\eta^1_1) = \widehat{L}^1\sigma_1 L^8 - \overline{L}^8\sigma_1\widetilde{L}^1, \\
a_3 &= 2(\xi^1_1\overline{\eta}^8_1 + \xi^1_2\overline{\eta}^8_2) = R^1\widehat{L}^8 + \widetilde{L}^8\overline{R}^1, \\
a_4 &= 2(\xi^8_1\overline{\eta}^1_1 + \xi^8_2\overline{\eta}^1_2) = \widetilde{R}^8\overline{L}^1 + L^1\widehat{R}^8, \\
a_5 &= 2(\xi^1_1\xi^8_2 - \xi^1_2\xi^8_1) = \widetilde{R}^8\sigma_1\overline{R}^1 - R^1\sigma_1\widehat{R}^8, \\
a_6 &= 2(\xi^8_1\overline{\eta}^8_1 + \xi^8_2\overline{\eta}^8_2) = \widetilde{R}^8\widehat{L}^8 + \widetilde{L}^8\widehat{R}^8.
\end{aligned} \tag{2.31}$$

In the similitude D generated by any dilator M in Cl_3^* we have for $j = 1, 2, \dots, 6$:

$$a'_j = M a_j \overline{M} = M\overline{M} a_j = re^{i\theta} a_j, \tag{2.32}$$

$$a'_j a'^*_j = re^{i\theta} a_j r e^{-i\theta} a_j^* = r^2 a_j a_j^*. \tag{2.33}$$

4. It is thus surprising and disturbing that this significant mistake concerning the number of tensor densities built from the electron wave in the Dirac theory, pointed out since nearly thirty years ago [15] by one of the present authors, is not yet corrected. This proves that the control of errors by the same "science community" which propagated these errors, is imperfect, inefficient, self-assured [38], and much too conservative. Nevertheless this control may perhaps be the least bad of all possible control systems.

We may then generalize the invariant ρ of the wave of the electron into ρ_l such that:

$$\rho_l^2 = m^2\Big[\frac{a_1a_1^*}{\mathbf{r}^1\mathbf{l}^1} + \frac{a_2a_2^*}{\mathbf{l}^1\mathbf{l}^8} + \frac{a_3a_3^*}{\mathbf{r}^1\mathbf{l}^8} + \frac{a_4a_4^*}{\mathbf{l}^1\mathbf{r}^8} + \frac{a_5a_5^*}{\mathbf{r}^1\mathbf{r}^8} + \frac{a_6a_6^*}{\mathbf{r}^8\mathbf{l}^8}\Big]. \tag{2.34}$$

which satisfies as ρ previously:

$$\rho'^2_l = r^2\rho_l^2;\ \rho'_l = r\rho_l;\ m'\rho'_l = m\rho_l. \tag{2.35}$$

Both a_j and ρ_l thus have an invdim of 1 and m has an invdim of -1 (see 1.7). In weak interactions the neutrino appears without a mass term. However it is necessary to add a mass term when physicists try to understand the behavior of neutrinos changing generation. And we may remark that $m\rho_l$ is null when ρ_l is null, and this precisely happens in the case where the neutrino only has a left wave. The currents thus satisfy $J_l = \mathrm{D}^8_L$ and $\mathrm{D}^8_L\cdot\mathrm{D}^8_L = 0$, and hence $m\rho_l = 0$: **the alone left neutrino appears without mass**. The mass term again appears as soon as R^8 is not null and as soon as the wave of the electron is not null. The behavior of this mass which appears and disappears *ad libitum* is not so mysterious if we consider that we are able to see a neutrino only when it interacts with a charged lepton or a quark. The mass term comes back as soon as R^8 does not cancel or as soon as the electron wave is non zero. For the electron we have $\rho^2 = a_1a_1^* = 2\mathrm{D}^1_R\cdot\mathrm{D}^1_L$. Similarly we obtain:

$$a_1a_1^* = 2\mathrm{D}^1_R\cdot\mathrm{D}^1_L, \tag{2.36}$$
$$a_5a_5^* = 2\mathrm{D}^1_R\cdot\mathrm{D}^8_R, \tag{2.37}$$

since we have:

$$2\mathrm{D}^1_R\cdot\mathrm{D}^8_R = \mathrm{D}^1_R\widehat{\mathrm{D}}^8_R + \mathrm{D}^8_R\widehat{\mathrm{D}}^1_R = R^1\widetilde{R}^1\overline{R}^8\widehat{R}^8 + \widetilde{R}^8R^8\widehat{R}^1\overline{R}^1,$$
$$\widetilde{R}^1\overline{R}^8 = \begin{pmatrix}0 & -a_5^*\\ 0 & 0\end{pmatrix};\ R^8\widehat{R}^1 = \begin{pmatrix}0 & a_5^*\\ 0 & 0\end{pmatrix}, \tag{2.38}$$
$$2\mathrm{D}^1_R\cdot\mathrm{D}^8_R = R^1\begin{pmatrix}0 & -a_5^*\\ 0 & 0\end{pmatrix}\widehat{R}^8 + \widetilde{R}^8\begin{pmatrix}0 & a_5^*\\ 0 & 0\end{pmatrix}\overline{R}^1$$
$$= (\widetilde{R}^8\sigma_1\overline{R}^1 - R^1\sigma_1\widehat{R}^8)a_5^* = a_5a_5^*. \tag{2.39}$$

We can likewise establish:

$$2\mathrm{D}^1_R\cdot\mathrm{D}^8_L = a_3a_3^*;\ 2\mathrm{D}^1_L\cdot\mathrm{D}^8_R = a_4a_4^*,$$
$$2\mathrm{D}^1_L\cdot\mathrm{D}^8_L = a_2a_2^*;\ 2\mathrm{D}^8_R\cdot\mathrm{D}^8_L = a_6a_6^*. \tag{2.40}$$

We then derive from these equations and from (2.30), (2.32) and (2.34) that we have, with the signature $+\ -\ -\ -$ of space-time:

$$\mathbf{J}_l\cdot\mathbf{J}_l = \rho_l^2;\ ||\mathbf{J}_l|| = \rho_l. \tag{2.41}$$

As for the single electron we may define a unitary vector v as

$$\mathrm{v} = \frac{\mathbf{J}_l}{\rho_l} = \mathrm{v}^\mu \sigma_\mu;\ \mathrm{v}\widehat{\mathrm{v}} = \widehat{\mathrm{v}}\mathrm{v} = 1;\ \widehat{\mathrm{v}} = \mathrm{v}^{-1}. \tag{2.42}$$

We retain the same notation v as for the case of the electron because this vector is exactly that of Chapter 1 when the wave of the neutrino is null. The natural generalization of the Lagrangian density of the electron is thus able to contain the same mass terms, and the wave equation is able to contain the same v in mass terms.

We must never forget that the previous tensors are only a small part of the many tensor densities that we are able to construct from the spinor wave. The differentiation of these tensors indeed yields new ones, which in turn give others by deriving again, *ad infinitum* [15]. We will also use some of these other tensor densities, for instance the energy–momentum tensors.

2.1.2 The electroweak gauge invariance

We begin with the case of the electron following [70]. We modify nothing to the wave of the electron which we denote as ψ^1 in the usual Dirac formalism and ϕ^1 in Cl_3. The wave of the electron neutrino is denoted as ψ^8 in the Dirac formalism and $\widetilde{\phi}^8$ in Cl_3. The wave of the positron is denoted as ψ_p in the Dirac formalism and the wave of the electron antineutrino is denoted as ψ_a. The link between the particle and antiparticle wave remains the previous link seen in 1.4.1. We start with the particle waves. The right spinors are denoted ξ^n and the left spinors are denoted η^n:

$$\psi^1 = \begin{pmatrix} \xi^1 \\ \eta^1 \end{pmatrix}\ ;\ \psi^8 = \begin{pmatrix} \xi^8 \\ \eta^8 \end{pmatrix}. \tag{2.43}$$

We again use the notation of (2.11) which gives us

$$\phi^1 = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix};\ \widehat{\phi}^1 = \sqrt{2}\begin{pmatrix} \eta^1 & \widehat{\xi}^1 \end{pmatrix}, \tag{2.44}$$

$$\widetilde{\phi}^8 = \sqrt{2}\begin{pmatrix} \xi^8 & \widehat{\eta}^8 \end{pmatrix};\ \overline{\phi}^8 = \sqrt{2}\begin{pmatrix} \eta^8 & \widehat{\xi}^8 \end{pmatrix}. \tag{2.45}$$

With the link that the Standard Model makes between the particle and the antiparticle wave, using $Cl_{1,3}$ and the left modulus $Cl_3 \times Cl_3$, we use:

$$\Psi^1 = \begin{pmatrix} \phi^1 & \widetilde{\phi}^8 \\ \overline{\phi}^8 & \widehat{\phi}^1 \end{pmatrix} = \begin{pmatrix} \phi^1 & \widetilde{\phi}^8 \end{pmatrix}. \tag{2.46}$$

And the internal multiplication in $Cl_{1,3}$ becomes in $Cl_3 \times Cl_3$:

$$\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} AC + B\widehat{D} & AD + B\widehat{C} \end{pmatrix}. \tag{2.47}$$

So the order in the multiplications is conserved and it is necessary to use the operator parity $P : M \mapsto \widehat{M}$ which is the main automorphism in $Cl_{1,3}$. Moreover the differential operator $\boldsymbol{\partial} = \gamma^\mu \partial_\mu$ becomes, with the modulus $Cl_3 \times Cl_3$:

$$\boldsymbol{\partial} = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla \end{pmatrix}; \quad \nabla = \sigma^\mu \partial_\mu \tag{2.48}$$

$$\boldsymbol{\partial}\Psi^1 = \begin{pmatrix} 0 & \nabla \end{pmatrix} \begin{pmatrix} \phi^1 & \phi^{8\dagger} \end{pmatrix} = \begin{pmatrix} \nabla \widehat{\phi}^{8\dagger} & \nabla \widehat{\phi}^1 \end{pmatrix}. \tag{2.49}$$

The Weinberg–Salam model uses ξ^1, η^1 and η^8 and supposes $\xi^8 = 0$. We use here the complete wave for Lochak's magnetic monopole while the neutrino itself has no right wave. So we consider the magnetic monopole as a complete neutrino with a left and right wave and we consider the neutrino of the SM as a monopole in which the right wave is absent. For the separation of ξ^1, η^1 and η^8 the Weinberg–Salam model uses the $\frac{1}{2}(1 \pm \gamma_5)$ projectors that can be presented as follows, with our choice (1.4) of Dirac matrices:

$$\frac{1}{2}(1 - \gamma_5)\psi = \psi_L = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tag{2.50}$$

$$\frac{1}{2}(1 + \gamma_5)\psi = \psi_R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \tag{2.51}$$

Thus for the particles the left waves are η waves and the right waves are ξ waves. This is invariant under Cl_3^* and therefore relativistically invariant, since under a similitude D generated by a dilator M such that $D : \mathrm{x} \mapsto \mathrm{x}' = M\mathrm{x}M^\dagger$, we have (1.62): $\xi' = M\xi$, $\eta' = \widehat{M}\eta$. So we use

$$R^1 = \sqrt{2}\begin{pmatrix} \xi^1 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix} \frac{1}{2}(1 + \sigma_3),$$

$$L^1 = \sqrt{2}\begin{pmatrix} 0 & \widehat{\eta}^1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix} \frac{1}{2}(1 - \sigma_3). \tag{2.52}$$

And we get similar formulas for $\widetilde{R}^8$ and $\widetilde{L}^8$. We now define two projectors $P_\pm$ and four operators P_0, P_1, P_2, P_3 acting as follows on $\Psi \in Cl_3 \times Cl_3$ as:

$$P_\pm(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{21}) \; ; \; \mathbf{i} = (i \;\; 0); \; \gamma_{21} = (\sigma_{12} \;\; 0), \tag{2.53}$$

Thus we get:

$$P_+(\Psi_l) = \begin{pmatrix} L^1 & \widetilde{L}^8 \end{pmatrix}; \; P_-(\Psi_l) = \begin{pmatrix} R^1 & \widetilde{R}^8 \end{pmatrix}. \tag{2.54}$$

So P_+ is the projector on the left part of the wave and P_- is the projector on the right part of the wave. We let:

$$P_0(\Psi) = \Psi\gamma_{21} + (1-p)P_-(\Psi)\mathbf{i} + p\mathbf{i}P_-(\Psi), \tag{2.55}$$

$$P_1(\Psi) = \mathbf{i}P_+(\Psi)\gamma_3\gamma_5, \tag{2.56}$$

$$P_2(\Psi) = \mathbf{i}P_+(\Psi)(-i\gamma_3), \tag{2.57}$$

$$P_3(\Psi) = P_+(\Psi)(-\mathbf{i}). \tag{2.58}$$

We introduce here a number p which is linked to the charge of the magnetic monopole and which acts only on the right wave of the neutrino, which is unknown in the Standard Model.[5] Noting $P_\mu P_\nu(\Psi) = P_\mu[P_\nu(\Psi)]$, they satisfy:

$$P_1P_2 = P_3 = -P_2P_1;\ P_2P_3 = P_1 = -P_3P_2;\ P_3P_1 = P_2 = -P_1P_3, \quad (2.59)$$
$$P_1^2 = P_2^2 = P_3^2 = -P_+;\ P_0P_j = P_jP_0 = -\mathbf{i}P_j\ ,\quad j = 1,\ 2,\ 3.$$

The Weinberg–Salam model replaces the ∂_μ derivatives with the covariant derivatives:

$$\mathrm{D}_\mu = \partial_\mu - ig_1\frac{Y}{2}B_\mu - ig_2T_jW^j_\mu, \quad (2.60)$$

with $T_j = \tau_j/2$ for a doublet of left particles and $T_j = 0$ for a singlet of right[6] particles. Y is the weak hypercharge, with $Y_L = -1$, $Y_R = -2$ for the electron. For transposing this to $Cl_3 \times Cl_3$ we let:

$$\mathrm{D} = \sigma^\mu\mathrm{D}_\mu;\ B = \sigma^\mu B_\mu;\ W^j = \sigma^\mu W^j_\mu,\ j = 1,2,3, \quad (2.61)$$
$$\mathbf{D} = \gamma^\mu\mathrm{D}_\mu;\ \mathbf{B} = \gamma^\mu B_\mu;\ \mathbf{W}^j = \gamma^\mu W^j_\mu,\ j = 1,2,3;\ \boldsymbol{\partial} = \gamma^\mu\partial_\mu. \quad (2.62)$$

We now replace (2.60) by:

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_1}{2}\mathbf{B}P_0 + \frac{g_2}{2}(\mathbf{W}^1P_1 + \mathbf{W}^2P_2 + \mathbf{W}^3P_3). \quad (2.63)$$

First we have

$$\boldsymbol{\partial}\Psi_l = \begin{pmatrix}-\nabla\overline{\phi}^8 & \nabla\widehat{\phi}^1\end{pmatrix}, \quad (2.64)$$
$$\mathbf{D}\Psi_l = \begin{pmatrix}-\mathrm{D}\overline{\phi}^8 & \mathrm{D}\widehat{\phi}^1\end{pmatrix}. \quad (2.65)$$

And we get:

$$P_0(\Psi_l) = i\begin{pmatrix}2R^1 - L^1 & 2p\widetilde{R}^8 - \widetilde{L}^8\end{pmatrix}. \quad (2.66)$$

From the form of these P_μ we may see that the Weinberg–Salam model of weak interactions using only R^1, L^1 and L^8 does not depend on the value of p that may be any number. We will later see how this value is linked to the charge of the magnetic monopole. We next obtain:

$$\mathbf{B}P_0(\Psi_l) = \begin{pmatrix}iB(2p\overline{R}^8 - \overline{L}^8) & iB(-2\widehat{R}^1 + \widehat{L}^1)\end{pmatrix}. \quad (2.67)$$

Next we have

$$P_1(\Psi_l) = \begin{pmatrix}i\widetilde{L}^8 & iL^1\end{pmatrix};\ \mathbf{W}^1P_1(\Psi_l) = \begin{pmatrix}-iW^1\widehat{L}^1 & -iW^1\overline{L}^8\end{pmatrix}, \quad (2.68)$$
$$P_2(\Psi_l) = \begin{pmatrix}\widetilde{L}^8 & -L^1\end{pmatrix};\ \mathbf{W}^2P_2(\Psi_l) = \begin{pmatrix}-W^2\widehat{L}^1 & W^2\overline{L}^8\end{pmatrix}. \quad (2.69)$$

---

5. If $R^8 = 0$, what we work is part of the Standard Model. If $R^8 = 0$, what we get is considered, probably wrongly, as beyond the Standard Model.

6. This preference for the left waves is presupposed and not explained in the Weinberg–Salam model. We will explain the origin of this preference in 3.8 .

We get for $j = 3$:

$$P_3(\Psi_l) = \begin{pmatrix} -iL^1 & i\widetilde{L}^8 \end{pmatrix}; \ \mathbf{W}^3 P_3(\Psi_l) = \begin{pmatrix} -iW^3\overline{L}^8 & iW^3\widehat{L}^1 \end{pmatrix}. \tag{2.70}$$

Therefore (2.63) is equivalent to the system:

$$\begin{aligned} D\widehat{\phi}^1 &= \nabla\widehat{\phi}^1 - i\frac{g_1}{2}B(2\widehat{R}^1 - \widehat{L}^1) - i\frac{g_2}{2}[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1], \\ D\overline{\phi}^8 &= \nabla\overline{\phi}^8 - \frac{g_1}{2}B(2p\overline{R}^8 - \overline{L}^8) - \frac{g_2}{2}[(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8. \end{aligned} \tag{2.71}$$

The gauge term of the electron contains the left part L^8 of the neutrino wave, while the gauge term of the left part L^8 contains the left part L^1 ot the electron wave. We hence see how weak interactions mix the left waves of the electron and its neutrino. Since ξ^1 is the left column of R^1, and ξ^8 is the left column of $\widetilde{R}^8$, while η^1 is the left column of $\widehat{L}^1$, and η^8 is the left column of $\overline{L}^8$ (not forgetting a $\sqrt{2}$ factor), this system gives for the particles (electrons and neutrinos) and using the main automorphism $P : M \mapsto \widehat{M}$ for the right waves:

$$\begin{aligned} \widehat{D}R^1 &= \widehat{\nabla}R^1 + ig_1\widehat{B}R^1, \\ D\widehat{L}^1 &= \nabla\widehat{L}^1 + i\frac{g_1}{2}B\widehat{L}^1 - \frac{ig_2}{2}[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1], \\ D\overline{L}^8 &= \nabla\overline{L}^8 + \frac{ig_1}{2}B\overline{L}^8 - \frac{ig_2}{2}[(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8]; \\ \widehat{D}\widetilde{R}^8 &= \widehat{\nabla}\widetilde{R}^8 + ipg_1\widehat{B}\widetilde{R}^8. \end{aligned} \tag{2.72}$$

For the waves of the positron and the antineutrino we similarly obtain

$$\begin{aligned} D\widehat{L}^{\overline{1}} &= \nabla\widehat{L}^{\overline{1}} - ig_1B\widehat{L}^{\overline{1}}, \\ \widehat{D}R^{\overline{1}} &= \widehat{\nabla}R^{\overline{1}} - \frac{ig_1}{2}\widehat{B}R^{\overline{1}} - \frac{ig_2}{2}[(\widehat{W}^1 - i\widehat{W}^2)\widetilde{R}^{\overline{8}} + \widehat{W}^3R^{\overline{1}}], \\ \widehat{D}\widetilde{R}^{\overline{8}} &= \widehat{\nabla}\widetilde{R}^{\overline{8}} - i\frac{g_1}{2}\widehat{B}\widetilde{R}^{\overline{8}} - i\frac{g_2}{2}[(\widehat{W}^1 + i\widehat{W}^2)R^{\overline{1}} - \widehat{W}^3\widetilde{R}^{\overline{8}}], \\ D\overline{L}^{\overline{8}} &= \nabla\overline{L}^{\overline{8}} - ig_1pB\overline{L}^{\overline{8}}. \end{aligned} \tag{2.73}$$

The system (2.72) is equivalent to:

$$D_\mu\xi^1 = \partial_\mu\xi^1 + ig_1B_\mu\xi^1, \tag{2.74}$$
$$D_\mu\eta^1 = \partial_\mu\eta^1 + i\frac{g_1}{2}B_\mu\eta^1 - i\frac{g_2}{2}[(W^1_\mu + iW^2_\mu)\eta^8 - W^3_\mu\eta^1], \tag{2.75}$$
$$D_\mu\eta^8 = \partial_\mu\eta^8 + i\frac{g_1}{2}B_\mu\eta^8 - i\frac{g_2}{2}[(W^1_\mu - iW^2_\mu)\eta^1 + W^3_\mu\eta^8], \tag{2.76}$$
$$D_\mu\xi^8 = \partial_\mu\xi^8 + ig_1pB_\mu\xi^8, \ \mu = 0, 1, 2, 3, \tag{2.77}$$

for the particle waves (2.73) is likewise equivalent to:

$$D_\mu\xi^{\overline{8}} = \partial_\mu\xi^{\overline{8}} - i\frac{g_1}{2}B_\mu\xi^{\overline{8}} - i\frac{g_2}{2}[(W^1_\mu + iW^2_\mu)\xi^{\overline{1}} - W^3_\mu\xi^{\overline{8}}], \tag{2.78}$$

$$D_\mu\eta^{\overline{8}} = \partial_\mu\eta^{\overline{8}} - ig_1pB_\mu\eta^{\overline{8}}, \tag{2.79}$$

$$D_\mu\eta^{\overline{1}} = \partial_\mu\eta^{\overline{1}} - ig_1B_\mu\eta^{\overline{1}},\ \mu = 0,1,2,3, \tag{2.80}$$

$$D_\mu\xi^{\overline{1}} = \partial_\mu\xi^{\overline{1}} - i\frac{g_1}{2}B_\mu\xi^{\overline{1}} - i\frac{g_2}{2}[(W^1_\mu - iW^2_\mu)\xi^{\overline{8}} + W^3_\mu\xi^{\overline{1}}], \tag{2.81}$$

for the antiparticles waves. For the doublet $\psi_L = \begin{pmatrix}\eta^8\\ \eta^1\end{pmatrix}$ of weak isospin $Y = -1$ the operators in (2.75) and (2.76) give

$$D_\mu\psi_L = \partial_\mu\psi_L - ig_1\frac{Y}{2}B_\mu\psi_L - i\frac{g_2}{2}W^j_\mu\tau_j\psi_L,$$
$$\tau_1 = \gamma_0\ ;\ \tau_2 = \gamma_{123}\ ;\ \tau_3 = \gamma_5. \tag{2.82}$$

Only the operator in (2.77) is not accounted for the Weinberg–Salam model because this model does not use the right wave of the electron neutrino (due to the mass term of the Dirac equation). We hence success from our improved wave equation, precisely to the same result as the Weinberg-Salam model when the right neutrino wave is null. Next the operator in (2.74) is interpreted as a singlet under $SU(2)$: $\psi_R = \xi^1$ with weak isospin $Y = -2$:

$$D_\mu\psi_R = \partial_\mu\psi_R - ig_1\frac{Y}{2}B_\mu\psi_R. \tag{2.83}$$

Finally we get here all aspects of the weak interactions in the case electron–neutrino: a doublet of left waves, a right wave which is a singlet, a right neutrino unable to interact, a charge conjugation exchanging right and left waves, all this is obtained here from simple hypotheses:

1 - The wave of all components of the lepton sector (electron, positron, electron neutrino and its antineutrino) is the function (**??**) of space and time with values in the Clifford algebra of space-time.

2 - Four operators P_0, P_1, P_2 and P_3 are defined in (2.55) to (2.58).

3 - A covariant derivative is defined in (2.63).

For the antiparticles, in the case where the wave of the magnetic monopole is reduced to the neutrino wave, we have a singulet of left wave and a doublet of right waves. By letting:

$$\psi_{\overline{L}} = \eta^{\overline{1}};\ \psi_{\overline{R}} = \begin{pmatrix}\xi^{\overline{8}}\\ \xi^{\overline{1}}\end{pmatrix};\ \overline{\tau}_1 = -\gamma_0;\ \overline{\tau}_2 = \gamma_{123};\ \overline{\tau}_3 = \gamma_5, \tag{2.84}$$

we get

$$D_\mu\psi_{\overline{L}} = \partial_\mu\psi_{\overline{L}} - ig_1\frac{Y}{2}B_\mu\psi_{\overline{L}}, \tag{2.85}$$

with a weak isospin $Y = 2$, in accordance with the usual rule changing charge signs. For the doublet of right waves we get:

$$D_\mu \psi_{\overline{R}} = \partial_\mu \psi_{\overline{R}} - ig_1 \frac{Y}{2} B_\mu \psi_{\overline{R}} + i\frac{g_2}{2} W_\mu^j \overline{\tau}_j \psi_{\overline{R}}. \tag{2.86}$$

The rule of the change of signs for all charges is equivalent to the change of sign for $g_1 Y$ and g_2. But these rules are not sufficient; another change of sign concerns $\overline{\tau}_1 = -\tau_1$. This calls for two remarks: First, the $SU(2)$ gauge group thought of by quantum theory as an "internal symmetry" is indeed a geometrical invariance group. This is lost from sight when the explanation passes from (2.73), where equations contain space-time vectors, to (2.74)–(2.81) where equations only have components of tensors: these tensors are no longer constrained by invariance under Cl_3^* but only by invariance under the Lorentz group. The relativistic invariance works there classically: the Lorentz transformation R defines a 4×4 real R^μ_ν matrix which changes the x^μ into $\mathrm{x}'^\mu = R^\mu_\nu \mathrm{x}^\nu$, and changes the electromagnetic field $F_{\mu\nu}$ into $F'_{\rho\sigma} = R^\mu_\rho R^\nu_\sigma F_{\mu\nu}$. And on the contrary, relativistic invariance works with quantum mechanics rules for spinor waves: a dilator M induces a similitude R which changes F into $F' = MFM^{-1}$. Indeed the two points of view cannot be equivalent. We establish hence here a strong new link between the fermion wave of the electron and of its neutrino with the tensor densities of the quantum wave. That allows us to build a unified synthesis of the different parts of relativistic physics, a synthesis impossible from the old tensor–based theory.

With charge conjugation simply acting like PT symmetry, space changes orientation. Thus the three τ_j rotate inversely from the $\overline{\tau}_j$, as is shown by the sign change of τ_1.

2.2 Retaining mass terms

The first improvement that Cl_3 brings to quantum mechanics being restricted to Dirac matrices, is the possibility of also using the right R^8 spinor that we associate with the magnetic monopole. A second and major improvement: we no longer need to suppress mass terms in wave equations. This deletion was necessary when using the linear Dirac equation, because mass terms link the left and the right wave, while ξ and η change in very different ways under the gauge transformations of the electroweak group [39] [40] [41] [44]. The mass suppression was also an acceptable lesser evil, from the experimental point of view, because proper masses [7] of the electron and, still more, proper masses of the electron neutrino are very small in comparison with the mass-energy of the W and Z^0 bosons. Nevertheless this

7. The improved wave equation in Chapter 1 tolerates two different mass terms for the electron: $\mathbf{r}^1$ for the right wave and $\mathbf{l}^1$ for the left wave of the electron. The origin of this hypothesis is the experimental study of neutrinos–monopoles [52].

deletion is necessarily an approximation since the electron *has* mass-energy, and since the wave equation of the neutrino probably also has mass terms. Since it was impossible to account for both proper mass and electroweak gauge, a mechanism of spontaneous symmetry breaking was constructed. The Higgs boson (which was thought of as able to reintroduce masses into wave equations) was finally observed at very high energy ($\approx$ 126 GeV). However, this still does not transform the electroweak gauge into a theory compatible with mass and gravitation. The existence of such a scalar field with high mass was suspected as early as the 30,s by de Broglie's theory of the photon [57, 58].

What we will establish restore the compatibility of the covariant electroweak derivation that we just studied with equations (2.12) to (2.15). The improved wave equation of the electron, non linear, may be recast into a form that seems uncrossed and acting on each chiral spinor. This is easily generalized. Maintaining mass terms in the wave equation will allow us in Chapter 4 to directly put together gravitation and other forces in the wave equations. It is thus an important improvement towards the unification of all interactions. The form of mass terms $\mathrm{v}\widehat{\phi}^1\mathbf{m}$ of the improved wave equation is conserved:

$$\begin{aligned} 0 &= -iD\widehat{L}^1 + \mathbf{l}^1\mathrm{v}\widehat{L}^1;\ 0 = -i\widehat{D}R^1 + \mathbf{r}^1\widehat{\mathrm{v}}R^1, \\ 0 &= -i\widetilde{D}\overline{L}^8 + \mathbf{l}^8\mathrm{v}\overline{L}^8;\ 0 = -i\overline{D}\widetilde{R}^8 + \mathbf{r}^8\widehat{\mathrm{v}}\widetilde{R}^8. \end{aligned} \tag{2.87}$$

We do not suppose that the $\mathbf{l}$, $\mathbf{r}$, m_l and m_r coefficients have particular properties. The unitary vector v remains defined from the four spinors of the lepton wave by (2.42). We simplify the following study by considering at first only the wave of the electron and of the neutrino–monopole, as a beginning. We will derive the properties of the positron and antineutrino–monopole by changing the sign of the differential terms of the wave equation, and exchanging η and ξ terms. With the form obtained in (2.72) for the derivation with gauge terms, the wave equation (2.87) becomes:

$$0 = \widehat{\nabla}R^1 + ig_1\widehat{B}R^1 + i\mathbf{r}^1\widehat{\mathrm{v}}R^1, \tag{2.88}$$

$$0 = \nabla\widehat{L}^1 + i\frac{g_1}{2}B\widehat{L}^1 - i\frac{g_2}{2}[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1] + i\mathbf{l}^1\mathrm{v}\widehat{L}^1, \tag{2.89}$$

$$0 = \widetilde{\nabla}\overline{L}^8 + i\frac{g_1}{2}B\overline{L}^8 - i\frac{g_2}{2}[(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8] + i\mathbf{l}^8\mathrm{v}\overline{L}^8, \tag{2.90}$$

$$0 = \overline{\nabla}\widetilde{R}^8 + ig_1p\widehat{B}\widetilde{R}^8 + i\mathbf{r}^8\widehat{\mathrm{v}}\widetilde{R}^8. \tag{2.91}$$

We may remark that the coefficients of B are the same only for L^1 and L^8. So left waves, turning in the same manner in the chiral gauge, can be mixed in the $SU(2)$ gauge group. Comparing with potential terms we may see that these equations are indeed wave equations with two different spinors. We now see how it is possible to use the invariance of each wave equation.

2.3 Extended invariance

With the similitude induced by any dilator non zero M in Cl_3 such as $\det(M) = re^{i\theta}$, we have:

$$\mathrm{D}'^1_R = R'^1\widetilde{R}'^1 = MR^1\widetilde{MR}^1 = MR^1\widetilde{R}^1\widetilde{M} = M\mathrm{D}^1_R\widetilde{M},$$
$$\mathrm{J}'_l = M\mathrm{J}_l\widetilde{M};\ \rho' = r\rho;\ m = m'r;\ (m = \mathbf{l}^1, \mathbf{r}^1, \mathbf{l}^8, \mathbf{r}^8), \tag{2.92}$$
$$m'\mathrm{v}' = \frac{m}{r}\frac{\mathrm{J}'_l}{\rho'} = \frac{m}{r}\frac{M\mathrm{J}_l\widetilde{M}}{r\rho} = m\frac{M}{re^{i\theta}}\mathrm{v}\frac{\widetilde{M}}{re^{-i\theta}} = m\overline{M}^{-1}\mathrm{v}\widehat{M}^{-1},$$
$$\overline{M}m'\mathrm{v}'\widehat{L}'^1 = m\mathrm{v}\widehat{M}^{-1}\widehat{M}\widehat{L}^1 = m\mathrm{v}\widehat{L}^1, \tag{2.93}$$

we simplify the wave equations (2.88)–(2.91) with:

$$\mathrm{p}^1_L := \frac{g_1}{2}B + \mathbf{l}^1\mathrm{v} = \mathrm{b} + \mathbf{l}^1\mathrm{v};\ \mathrm{w}^j := \frac{g_2}{2}W^j,\ j = 1, 2, 3,$$
$$\mathrm{p}^1_R := g_1B + \mathbf{r}^1\mathrm{v} = 2\mathrm{b} + \mathbf{r}^1\mathrm{v};\ \mathrm{p}^8_L := \frac{g_1}{2}B + \mathbf{l}^8\mathrm{v} = \mathrm{b} + \mathbf{l}^8\mathrm{v}, \tag{2.94}$$
$$\mathrm{p}^8_R := g_1pB + \mathbf{r}^8\mathrm{v} = 2p\mathrm{b} + \mathbf{r}^8\mathrm{v}.$$

To obtain the relativistic invariance of the equation of L^1, for instance, we must have for the gauge potentials the same variance as the differential term. And this term is covariant which means it satisfies $\nabla = \overline{M}\nabla'\widehat{M}$. It must be the same with p, b and w^j which are also covariant vectors because these vectors incorporate the g_1 and g_2 coefficients. We have:

$$0 = \widehat{\nabla}R^1 + i\widehat{\mathrm{p}}^1_R R^1 = \widetilde{M}[\widehat{\nabla}'R'^1 + i\widehat{\mathrm{p}}'^1_R R'^1], \tag{2.95}$$
$$0 = \nabla\widehat{L}^1 + i\mathrm{p}^1_L\widehat{L}^1 - i[(\mathrm{w}^1 + i\mathrm{w}^2)\overline{L}^8 - \mathrm{w}^3\widehat{L}^1)]$$
$$= \overline{M}\Big[\nabla\widehat{L}'^1 + i\mathrm{p}'^1_L\widehat{L}'^1 - i[(\mathrm{w}'^1 + i\mathrm{w}'^2)\overline{L}'^8 - \mathrm{w}'^3\widehat{L}'^1]\Big]; \tag{2.96}$$
$$0 = \nabla\overline{L}^8 + i\mathrm{p}^8_L\overline{L}^8 - i[(\mathrm{w}^1 - i\mathrm{w}^2)\widehat{L}^1 + \mathrm{w}^3\overline{L}^8)];$$
$$= \overline{M}\Big[\nabla\overline{L}'^8 + i\mathrm{p}'^8_L)\overline{L}'^8 - i[(\mathrm{w}'^1 - i\mathrm{w}'^2)\widehat{L}'^1 + \mathrm{w}'^3\overline{L}^8]\Big]; \tag{2.97}$$
$$0 = \widehat{\nabla}\widetilde{R}^8 + i\widehat{\mathrm{p}}^8_R\widetilde{R}^8 = \widetilde{M}\big[\widehat{\nabla}'\widetilde{R}'^8 + i\widehat{\mathrm{p}}'^8_R\widetilde{R}'^8\big]. \tag{2.98}$$

This provides the form invariance of the wave equations, as in the case of the lone electron that we studied in Chapter 1.

The gauge transformations are generated by P_0, P_1, P_2 and P_3. This gives to us a group with four parameters a^0, a^1, a^2 and a^3. We recall the definition of the exponential function:

$$\exp(a^0P_0) = \sum_{n=0}^{\infty}\frac{(a^0P_0)^n}{n!};\ \ \exp(a^jP_j) = \sum_{n=0}^{\infty}\frac{(a^1P_1 + a^2P_2 + a^3P_3)^n}{n!}. \tag{2.99}$$

Since these operators were defined in $Cl_3 \times Cl_3$ and since they are different for right and left waves we will study them with the form (see B.1.3):

$$\Psi_R^1 = \begin{pmatrix} R^1 & 0 \end{pmatrix}; \ \Psi_L^1 = \begin{pmatrix} L^1 & 0 \end{pmatrix}; \ \Psi_R^8 = \begin{pmatrix} 0 & \widetilde{R}^8 \end{pmatrix}; \ \Psi_L^8 = \begin{pmatrix} 0 & \widetilde{L}^8 \end{pmatrix}. \tag{2.100}$$

With P_0 we have:

$$\begin{aligned} P_0(\Psi_R^8) &= 2p\Psi_R^8\gamma_{21}; \ \exp(a^0P_0)(\Psi_R^8) = \Psi_R^8\exp[2pa^0\gamma_{21}],\\ P_0(\Psi_R^1) &= 2\Psi_R^1\gamma_{21}; \ \exp(a^0P_0)(\Psi_R^1) = \Psi_R^1\exp[2a^0\gamma_{21}],\\ P_0(\Psi_L) &= \Psi_L\gamma_{21}; \ \exp(a^0P_0)(\Psi_L) = \Psi_L\exp[a^0\gamma_{21}]. \end{aligned} \tag{2.101}$$

Next we let:

$$\begin{aligned} s &:= \theta u = \theta(s_1P_1 + s_2P_2 + s_3P_3); \ u^2 = s_1^2 + s_2^2 + s_3^2 = 1,\\ U &:= e^s = e^{\theta u}, \end{aligned} \tag{2.102}$$

and we have:

$$\begin{aligned} \Psi_R'^1 &= U\Psi_R^1 = \Psi_R^1; \ \Psi_R'^8 = U\Psi_R^8 = \Psi_R^8,\\ \Psi_L' &= U\Psi_L = \cos(\theta)\Psi_L + \sin(\theta)u(\Psi_L),\\ u(\Psi_L) &= s_1\Psi_L\gamma_3\mathbf{i} + s_2\Psi_L\gamma_3 + s_3\Psi_L(-\mathbf{i}),\\ \Psi_L &= U^{-1}\Psi_L' = \cos(\theta)\Psi_L' - \sin(\theta)u(\Psi_L')\\ &= \cos(\theta)\Psi_L' - \sin(\theta)[s_1\Psi_L'\gamma_3\mathbf{i} + s_2\Psi_L'\gamma_3 + s_3\Psi_L'(-\mathbf{i})]. \end{aligned} \tag{2.103}$$

Since P_0 commutes with s we get :

$$\exp(S) = \exp(a^0P_0)e^s = e^s\exp(a^0P_0); \ \exp(-S) = \exp(S)^{-1}. \tag{2.104}$$

The set of the $\exp(S)$ is a $U(1)\times SU(2)$ Lie group. The gauge transformation uses the derivative of the exponential function and satisfies

$$\Psi' = [\exp(S)](\Psi) \ ; \ \mathrm{D} = \sigma^\mu D_\mu \ ; \ \mathrm{D}' = \sigma^\mu D_\mu', \tag{2.105}$$

and so $D_\mu\Psi$ is replaced by $D_\mu'\Psi'$ such that:

$$(0 \ \ \mathrm{D}')\Psi' = (0 \ \ \nabla)\Psi' + G'(\Psi') = \exp(S)[(0 \ \ \nabla)\Psi + G(\Psi)], \tag{2.106}$$

$$G'(\Psi') = \exp(S)(X+Y); \ X = \Big[(0 \ \ \nabla)[\exp(-S)]\Big](\Psi'); \ Y = G(\Psi). \tag{2.107}$$

The transformation of the gauge potentials thus has two parts: a part that comes from the derivative of the exponential function and another that comes from the non-commutation of $\exp(S)$ with P_j.

2.3.1 The $U(1)$ gauge group generated by P_0

Since P_0 commutes with s, the relation between $\mathrm{w}'^0 = \mathrm{b}'$ and $\mathrm{w}^0 = \mathrm{b}$ is reduced to only the part coming from the derivative, and we get:

$$\mathrm{b}'_\mu = \mathrm{b}_\mu - \partial_\mu a^0. \tag{2.108}$$

The different space-time vectors that we may form from the wave with spin 1/2 to obtain the gauge potentials w^j, where $j = 0, 1, 2, 3$, must behave in a similitude like J_l which is the sum of D^1_R, D^8_R, D^1_L and D^8_L. In addition to these four vectors we may consider the following vectors:

$$\begin{aligned}
\mathrm{D}^1_{RL} &= \mathrm{D}_1 = R^1\sigma_1\widetilde{L}^1 + L^1\sigma_1\widetilde{R}^1;\ d^1_{RL} = \mathrm{D}_2 = i(R^1\sigma_1\widetilde{L}^1 - L^1\sigma_1\widetilde{R}^1),\\
\mathrm{D}^{18}_L &= L^1L^8 + \widetilde{L}^8\widetilde{L}^1;\ d^{18}_L = i(L^1L^8 - \widetilde{L}^8\widetilde{L}^1),\\
\mathrm{D}^{18}_{RL} &= R^1\sigma_1L^8 + \widetilde{L}^8\sigma_1\widetilde{R}^1;\ d^{18}_{RL} = i(R^1\sigma_1L^8 - \widetilde{L}^8\sigma_1\widetilde{R}^1),\\
\mathrm{D}^{18}_R &= R^1R^8 + \widetilde{R}^8\widetilde{R}^1;\ d^{18}_R = i(R^1R^8 - \widetilde{R}^8\widetilde{R}^1),\\
\mathrm{D}^{81}_{RL} &= \widetilde{R}^8\sigma_1\widetilde{L}^1 + L^1\sigma_1R^8;\ d^{81}_{RL} = i(\widetilde{R}^8\sigma_1\widetilde{L}^1 + L^1\sigma_1R^8),\\
\mathrm{D}^8_{RL} &= \widetilde{R}^8\sigma_1L^8 + \widetilde{L}^8\sigma_1R^8;\ d^8_{RL} = i(\widetilde{R}^8\sigma_1L^8 - \widetilde{L}^8\sigma_1R^8).
\end{aligned} \tag{2.109}$$

All these D vectors transform, like J_l, in a the similitude defined by a dilator M into $\mathrm{D}' = M\mathrm{D}\widetilde{M}$. With (2.101) we have:

$$\begin{aligned}
\widetilde{L}'^8 &= e^{-ia^0}\widetilde{L}^8;\ \widetilde{R}'^8 = e^{2ipa^0}\widetilde{R}^8;\ L'^1 = e^{-ia^0}L^1;\ R'^1 = e^{2ia^0}R^1, \qquad (2.110)\\
\widetilde{R}'^{\overline{8}} &= e^{ia^0}\widetilde{R}^{\overline{8}};\ \widetilde{L}'^{\overline{8}} = e^{-2ipa^0}\widetilde{L}^{\overline{8}};\ R'^{\overline{1}} = e^{ia^0}R^{\overline{1}};\ L'^{\overline{1}} = e^{-2ia^0}L^{\overline{1}}.
\end{aligned}$$

This gives:

$$\begin{aligned}
R'^1\sigma_1\widetilde{L}'^1 &= e^{3ia^0}R^1\sigma_1\widetilde{L}^1;\ R'^1\sigma_1L'^8 = e^{3ia^0}R^1\sigma_1L^8,\\
L'^1L'^8 &= L^1L^8;\ \widetilde{R}'^8\sigma_1\widetilde{L}'^1 = e^{i(1+2p)a^0}\widetilde{R}^8\sigma_1\widetilde{L}^1,\\
R'^1R'^8 &= e^{2i(1-p)a^0}R^1R^8;\ \widetilde{R}'^8\sigma_1L'^8 = e^{i(1+2p)a^0}\widetilde{R}^8\sigma_1L^8.
\end{aligned} \tag{2.111}$$

We then have:

$$\mathrm{D}'^1_R = R'^1\widetilde{R}'^1 = e^{2ia^0}R^1e^{-2ia^0}\widetilde{R}^1 = R^1\widetilde{R}^1 = \mathrm{D}^1_R, \tag{2.112}$$

and similarly:

$$\mathrm{D}'^8_R = \mathrm{D}^8_R;\ \mathrm{D}'^1_L = \mathrm{D}^1_L;\ \mathrm{D}'^8_L = \mathrm{D}^8_L;\ \mathrm{D}'^8_R = \mathrm{D}^8_R;\ \mathrm{J}'_l = \mathrm{J}_l;\ \mathrm{v}' = \mathrm{v}. \tag{2.113}$$

We hence get:

$$\begin{aligned}
\widehat{\nabla}R'^1 &= \widehat{\nabla}(e^{2ia^0}R^1) = e^{2ia^0}[2i(\widehat{\nabla}a^0)R^1 + \widehat{\nabla}R^1]\\
&= e^{2ia^0}[2i(\widehat{\nabla}a^0)R^1 - i(2\widehat{\mathrm{b}} + \mathbf{r}^1\widehat{\mathrm{v}})R^1]\\
&= i[2e^{2ia^0}(\widehat{\nabla}a^0 - \widehat{\mathrm{b}}) - \mathbf{r}^1\widehat{\mathrm{v}}e^{2ia^0}]R^1 = -i(2\widehat{\mathrm{b}}' + \mathbf{r}^1\widehat{\mathrm{v}})R'^1, \qquad (2.114)\\
\widehat{\mathrm{b}}' &= \widehat{\mathrm{b}} - \widehat{\nabla}a^0. \qquad (2.115)
\end{aligned}$$

The gauge invariance with P_0 of the other parts of the leptonic wave acts in the same way; this is the case with (2.115) as well as for Ψ^8_R, Ψ^1_R or Ψ^n_L.

2.3.2 The $SU(2)$ gauge group

This gauge group acts differently on the left and right parts of the waves. For the left parts we only need to think about $\Psi_L = \Psi_L^1 + \Psi_L^8$. The gauge transformation reads:

$$\Psi'_L = U(\Psi_L) = \cos(\theta)\Psi_L + \sin(\theta)[s_1\Psi_L\gamma_3 + s_2\Psi_L\gamma_3(-\mathbf{i}) + s_3\Psi_L(-\mathbf{i})]. \tag{2.116}$$

The gauge invariance means that with:

$$\mathbf{D} = (0 \;\; \mathrm{D}) = \boldsymbol{\partial} + \mathbf{G};\; \boldsymbol{\partial} = (0 \;\; \nabla);\; (0 \;\; \mathrm{D}') = \mathbf{D}' = \boldsymbol{\partial} + \mathbf{G}', \tag{2.117}$$

we must have:

$$\mathbf{D}'\Psi'_L = U(\mathbf{D}\Psi_L), \tag{2.118}$$

which needs:

$$\mathbf{G}'(\Psi'_L) = U(X) + U(Y);\; X = [\boldsymbol{\partial}(U^{-1})](\Psi'_L);\; Y = \mathbf{G}(\Psi_L) \tag{2.119}$$
$$X = \boldsymbol{\partial}(\cos\theta)\Psi'_L - [\boldsymbol{\partial}(s_1\sin\theta)\Psi'_L\gamma_3 + \boldsymbol{\partial}(s_2\sin\theta)\Psi'_L\gamma_3(-\mathbf{i})$$
$$+ \boldsymbol{\partial}(s_3\sin\theta)\Psi'_L(-\mathbf{i})].$$

We thus get:

$$\mathbf{w}^j = (0 \;\; \mathrm{w}^j);\; j = 1,2,3, \tag{2.120}$$

$$\begin{aligned} U(X) = &-[s_1\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_1 + \sin^2(\theta)(s_2\boldsymbol{\partial}s_3 - s_3\boldsymbol{\partial}s_2)]\Psi'_L\gamma_3 \\ &- [s_2\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_2 + \sin^2(\theta)(s_3\boldsymbol{\partial}s_1 - s_1\boldsymbol{\partial}s_3)]\Psi'_L\gamma_3(-\mathbf{i}) \\ &- [s_3\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_3 + \sin^2(\theta)(s_1\boldsymbol{\partial}s_2 - s_2\boldsymbol{\partial}s_1)]\Psi'_L(-\mathbf{i}). \end{aligned} \tag{2.121}$$

$$\begin{aligned} U(Y) = &\cos(2\theta)[\mathbf{w}^1\Psi'_L\gamma_3 + \mathbf{w}^2\Psi'_L\gamma_3(-\mathbf{i}) + \mathbf{w}^3\Psi'_L(-\mathbf{i})] + \sin(2\theta) \\ &\times [(s_2\mathbf{w}^3 - s_3\mathbf{w}^2)\Psi'_L\gamma_3\mathbf{i} + (s_3\mathbf{w}^1 - s_1\mathbf{w}^3)\Psi'_L\gamma_3(-\mathbf{i}) \\ &\quad + (s_1\mathbf{w}^2 - s_2\mathbf{w}^1)\Psi'_L(-\mathbf{i})] \\ &+ 2\sin^2(\theta)(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3)[s_1\Psi'_L\gamma_3 + s_2\Psi'_L\gamma_3(-\mathbf{i}) + s_3\Psi'_L(-\mathbf{i})]. \end{aligned} \tag{2.122}$$

Finally we thus have:

$$\begin{aligned} \mathbf{w}'^1 = &-[s_1\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_1 + \sin^2(\theta)(s_2\boldsymbol{\partial}s_3 - s_3\boldsymbol{\partial}s_2)] \\ &+ \cos(2\theta)\mathbf{w}^1 + \sin(2\theta)(s_2\mathbf{w}^3 - s_3\mathbf{w}^2) + 2\sin^2(\theta)s_1(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3), \end{aligned} \tag{2.123}$$

$$\begin{aligned} \mathbf{w}'^2 = &-[s_2\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_2 + \sin^2(\theta)(s_3\boldsymbol{\partial}s_1 - s_1\boldsymbol{\partial}s_3)] \\ &+ \cos(2\theta)\mathbf{w}^2 + \sin(2\theta)(s_3\mathbf{w}^1 - s_1\mathbf{w}^3) + 2\sin^2(\theta)s_2(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3), \end{aligned} \tag{2.124}$$

$$\begin{aligned} \mathbf{w}'^3 = &-[s_3\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_3 + \sin^2(\theta)(s_1\boldsymbol{\partial}s_2 - s_2\boldsymbol{\partial}s_1)] \\ &+ \cos(2\theta)\mathbf{w}^3 + \sin(2\theta)(s_1\mathbf{w}^2 - s_2\mathbf{w}^1) + 2\sin^2(\theta)s_3(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3). \end{aligned} \tag{2.125}$$

We may note the three-fold symmetry of these equalities: $SU(2)$ is a 3-dimensional Lie group. This symmetry is no longer an "internal" symmetry but an invariance under a geometrical group emerging from the properties of multiplication in the Cl_3 algebra.

Gauge generated by P_3

We arrive at the one-parameter group generated by P_3 with $a^0 = s_1 = s_2 = 0$ and $s_3 = 1$. We thus get:

$$\begin{aligned} S &= s = \theta P_3, \\ \Psi' &= [\exp(S)](\Psi) = P_-(\Psi) + \cos(\theta)P_+(\Psi) + \sin(\theta)P_3(\Psi) \\ &= \begin{pmatrix} R^1 + e^{-i\theta}L^1 & \widetilde{R}^8 + e^{i\theta}\widetilde{L}^8 \end{pmatrix}. \end{aligned} \tag{2.126}$$

So we have:

$$\begin{aligned} &R'^1 = R^1;\ \widetilde{R}'^8 = \widetilde{R}^8;\ L'^1 = e^{-i\theta}L^1;\ \widetilde{L}'^8 = e^{i\theta}\widetilde{L}^8;\ \mathrm{J}'_l = \mathrm{J}_l, \\ &L'^{\overline{1}} = L^{\overline{1}};\ \widetilde{L}'^{\overline{8}} = \widetilde{L}^{\overline{8}};\ R'^{\overline{1}} = e^{-i\theta}R^{\overline{1}};\ \widetilde{R}'^{\overline{8}} = e^{i\theta}\widetilde{R}^{\overline{8}}. \end{aligned} \tag{2.127}$$

We also have $\mathrm{w}'^0_\mu = \mathrm{w}^0_\mu$, which means $\mathrm{b}' = \mathrm{b}$. The equations (2.124) become

$$\begin{aligned} \mathrm{w}'^1 &= \cos(2\theta)\mathrm{w}^1 - \sin(2\theta)\mathrm{w}^2, \\ \mathrm{w}'^2 &= \cos(2\theta)\mathrm{w}^2 + \sin(2\theta)\mathrm{w}^1, \\ \mathrm{w}'^3 &= -i\nabla\theta + \mathrm{w}^3. \end{aligned} \tag{2.128}$$

And we have:

$$\begin{aligned} \mathrm{D}'^{18}_L - id'^{18}_L &= 2L'^1L'^8 = 2e^{-i\theta}L^1e^{-i\theta}L^8 = e^{-2i\theta}(\mathrm{D}^{18}_L - id^{18}_L) \\ &= \cos(2\theta)\mathrm{D}^{18}_L - \sin(2\theta)d^{18}_L - i[\sin(2\theta)\mathrm{D}^{18}_L + \cos(2\theta)d^{18}_L], \\ \mathrm{D}'^{18}_L &= \cos(2\theta)\mathrm{D}^{18}_L - \sin(2\theta)d^{18}_L, \\ d'^{18}_L &= \sin(2\theta)\mathrm{D}^{18}_L + \cos(2\theta)d^{18}_L. \end{aligned} \tag{2.129}$$

This is compatible with:

$$\begin{aligned} &(\Box + m_W^2)W^1 = \mathrm{D}^{18}_L;\ (\Box + m_W^2)W^2 = d^{18}_L;\ (\Box + m_W^2)\mathrm{w}^1 = \frac{g_2}{2}\mathrm{D}^{18}_L, \\ &(\Box + m_W^2)\mathrm{w}^2 = \frac{g_2}{2}d^{18}_L;\ W^1 = (\Box + m_W^2)^{-1}\mathrm{D}^{18}_L;\ W^2 = (\Box + m_W^2)^{-1}d^{18}_L, \end{aligned} \tag{2.130}$$

where $\Box$ is the Dalembertian operator ($\Box = \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3$). The g_2 coefficient named "coupling constant" is necessary to transform the contravariant vector W^j into a covariant vector w_j (see 1.7).

Gauge generated by P_1

We now have $a^0 = s_2 = s_3 = 0$ and $s_1 = 1$. We then have:

$$\begin{aligned} S &= s = \theta P_1, \\ \Psi' &= [\exp(S)](\Psi) = P_-(\Psi) + \cos(\theta)P_+(\Psi) + \sin(\theta)P_1(\Psi) \qquad (2.131) \\ &= \left(R^1 + \cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8 \quad \widetilde{R}^8 + \cos(\theta)\widetilde{L}^8 + i\sin(\theta)L^1 \right). \end{aligned}$$

So we get:

$$\begin{aligned} &R'^1 = R^1;\;\; R'^8 = R^8;\;\; L'^{\overline{1}} = L^{\overline{1}};\;\; L'^{\overline{8}} = L^{\overline{8}}, \\ &L'^1 = \cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8;\;\; R'^{\overline{1}} = \cos(\theta)R^{\overline{1}} + i\sin(\theta)\widetilde{R}^{\overline{8}}, \qquad (2.132) \\ &\widetilde{L}'^8 = \cos(\theta)\widetilde{L}^8 + i\sin(\theta)L^1;\;\; \widetilde{R}'^{\overline{8}} = \cos(\theta)\widetilde{R}^{\overline{8}} + i\sin(\theta)R^{\overline{1}}. \end{aligned}$$

Hence we have:

$$\begin{aligned} \mathrm{D}_L'^1 &= L'^1\widetilde{L}'^1 = [\cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8][\cos(\theta)\widetilde{L}^1 - i\sin(\theta)L^8] \\ &= \cos^2(\theta)L^1\widetilde{L}^1 + i\sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 - L^1L^8) + \sin^2(\theta)\widetilde{L}^8L^8, \\ \mathrm{D}_L'^8 &= \widetilde{L}'^8L'^8 = [\cos(\theta)\widetilde{L}^8 + i\sin(\theta)L^1][\cos(\theta)L^8 - i\sin(\theta)\widetilde{L}^1] \qquad (2.133) \\ &= \sin^2(\theta)L^1\widetilde{L}^1 - i\sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 - L^1L^8) + \cos^2(\theta)\widetilde{L}^8L^8. \end{aligned}$$

We deduce the following:

$$\mathrm{D}_L'^1 + \mathrm{D}_L'^8 = \mathrm{D}_L^1 + \mathrm{D}_L^8;\;\; \mathrm{J}_l' = \mathrm{J}_l;\; \rho_l' = \rho_l;\; \mathrm{v}' = \mathrm{v}. \qquad (2.134)$$

The mass term is thus invariant under the gauge transformation. We also derive from (2.133):

$$\mathrm{D}_L'^8 - \mathrm{D}_L'^1 = \cos(2\theta)(\mathrm{D}_L^8 - \mathrm{D}_L^1) + \sin(2\theta)d_L^{18}. \qquad (2.135)$$

Next we have:

$$\begin{aligned} 2L'^1L'^8 &= \mathrm{D}_L'^{18} - id_L'^{18} \qquad (2.136) \\ &= 2[\cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8][\cos(\theta)L^8 - i\sin(\theta)\widetilde{L}^1] \\ &= \mathrm{D}_L^{18} - i[\cos(2\theta)d_L^{18} - \sin(2\theta)(\mathrm{D}_L^8 - \mathrm{D}_L^1)]. \end{aligned}$$

Equations (2.124) become:

$$\begin{aligned} \mathrm{w}'^2 &= \cos(2\theta)\mathrm{w}^2 - \sin(2\theta)\mathrm{w}^3, \\ \mathrm{w}'^3 &= \cos(2\theta)\mathrm{w}^3 + \sin(2\theta)\mathrm{w}^2, \\ \mathrm{w}'^1 &= -\nabla\theta + \mathrm{w}^1. \qquad (2.137) \end{aligned}$$

All that is thus compatible with:

$$(\Box + m_W^2)W^3 = (\mathrm{D}_L^8 - \mathrm{D}_L^1);\; W^3 = (\Box + m_W^2)^{-1}(\mathrm{D}_L^8 - \mathrm{D}_L^1). \qquad (2.138)$$

Gauge generated by P_2

We now have $a^0 = s_1 = s_3 = 0$ and $s_2 = 1$. We then get:

$$\Psi_L = \begin{pmatrix} L^1 & \widetilde{L}^8 \end{pmatrix};\ P_2(\Psi_L) = \Psi_L \gamma_3 = \begin{pmatrix} \widetilde{L}^8 & -L^1 \end{pmatrix}, \tag{2.139}$$

$$S = s = \theta P_2,$$

$$\Psi' = [\exp(S)](\Psi) = P_-(\Psi) + \cos(\theta)P_+(\Psi) + \sin(\theta)P_2(\Psi) \tag{2.140}$$

$$= \begin{pmatrix} R^1 + \cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8 & \widetilde{R}^8 + \cos(\theta)\widetilde{L}^8 - \sin(\theta)L^1 \end{pmatrix}.$$

Hence we have:

$$R'^1 = R^1;\quad R'^8 = R^8,\ \mathrm{D}'^1_R = \mathrm{D}^1_R;\quad \mathrm{D}'^8_R = \mathrm{D}^8_R;\quad L'^{\overline{1}} = L^{\overline{1}};\quad L'^{\overline{8}} = L^{\overline{8}},$$
$$L'^1 = \cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8;\quad R'^{\overline{1}} = \cos(\theta)R^{\overline{1}} - \sin(\theta)\widetilde{R}^{\overline{8}}, \tag{2.141}$$
$$\widetilde{L}'^8 = \cos(\theta)\widetilde{L}^8 - \sin(\theta)L^1;\quad \widetilde{R}'^{\overline{8}} = \cos(\theta)\widetilde{R}^{\overline{8}} + \sin(\theta)R^{\overline{1}}.$$

We may notice that the changes of sign when we pass from the wave of the particle to the wave of the antiparticle are the origin of what we saw in 2.1.2: the charge conjugation changes the rotation of the matrix indices for the doublet of right waves. We then get:

$$\mathrm{D}'^1_L = L'^1\widetilde{L}'^1 = [\cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8][\cos(\theta)\widetilde{L}^1 + \sin(\theta)L^8]$$
$$= \cos^2(\theta)L^1\widetilde{L}^1 + \sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 + L^1L^8) + \sin^2(\theta)\widetilde{L}^8L^8,$$
$$\mathrm{D}'^8_L = \widetilde{L}'^8L'^8 = [\cos(\theta)\widetilde{L}^8 - \sin(\theta)L^1][\cos(\theta)L^8 - \sin(\theta)\widetilde{L}^1] \tag{2.142}$$
$$= \sin^2(\theta)L^1\widetilde{L}^1 - \sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 + L^1L^8) + \cos^2(\theta)\widetilde{L}^8L^8.$$

We thus comes to:

$$\mathrm{D}'^1_L + \mathrm{D}'^8_L = \mathrm{D}^1_L + \mathrm{D}^8_L;\ \mathrm{J}'_l = \mathrm{J}_l. \tag{2.143}$$

We deduce first from (2.142) :

$$\mathrm{D}'^8_L - \mathrm{D}'^1_L = \cos(2\theta)(\mathrm{D}^8_L - \mathrm{D}^1_L) - \sin(2\theta)\mathrm{D}^{18}_L. \tag{2.144}$$

Next we have:

$$2L'^1L'^8 = \mathrm{D}'^{18}_L - id'^{18}_L \tag{2.145}$$
$$= 2[\cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8][\cos(\theta)L^8 - \sin(\theta)\widetilde{L}^1]$$
$$= -id^{18}_L + [\cos(2\theta)\mathrm{D}^{18}_L + \sin(2\theta)(\mathrm{D}^8_L - \mathrm{D}^1_L)].$$

Equations (2.124) become:

$$\mathrm{w}'^3 = \cos(2\theta)\mathrm{w}^3 - \sin(2\theta)\mathrm{w}^1,$$
$$\mathrm{w}'^1 = \cos(2\theta)\mathrm{w}^1 + \sin(2\theta)\mathrm{w}^3,$$
$$\mathrm{w}'^2 = -\nabla\theta + \mathrm{w}^2. \tag{2.146}$$

That is compatible with (2.130) and (2.138).

2.3.3 Simplification of the equations

Since W^3 and B are, like J_l, linear combinations of the chiral currents D^1_R, D^8_R, D^1_L and D^8_L, and since $\widetilde{L}^1\widehat{L}^1 = 0$, we have:

$$(\Box + m_W^2)[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1] = i[2\widetilde{L}^8\widetilde{L}^1\overline{L}^8 - (\widetilde{L}^8L^8 - L^1\widetilde{L}^1)\widehat{L^1}]$$
$$= i[\widetilde{L}^8(2\widetilde{L}^1\overline{L}^8 - L^8\widehat{L}^1)]. \qquad (2.147)$$

And we have:

$$\widetilde{L}^1\overline{L}^8 = 2\begin{pmatrix} 0 & 0 \\ -\eta^1_2 & \eta^1_1 \end{pmatrix}\begin{pmatrix} \eta^8_1 & 0 \\ \eta^8_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -a_2 & 0 \end{pmatrix} = -a_2\frac{\sigma_1 - i\sigma_2}{2},$$
$$L^8\widehat{L}^1 = \overline{\widetilde{L}^1\overline{L}^8} = a_2\frac{\sigma_1 - i\sigma_2}{2} = -\widetilde{L}^1\overline{L}^8, \qquad (2.148)$$
$$(\Box + m_W^2)[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1] = [-3\widetilde{L}^8L^8\widehat{L}^1] = [-3\mathrm{D}^8_L\widehat{L}^1]$$
$$= [-3(\mathrm{D}^8_L - \mathrm{D}^1_L)\widehat{L}^1] = (\Box + m_W^2)[-3W^3\widehat{L}^1]. \qquad (2.149)$$

And similarly we get:

$$(\Box + m_W^2)[(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8] = [2L^1L^8\widehat{L}^1 + (\widetilde{L}^8L^8 - L^1\widetilde{L}^1)\overline{L}^8]$$
$$= [2L^1L^8\widehat{L}^1 - L^1\widetilde{L}^1\overline{L}^8] = [-3L^1\widetilde{L}^1\overline{L}^8] = [-3\mathrm{D}^1_L\overline{L}^8]$$
$$= [3(\widetilde{L}^8L^8 - L^1\widetilde{L}^1)\overline{L}^8] = (\Box + m_W^2)[3W^3\overline{L}^8]. \qquad (2.150)$$

Then the equations of the left waves can be expressed as follows, by simplifying the three terms W^j:

$$0 = (\nabla + i\mathrm{b} + 3i\mathrm{w}^3 + i\mathbf{l}^1\mathrm{v})\widehat{L}^1;\ 0 = (\nabla + i\mathrm{b} - 3i\mathrm{w}^3 + i\mathbf{l}^8\mathrm{v})\widehat{L}^8. \qquad (2.151)$$

The four spinor wave equations of the lepton wave become

$$i\nabla\eta^1 = (\mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}^1\mathrm{v})\eta^1;\ i\widehat{\nabla}\xi^1 = (2\widehat{\mathrm{b}} + \mathbf{r}^1\widehat{\mathrm{v}})\xi^1, \qquad (2.152)$$
$$i\widetilde{\nabla}\eta^8 = (\mathrm{b} - 3\mathrm{w}^3 + \mathbf{l}^8\mathrm{v})\eta^8;\ i\overline{\nabla}\xi^8 = (2p\widehat{\mathrm{b}} + \mathbf{r}^8\widehat{\mathrm{v}})\xi^8. \qquad (2.153)$$

With (2.12) to (2.15), that corresponds to:

$$\mathrm{a}^1 = \mathrm{b} + 3\mathrm{w}^3;\ l^1 = \mathrm{a}^1 + \mathbf{l}^1\mathrm{v} = \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}^1\mathrm{v};\ 0 = -i\nabla\eta^1 + l^1\eta^1, \quad (2.154)$$
$$\mathrm{a}^2 = 2\mathrm{b};\ r^1 = \mathrm{a}^2 + \mathbf{r}^1\mathrm{v} = 2\mathrm{b} + \mathbf{r}^1\mathrm{v};\ 0 = -i\widehat{\nabla}\xi^1 + \widehat{r}^1\xi^1, \qquad (2.155)$$
$$\mathrm{a}^3 = \mathrm{b} - 3\mathrm{w}^3;\ l^8 = \mathrm{a}^3 + \mathbf{l}^8\mathrm{v} = \mathrm{b} - 3\mathrm{w}^3 + \mathbf{l}^8\mathrm{v};\ 0 = -i\widetilde{\nabla}\eta^8 + l^8\eta^8, \quad (2.156)$$
$$\mathrm{a}^4 = 2p\mathrm{b};\ r^8 = \mathrm{a}^4 + \mathbf{r}^8\mathrm{v} = 2p\mathrm{b} + \mathbf{r}^8\mathrm{v};\ 0 = -i\overline{\nabla}\xi^8 + \widehat{r}^8\xi^8. \qquad (2.157)$$

Equations (2.154) and (2.155) join into:

$$0 = \nabla\widehat{\phi}^1\sigma_{21} + (\mathbf{b} + 3\mathbf{w}^3)\widehat{\phi}^1p_r + 2\mathbf{b}\widehat{\phi}^1p_l + \mathrm{v}\widehat{\phi}^1\mathbf{m}^1, \qquad (2.158)$$
$$p_r = \frac{1 + \sigma_3}{2};\quad p_l = \frac{1 - \sigma_3}{2};\quad \mathbf{m}^1 := \begin{pmatrix} \mathbf{l}^1 & 0 \\ 0 & \mathbf{r}^1 \end{pmatrix} = \mathbf{l}^1p_r + \mathbf{r}^1p_l.$$

Equations (2.156) and (2.157) join into:

$$0 = \widetilde{\nabla}\widehat{\phi}^8\sigma_{21} + (\mathbf{b} - 3\mathbf{w}^3)\widehat{\phi}^8p_r + 2p\mathbf{b}\widehat{\phi}^8p_l + \mathrm{v}\widehat{\phi}^8\mathbf{m}^8;\ \mathbf{m}^8 = \mathbf{l}^8p_r + \mathbf{r}^8p_l. \quad (2.159)$$

2.3.4 Double link with the Lagrangian density

From the left side we multiply (2.154) by $\eta^{1\dagger}$, (2.155) by $\xi^{1\dagger}$, (2.156) by $\eta^{8\dagger}$ and (2.157) by $\xi^{8\dagger}$:

$$0 = \mathcal{L}^1 = -i\eta^{1\dagger}\nabla\eta^1 + \eta^{1\dagger}l^1\eta^1, \tag{2.160}$$

$$0 = \mathcal{L}^2 = -i\xi^{1\dagger}\widehat{\nabla}\xi^1 + \xi^{1\dagger}\widehat{r}^1\xi^1, \tag{2.161}$$

$$0 = \mathcal{L}^3 = -i\eta^{8\dagger}\widetilde{\nabla}\eta^8 + \eta^{8\dagger}l^8\eta^8, \tag{2.162}$$

$$0 = \mathcal{L}^4 = -i\xi^{8\dagger}\overline{\nabla}\xi^8 + \xi^{8\dagger}\widehat{r}^8\xi^8, \tag{2.163}$$

$$0 = \mathcal{L} = \frac{m}{k\mathbf{l}^1}\mathcal{L}^1 + \frac{m}{k\mathbf{r}^1}\mathcal{L}^2 + \frac{m}{k\mathbf{l}^8}\mathcal{L}^3 + \frac{m}{k\mathbf{r}^8}\mathcal{L}^4. \tag{2.164}$$

From the construction itself of that Lagrangian density each $\mathcal{L}^n$ is stationary, hence also the $\mathcal{L}$ density, since they are identically null at any point in space-time, and not only on average. No physical principle is used to obtain this result: the Lagrangian density is null as a sum of null terms. **The principle of least action is no longer a quasi-metaphysical principle**. We will read each of the terms of this Lagrangian density as the sum of a real part and an imaginary part. We may first remark that we will repeat the same procedure four times [8], and thus it is enough to completely work out the only $\mathcal{L}^1$ part. We have:

$$\begin{aligned}
\mathcal{L}^1 &= \frac{1}{2}(\mathcal{L}^1 + \mathcal{L}^{1\dagger}) + \frac{1}{2}(\mathcal{L}^1 - \mathcal{L}^{1\dagger}), \\
\mathcal{L}^1 + \mathcal{L}^{1\dagger} &= -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + \eta^{1\dagger}l^1\eta^1 + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}l^{1\dagger}\eta^1 \\
&= -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}(l^1 + l^{1\dagger})\eta^1. \qquad (2.165)\\
\mathcal{L}^1 - \mathcal{L}^{1\dagger} &= -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + \eta^{1\dagger}l^1\eta^1 - i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 - \eta^{1\dagger}l^{1\dagger}\eta^1 \\
&= -i\partial_\mu(\eta^{1\dagger}\sigma^\mu\eta^1) + \eta^{1\dagger}(l^1 - l^{1\dagger})\eta^1.
\end{aligned}$$

Each l^n and r^n is a sum of vectors, and space-time vectors form the self-adjoint part of the space algebra. We then have:

$$\begin{aligned}
&\frac{1}{2}(l^1 + l^{1\dagger}) = l^1 = l^1_\mu\sigma^\mu;\ \frac{1}{2}(l^1 - l^{1\dagger}) = 0, \\
0 &= \frac{1}{2}(\mathcal{L}^1 + \mathcal{L}^{1\dagger}) = \frac{1}{2}[-i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1] + \eta^{1\dagger}l^1\eta^1, \qquad (2.166)\\
0 &= \frac{1}{2}(\mathcal{L}^1 - \mathcal{L}^{1\dagger}) = \frac{1}{2}[-i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) - i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1] = -\frac{i}{2}\partial_\mu \mathrm{D}_L^{1\mu}.
\end{aligned}$$

This last relation means that the D^1_L current is conservative. And since the three other vectors behave similarly, the D^1_R, D^8_L and D^8_R currents are also conservative. Therefore the $\mathcal{L}^n$ terms which have a null imaginary part are

8. When we pass from η to ξ we simply replace σ^μ with $\widehat{\sigma}^\mu$, and so we have only three signs to change; otherwise all is similar.

real, and thus equal to their real part:

$$0 = \mathcal{L}^n = \Re(\mathcal{L}^n). \tag{2.167}$$

Now we completely develop this equation, with the help of the following real matrix representation:

$$\eta^1 = \begin{pmatrix} a+ib \\ c+id \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix}, \tag{2.168}$$

$$\eta^{1\dagger} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \end{pmatrix}; \; -i\eta^{1\dagger} = \begin{pmatrix} -b & a & -d & c \\ -a & -b & -c & -d \end{pmatrix}, \tag{2.169}$$

$$\nabla = \sigma^\mu \partial_\mu = \begin{pmatrix} \partial_0 - \partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & \partial_0 - \partial_3 & \partial_2 & -\partial_1 \\ -\partial_1 & \partial_2 & \partial_0 + \partial_3 & 0 \\ -\partial_2 & -\partial_1 & 0 & \partial_0 + \partial_3 \end{pmatrix}, \tag{2.170}$$

$$l^1 = \begin{pmatrix} l^{10} + l^{13} & 0 & l^{11} & l^{12} \\ 0 & l^{10} + l^{13} & -l^{12} & l^{11} \\ l^{11} & -l^{12} & l^{10} - l^{13} & 0 \\ l^{12} & l^{11} & 0 & l^{10} - l^{13} \end{pmatrix}. \tag{2.171}$$

The matrix of η^1 contains a, b, c, d once in each column, and it is the same for each row in $\eta^{1\dagger}$ and in $-i\eta^{1\dagger}$. There are only two $-$ signs in the right column of η^1. There are two $-$ signs and two $+$ signs in each row and either four $+$ or four $-$ in the other row of $\eta^{1\dagger}$ and of $-i\eta^{1\dagger}$. For the 4×4 matrices, each row and each column contains exactly once the ∂_μ or the $l^{1\mu}$. We count exactly eight $+$ signs and eight $-$ signs in the ∇ matrix, and only four $-$ signs in the p_1 matrix, and these two matrices are symmetric. All this is obviously not at random but emerges from the properties of multiplication in the Cl_3 algebra, which themselves come from the anticommutative property of orthogonal vectors. Thus the Lagrangian density $\mathcal{L}^1$ satisfies:

$$\begin{aligned} 0 = \mathcal{L}^1 = &+ a\delta_0 b + c\delta_0 d + (aa + bb + cc + dd)l^{10} \\ &+ b\delta_1 c + d\delta_1 a + (ac + bd + ca + db)l^{11} \\ &+ a\delta_2 c + b\delta_2 d + (ad - bc - cb + da)l^{12} \\ &+ b\delta_3 a + c\delta_3 d = (aa + bb - cc - dd)l^{13}. \end{aligned} \tag{2.172}$$

where we use the notation:

$$u\delta_\mu v := u(\partial_\mu v) - (\partial_\mu u)v. \tag{2.173}$$

We remark that all the differential terms in the Lagrangian density are δ_μ terms. Each variable a, b, c, d is present once and only once with each δ_μ. And similarly each variable a, b, c, d is present once and only once with each

p_1^μ. These are all the properties that are necessary and sufficient to allow us to obtain the wave equations from the Lagrange equations, which is what we see now. The Lagrange equation relative to the parameter a is:

$$\frac{\partial \mathcal{L}}{\partial a} = \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu a)} \Big), \tag{2.174}$$

That gives the wave equation :

$$\begin{aligned} &\partial_0 b - \partial_1 d + \partial_2 c - \partial_3 b + 2(al^{10} + cl^{11} + dl^{12} + al^{13}) \\ &\qquad = \partial_0(-b) + \partial_1 d + \partial_2(-c) + \partial_3 b. \end{aligned} \tag{2.175}$$

The differential terms of the right part are exactly the opposites of the differential terms of the left part because the Lagrangian density contains only δ_μ. And there is exactly one term of each variable because each variable is contained once and only once with each value of μ. The origin of these properties is indeed the structure of the Cl_3 algebra. The factor of 2 comes from the fact that each variable is present twice as a factor of each $l^{1\mu}$, for the same reasons of structure and signs that result from the anticommutation. As a result we can simplify this wave equation:

$$0 = \partial_0 b - \partial_1 d + \partial_2 c - \partial_3 b + (al^{10} + cl^{11} + dl^{12} + al^{13}). \tag{2.176}$$

We can indeed use the same method to derive each Lagrange equation. For the b variable we obtain:

$$\begin{aligned} &-\partial_0 a + \partial_1 c + \partial_2 d + \partial_3 a + 2(bl^{10} + dl^{11} - cl^{12} + bl^{13}) \\ &= \partial_0(a) + \partial_1(-c) + \partial_2(-d) + \partial_3(-a), \\ 0 &= -\partial_0 a + \partial_1 c + \partial_2 d + \partial_3 a + (bl^{10} + dl^{11} - cl^{12} + bl^{13}). \end{aligned} \tag{2.177}$$

Next, these two wave equations may be combined into a single one by letting $a + ib = \eta_1^1$ and $c + id = \eta_2^1$:

$$0 = (\partial_0 - \partial_3)\eta_1^1 + (-\partial_1 + i\partial_2)\eta_2^1 + i[(l^{10} + l^{13})\eta_1^1 + (l^{11} - il^{12})\eta_2^1]. \tag{2.178}$$

We continue the calculation with the Lagrange equations relative to the c and d parameters. We simplify and group these equations and that gives:

$$0 = (-\partial_1 - i\partial_2)\eta_1^1 + (\partial_0 + \partial_3)\eta_2^1 + i[(l^{11} + il^{12})\eta_1^1 + (l^{10} - l^{13})\eta_2^1]. \tag{2.179}$$

Finally, we combine these two equations into a single one:

$$\begin{aligned} 0 &= \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} \eta_1^1 \\ \eta_2^1 \end{pmatrix} + i \begin{pmatrix} l^{10} + l^{13} & l^{11} - il^{12} \\ l^{11} + il^{12} & l^{10} - l^{13} \end{pmatrix} \begin{pmatrix} \eta_1^1 \\ \eta_2^1 \end{pmatrix} \\ 0 &= -i\nabla\eta^1 + l^1\eta^1. \end{aligned} \tag{2.180}$$

This calculation is often presented in a very concise manner, using the function ψ where $\psi(x, y, z, t)$ is a column matrix with four complex components, as if it could be a real number! This ultra-concise calculation is nevertheless always correct, and our complete and detailed calculation above is precisely the proof.

In sum, the equality $0 = \mathcal{L}$ is the necessary consequence of the wave equations of the four η^1, ξ^1, η^8 and ξ^8. And vice versa this real $\mathcal{L}$ density identically null, as a result of the anticommutation rule of the basis vectors that is itself a result of the equality $\mathbf{uu} = \mathbf{u} \cdot \mathbf{u}$ for any vector $\mathbf{u}$, from the dimension 3 of the physical space, from the fact that the space-time of special relativity is the self-adjoint part of the Clifford algebra, well this equation, by the Lagrange equations, gives exactly the four wave equations where we began. [9] . Moreover, when we vary the density with regard to one of the spinor variables, we calculate as if the potentials do not depend on these spinors. And yet they depend on these spinors as in any coherent field theory. We will also see how all terms (potentials and mass) present in the Lagrangian density come down *in fine* to linear combinations of the chiral currents D_R^n and D_L^n. It then happens that the calculation of the Lagrange equations is nevertheless correct as a result of identities that suppress consequences of this dependence. It is the reason why the gauge potentials seem exterior to the wave even if they strictly depend on the spinors. To see this we consider for instance the term: $B_\mu \eta^{1\dagger}\sigma^\mu\eta^1 = \eta^{1\dagger}B\eta^1$. When we derive this term in η^1, we suppose that this derivation does not affect B. Nevertheless this potential B may include a term depending on $\mathrm{D}_L^1 = L^1\widetilde{L}^1$. In practice this term gives no supplementary contribution to the Lagrange equation because:

$$\eta^{1\dagger}L^1 = \sqrt{2}\begin{pmatrix}\overline{\eta}_1^1 & \overline{\eta}_2^1\end{pmatrix}\begin{pmatrix}0 & -\overline{\eta}_2^1 \\ 0 & \overline{\eta}_1^1\end{pmatrix} = \sqrt{2}\begin{pmatrix}0 & -\overline{\eta}_1^1\overline{\eta}_2^1 + \overline{\eta}_2^1\overline{\eta}_1^1\end{pmatrix} = 0. \quad (2.181)$$

$$\eta^{1\dagger}\mathrm{D}_L^1\eta^1 = \eta^{1\dagger}L^1\widetilde{L}^1\eta^1 = 0.$$

In Lagrangian physics, the Lagrange equation is obtained by neglecting the term remaining after an integration by parts, which supposes that these terms may be neglected. In fact the thing that is neglected is just the proof that we may neglect the remaining terms! It was never really understood **why** Maxwell's equations governing the electromagnetic field or Einstein's equations governing the gravitational field must necessarily be derived from a Lagrangian formalism. The Lagrange equations are indeed usable to obtain a part of laws of electromagnetism, the part that links derivatives of fields to currents (details in **??**). This calculation is perfectly correct but

9. This double link between the wave equation and Lagrangian density is not a general property of Clifford algebra. There are dimensions and signatures that ensure an automatic derivation of wave equations from the only equation corresponding to the Clifford real part, but other dimensions or signatures do not ensure the same. Then there is no general principle behind the automatic behaviour of the Lagrangian formalism in relativistic quantum physics: it is only an inevitable consequence of the particular properties of space-time. These special features come from the dimension of time and space that are always respectively one and three in any tangent space to the space-time manifold.

those laws linking fields to potentials do not come from Lagrange equations. Consequently the potentials are considered as non physical tools of calculation, when they are part of the physical quantities in the electromagnetism of the de Broglie's photon [57] [58]. Moreover the propagation of the fields as far from the sources as desired, is not accounted for. This propagation *ad infinitum* without attenuation other than due to the distance from the source could invalidate the cancellation of the terms remaining after integration by parts: the increase of the volume exactly offsets the decrease in the magnitude of the potential terms.

If this Lagrangian physics acts perfectly, if the terms which must disappear are fortunately suppressed, that does not come from a metaphysical principle but rather results from the particular properties of the Cl_3 algebra. The double logical link has a purely algebraic origin. It is always valid for all interactions of physics, since it is directly linked to the dimension 3 of space and the dimension 1 of time, and also to the signature of space-time geometry. The equivalence between the usual form and the completely invariant form of the wave equations implies the logical deduction of the real equations forming the system of numerical equations of either linear or improved Dirac equation. That always possible logical deduction takes the form of Lagrange equations.

Therefore we are always correct when we use Noether's theorem which links conservative quantities to invariances of the Lagrangian density. We must hence note that it is enough to change the dimension or the signature of space-time to eventually lose the Lagrangian mechanism that is essential for the functioning of matter dynamics. [10]

The detailed study that we carried out on the $\mathcal{L}^1$ part of the Lagrangian density may indeed be extended to the three other parts of the leptonic wave and also to the antileptonic wave, which is the same wave simply with the change of ∇ into $-\nabla$ in the wave equation, and the exchange of left and right terms, and thus also in the Lagrangian density that is the real part of the wave equation.

2.3.5 Iteration and equations of gauge fields

For the leptonic wave the Lagrangian density of the Standard Model is made of two parts: a part describing the quantum wave of the electron and its neutrino and another part describing the gauge bosons. This means B and W^j, which are necessary for the gauge invariance. We saw that the building of the Lagrangian density, for the four spinors of the leptonic wave, was automatic. It is not at all the same for the boson part that is *not* relativistically linked to spinors. We saw in the first chapter that the

10. This is a sufficient reason for the general failure of theories with a great number of dimensions to obtain a realistic picture of the Standard Model of quantum physics. It is the same for any theory based on Riemannian manifolds, when using only an indeterminate n dimension.

classical link between potentials and fields is not the same as in classical physics, since the electromagnetic field is changed into a field of operators. With the Standard Model, a boson field is always a field of operators. The existence of a Lagrangian density for the boson part of the Standard Model is hence questionable. We have nevertheless the possibility of completely get around this not-fully-justified part of the Lagrangian density by the use of the fully-justified part and only this part. This part of the Lagrangian density comes from the fermionic wave equations which we will use with a functional recursive form. We again use the decomposition (2.100) of Ψ_l into its four chiral parts. We let:

$$\mathrm{p}_1 := l^1;\ \mathrm{p}_2 := r^1;\ \mathrm{p}_3 := l^8;\ \mathrm{p}_4 := r^8;\ \mathrm{p}_n^{-1} = \mathrm{p}_{n\mu}^{-1}\widehat{\sigma}^\mu. \tag{2.182}$$

With (2.154) to (2.157) we get:

$$\eta^1 = i\mathrm{p}_1^{-1}\nabla\eta^1;\ \nabla\eta^1 = -i\mathrm{p}_1\eta_1, \tag{2.183}$$
$$\xi^1 = i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}\xi^1;\ \widehat{\nabla}\xi^1 = -i\widehat{\mathrm{p}}_2\xi^1, \tag{2.184}$$
$$\eta^8 = i\mathrm{p}_3^{-1}\widetilde{\nabla}\eta^8;\ \widetilde{\nabla}\eta^8 = -i\mathrm{p}_3\eta^8, \tag{2.185}$$
$$\xi^8 = i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}\xi^8;\ \overline{\nabla}\xi^8 = -i\widehat{\mathrm{p}}_4\xi^8. \tag{2.186}$$

We then get by iterating these equations:

$$\eta^1 = i\mathrm{p}_1^{-1}\nabla(i\mathrm{p}_1^{-1}\nabla\eta^1) = i\mathrm{p}_1^{-1}\nabla[i\mathrm{p}_1^{-1}\nabla(i\mathrm{p}_1^{-1}\nabla\eta^1)] \tag{2.187}$$
$$\xi^1 = i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}(i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}\xi^1) = i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}[i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}(i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}\xi^1)], \tag{2.188}$$
$$\eta^8 = i\mathrm{p}_3^{-1}\widetilde{\nabla}(i\mathrm{p}_3^{-1}\widetilde{\nabla}\eta^8) = i\mathrm{p}_3^{-1}\widetilde{\nabla}[i\mathrm{p}_3^{-1}\widetilde{\nabla}(i\mathrm{p}_3^{-1}\widetilde{\nabla}\eta^8)], \tag{2.189}$$
$$\xi^8 = i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}(i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}\xi^8) = i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}[i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}(i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}\xi^8)]. \tag{2.190}$$

These equations are not optional; they are an obligatory consequence of the wave equation of each spinor. Iterated once, wave equations allow us to define gauge fields from potential and mass terms included in the p_n^{-1}. The Standard Model has problems with the Yang–Mills fields, and we can now see a reason for these difficulties: Yang–Mills fields are not independent from the quantum waves since they are defined from wave equations. And this definition includes four parts following the four kinds of representations of the Cl_3^* Lie group. We now replace the column-spinors ξ and η by the corresponding elements in Cl_3. We get:

$$\nabla\widehat{L}^1 = -i\mathrm{p}_1\widehat{L}^1;\ \widehat{\nabla}(\nabla\widehat{L}^1) = -i\widehat{\nabla}(\mathrm{p}_1\widehat{L}^1), \tag{2.191}$$
$$\widehat{\nabla}R^1 = -i\widehat{\mathrm{p}}_2R^1;\ \nabla(\widehat{\nabla}R^1) = -i\nabla(\widehat{\mathrm{p}}_2R^1), \tag{2.192}$$
$$\nabla\overline{L}^8 = -i\mathrm{p}_3\overline{L}^8;\ \widehat{\nabla}(\nabla\overline{L}^8) = -i\widehat{\nabla}(\mathrm{p}_3\overline{L}^8), \tag{2.193}$$
$$\widehat{\nabla}\widetilde{R}^8 = -i\widehat{\mathrm{p}}_4\widetilde{R}^8;\ \nabla(\widehat{\nabla}\widetilde{R}^8) = -i\nabla(\widehat{\mathrm{p}}_4\widetilde{R}^8). \tag{2.194}$$

We now define the gauge fields F as follows:

$$\widehat{\nabla}(\mathrm{p}_1\widehat{L}^1) = F_L^1(\widehat{L}^1) + \widehat{\mathrm{p}}_1\nabla\widehat{L}^1, \tag{2.195}$$
$$\nabla(\widehat{\mathrm{p}}_2 R^1) = \widehat{F}_R^1(R^1) + \mathrm{p}_2\widehat{\nabla}R^1, \tag{2.196}$$
$$\widehat{\nabla}(\mathrm{p}_3\overline{L}^8) = F_L^8(\overline{L}^8) + \widehat{\mathrm{p}}_3\nabla\overline{L}^8, \tag{2.197}$$
$$\nabla(\widehat{\mathrm{p}}_4\widetilde{R}^8) = \widehat{F}_R^8(R^8) + \mathrm{p}_4\widehat{\nabla}\widetilde{R}^8. \tag{2.198}$$

In any physical theory of fields, a link exists between potential terms and field terms. For instance, the gravitational potential of the sun is not postulated but calculated from the equations of the gravitational field. **The main novelty of the previous relations is that the gauge fields are different with left waves and with right waves**. And we must recall that a photon is always a purely left or purely right wave. Since:

$$\Box\widehat{L}^1 = -i[F_L^1(\widehat{L}^1) + \widehat{\mathrm{p}}_1\nabla\widehat{L}^1] = -iF_L^1(\widehat{L}^1) - i\widehat{\mathrm{p}}_1(-i\mathrm{p}_1\widehat{L}^1), \tag{2.199}$$

and with:

$$\mathrm{p}_n^2 = \mathrm{p}_n \cdot \mathrm{p}_n = \mathrm{p}_n\widehat{\mathrm{p}}_n = \widehat{\mathrm{p}}_n\mathrm{p}_n, \tag{2.200}$$

The second-order wave equations read:

$$0 = (\Box + \mathrm{p}_1^2 + iF_L^1)(\widehat{L}^1), \tag{2.201}$$
$$0 = (\Box + \mathrm{p}_2^2 + i\widehat{F}_R^1)(R^1), \tag{2.202}$$
$$0 = (\Box + \mathrm{p}_3^2 + iF_L^8)(\overline{L}^8), \tag{2.203}$$
$$0 = (\Box + \mathrm{p}_4^2 + i\widehat{F}_R^8)(\widetilde{R}^8). \tag{2.204}$$

We hence get:

$$F_L^1(\widehat{L}^1) = i(\Box + \mathrm{p}_1^2)(\widehat{L}^1), \tag{2.205}$$
$$\widehat{F}_R^1(R^1) = i(\Box + \mathrm{p}_2^2)(R^1), \tag{2.206}$$
$$F_L^8(\overline{L}^8) = i(\Box + \mathrm{p}_3^2)(\overline{L}^8), \tag{2.207}$$
$$\widehat{F}_R^8(\widetilde{R}^8) = i(\Box + \mathrm{p}_4^2)(\widetilde{R}^8). \tag{2.208}$$

2.3.6 Weinberg–Salam angle

This parameter of the Standard Model is an angle which measures the mixing between the photon and the other gauge bosons of the $U(1) \times SU(2)$ group. This θ_W angle satisfies:

$$g_1 = \frac{q}{\cos(\theta_W)} \; ; \; g_2 = \frac{q}{\sin(\theta_W)} \; ; \; q = \frac{e}{\hbar c}, \tag{2.209}$$
$$-g_1 B + g_2 W^3 = \sqrt{g_1^2 + g_2^2}\, Z^0 = \frac{2q}{\sin(2\theta_W)} Z^0, \tag{2.210}$$

$$B = \cos(\theta_W)\mathrm{A} - \sin(\theta_W)Z^0 \ ; \ W^3 = \sin(\theta_W)\mathrm{A} + \cos(\theta_W)Z^0, \qquad (2.211)$$

$$B + iW^3 = e^{i\theta_W}(\mathrm{A} + iZ^0) \ ; \ \ \mathrm{A} + iZ^0 = e^{-i\theta_W}(B + iW^3). \qquad (2.212)$$

With the equations (2.147) and (2.148), by grouping together the three terms W^j, we replace $3W^3$ by W, and in the place of the previous equations we then let:

$$g_1 = \frac{q}{\cos(\theta_W)} \ ; \ \ g_2 = \frac{q}{\sin(\theta_W)} \ ; \ \ q = \frac{e}{\hbar c}, \qquad (2.213)$$

$$\mathrm{A} = \cos(\theta_W)B + \sin(\theta_W)W; \ Z^0 = -\sin(\theta_W)B + \cos(\theta_W)W, \qquad (2.214)$$

$$B = \cos(\theta_W)\mathrm{A} - \sin(\theta_W)Z^0 \ ; \ W = \sin(\theta_W)\mathrm{A} + \cos(\theta_W)Z^0, \qquad (2.215)$$

$$B + iW = e^{i\theta_W}(\mathrm{A} + iZ^0) \ ; \ \ \mathrm{A} + iZ^0 = e^{-i\theta_W}(B + iW). \qquad (2.216)$$

The system of wave equations of the electron is now expressed as:

$$0 = (\nabla + i\frac{g_1}{2}B + i\frac{g_2}{2}W + i\mathbf{l}^1\mathrm{v})\widehat{L}^1,$$
$$0 = (\nabla - ig_1B - i\mathbf{r}^1\mathrm{v})\widehat{R}^1. \qquad (2.217)$$

With the previous definitions that is equivalent to

$$0 = [\nabla - i(q\mathrm{A} + \mathbf{r}^1\mathrm{v}) + iqTZ^0]\widehat{R}^1; \ T = \tan(\theta_W),$$
$$0 = \Big[\nabla + i(q\mathrm{A} + \mathbf{l}^1\mathrm{v}) + i\frac{q}{2}\Big(-T + \frac{1}{T}\Big)Z^0\Big]\widehat{L}^1. \qquad (2.218)$$

Since there is only one way to express the X and Y terms as sum and difference: $X = 1/2(X+Y)+1/2(X-Y)$ and $Y = 1/2(X+Y)-1/2(X-Y)$, we recast this system in the form:

$$0 = \Big[\nabla - i[q\mathrm{A} + \mathbf{r}^1\mathrm{v}] - i\frac{q}{4}\Big(\frac{1}{T} - 3T\Big)Z^0 + i\frac{q}{4}\Big(\frac{1}{T} + T\Big)Z^0\Big]\widehat{R}^1,$$
$$0 = \Big[\nabla + i[q\mathrm{A} + \mathbf{l}^1\mathrm{v}] + i\frac{q}{4}\Big(\frac{1}{T} - 3T\Big)Z^0 + i\frac{q}{4}\Big(\frac{1}{T} + T\Big)Z^0\Big]\widehat{L}^1. \qquad (2.219)$$

We can obtain the wave equation of the electron (1.149) only if the Z^0 terms have only one sign, positive, thus only if $3T - 1/T$ cancels. And this is just the case if $\theta_W = 30°$, which we obtained in [32] via another reasoning, independent of the previous one. Moreover this result was also obtained by Stoica in a completely different manner [110], which supports this result and gives:

$$T = \frac{1}{\sqrt{3}}; \ \frac{1}{T} = \sqrt{3} = 3T; \ 3T - \frac{1}{T} = 0; \ \frac{q}{4}\Big(T + \frac{1}{T}\Big) = \frac{q}{\sqrt{3}}. \qquad (2.220)$$

We hence have:

$$0 = \Big[\nabla - i(q\mathrm{A} + \mathbf{r}^1\mathrm{v}) + i\frac{q}{\sqrt{3}}Z^0\Big]\widehat{R}^1,$$
$$0 = \Big[\nabla + i(q\mathrm{A} + \mathbf{l}^1\mathrm{v}) + i\frac{q}{\sqrt{3}}Z^0\Big]\widehat{L}^1. \qquad (2.221)$$

The rotation of 30° that the Weinberg–Salam angle makes, is thus shown to be the simple rewriting of the gauge terms as sum and difference of terms that apply to the left spinor and the right spinor of the electron. Moreover it turns out that the calculation of this angle from the experimental data through the approximation method of quantum field theory gives a value near 30° and which gets closer for the data with low energy–momentum.

2.3.7 Consequence for the neutrino–monopole

The Weinberg–Salam angle links several properties: the Z^0 boson has a proper mass greater than that of W^n bosons. The experimental ratio of masses is in the vicinity of the $2/\sqrt{3}$ ratio resulting from the 30° value of the Weinberg–Salam angle. Two other properties are the null electric charge of the neutrino and the null proper mass of the photon. The equations of the left and right waves of the neutrino–monopole are now:

$$0 = (-i\widetilde{\nabla} + \mathrm{b} - 3\mathrm{w}^3 + \mathbf{l}^8\mathrm{v})\overline{L}^8, \tag{2.222}$$

$$0 = (i\widetilde{\nabla} + 2p\mathrm{b} + \mathbf{r}^8\mathrm{v})\overline{R}^8. \tag{2.223}$$

With the 30° value of the angle, this becomes:

$$0 = \Big[-i\nabla + \frac{q}{2}(A - Z'^0) - \frac{q}{2}(A + Z'^0) + \mathbf{l}^8\mathrm{v}\Big]\overline{L}^8, \tag{2.224}$$

$$0 = [i\nabla + pq(A - Z'^0) + \mathbf{r}^8\mathrm{v}]\overline{R}^8;\ Z'^0 := \frac{Z^0}{\sqrt{3}}. \tag{2.225}$$

The potential A cancels out in the wave equation of $\overline{L}^8$, and this is the reason for the neutrino being neutral, which means without electric interaction. We recall that this system is equivalent to the single equation summing the two equations of the system, because the non zero terms in $\overline{L}^8$ and $\overline{R}^8$ form two independent columns of the same $\overline{\phi}^8$ matrix:

$$0 = \nabla\overline{\phi}^8(-i\sigma_3) - 2qZ'^0\overline{L}^8 + pqA\overline{R}^8 - pqZ'^0\overline{R}^8 + \mathbf{l}^8\mathrm{v}\overline{L}^8 + \mathbf{r}^8\mathrm{v}\overline{R}^8 \tag{2.226}$$

If $p = -2$, a value that we will also explain in Chapter 4, we can put together the two terms containing the Z^0 boson:

$$0 = \nabla\overline{\phi}^8\sigma_{21} + qA\overline{\phi}^8(1-\sigma_3) - 2qZ'^0\overline{\phi}^8\sigma_3 + \mathrm{v}\overline{\phi}^8\begin{pmatrix}\mathbf{l}^8 & 0\\ 0 & \mathbf{r}^8\end{pmatrix}, \tag{2.227}$$

$$0 = \nabla\overline{\phi}^8\sigma_{21} + qA\overline{\phi}^8(1-\sigma_3) - 2qZ'^0\overline{\phi}^8\sigma_3 + \mathrm{v}\overline{\phi}^8\mathbf{m}^8;\ \mathbf{m}^8 := \begin{pmatrix}\mathbf{l}^8 & 0\\ 0 & \mathbf{r}^8\end{pmatrix}.$$

2.4 Energy–momentum tensor

The Dirac equation uses a unique Lagrangian density but in fact several different Lagrangian densities are possible, all stationary because identically null. In (2.164) we let

$$0 = \mathcal{L} = \frac{m}{k\mathbf{l}^1}\mathcal{L}^1 + \frac{m}{k\mathbf{r}^1}\mathcal{L}^2 + \frac{m}{k\mathbf{l}^8}\mathcal{L}^3 + \frac{m}{k\mathbf{r}^8}\mathcal{L}^4.$$

We encountered in Chapter 1 the other density that may be formed from the single electron wave. These two densities are generalized as:

$$0 = \mathcal{L}^+ = \mathcal{L} = \frac{m}{k\mathbf{l}^1}\mathcal{L}^1 + \frac{m}{k\mathbf{r}^1}\mathcal{L}^2 + \frac{m}{k\mathbf{l}^8}\mathcal{L}^3 + \frac{m}{k\mathbf{r}^8}\mathcal{L}^4, \tag{2.228}$$
$$0 = \mathcal{L}^- = \frac{m}{k\mathbf{l}^1}\mathcal{L}^1 - \frac{m}{k\mathbf{r}^1}\mathcal{L}^2 + \frac{m}{k\mathbf{l}^8}\mathcal{L}^3 - \frac{m}{k\mathbf{r}^8}\mathcal{L}^4.$$

Each of these Lagrangian densities is invariant under the extended invariance group Cl_3^* . To each of these invariances is associated a conservative current (Noether's theorem). The energy–momentum tensor is the tensor associated with invariance under translation. Tetrode's tensor T is the tensor associated with $\mathcal{L}^+$. The tensor associated with $\mathcal{L}^-$ generalizes the non-interpreted tensor V of O. Costa de Beauregard [53]. [11] They satisfy:

$$T^\mu_\lambda = \Re\Big[\Big(\frac{m}{k\mathbf{l}^1}\eta^{1\dagger}\sigma^\mu \mathrm{d}^1_{L\lambda}\eta^1 + \frac{m}{k\mathbf{r}^1}\xi^{1\dagger}\widehat{\sigma}^\mu \mathrm{d}^1_{R\lambda}\xi^1 + \frac{m}{k\mathbf{l}^8}\eta^{8\dagger}\sigma^\mu \mathrm{d}^8_{L\lambda}\eta^8 + \frac{m}{k\mathbf{r}^8}\xi^{8\dagger}\widehat{\sigma}^\mu \mathrm{d}^8_{R\lambda}\xi^8\Big)\Big], \tag{2.229}$$

$$V^\mu_\lambda = \Re\Big[-\Big(\frac{m}{k\mathbf{l}^1}\eta^{1\dagger}\sigma^\mu \mathrm{d}^1_{L\lambda}\eta^1 - \frac{m}{k\mathbf{r}^1}\xi^{1\dagger}\widehat{\sigma}^\mu \mathrm{d}^1_{R\lambda}\xi^1 + \frac{m}{k\mathbf{l}^8}\eta^{8\dagger}\sigma^\mu \mathrm{d}^8_{L\lambda}\eta^8 - \frac{m}{k\mathbf{r}^8}\xi^{8\dagger}\widehat{\sigma}^\mu \mathrm{d}^8_{R\lambda}\xi^8\Big)\Big]. \tag{2.230}$$

where d_λ operators are defined as:

$$\mathrm{d}^1_{L\lambda}\eta^1 = (-i\partial_\lambda + l^1_\lambda)\eta^1, \tag{2.231}$$
$$\mathrm{d}^1_{R\lambda}\xi^1 = (-i\partial_\lambda + r^1_\lambda)\xi^1, \tag{2.232}$$
$$\mathrm{d}^8_{L\lambda}\eta^8 = (-i\partial_\lambda + l^8_\lambda)\eta^8, \tag{2.233}$$
$$\mathrm{d}^8_{R\lambda}\xi^8 = (-i\partial_\lambda + r^8_\lambda)\xi^8. \tag{2.234}$$

The energy–momentum tensor T is thus the sum of four tensors, one for each spinor of the leptonic wave:

$$T = \frac{m}{k\mathbf{l}^1}T^1_L + \frac{m}{k\mathbf{r}^1}T^1_R + \frac{m}{k\mathbf{r}^8}T^8_R + \frac{m}{k\mathbf{l}^8}T^8_L, \tag{2.235}$$
$$T^{1\,\mu}_{L\lambda} = \Re(\eta^{1\dagger}\sigma^\mu \mathrm{d}^1_{L\lambda}\eta^1). \tag{2.236}$$

11. In this note O. Costa de Beauregard pointed out that the V_{ij} tensor is non-interpreted, which means it is without equivalent in classical physics. We may see that this tensor is obtained [21] by replacing γ_0 with γ_3 in the definition of Tetrode's tensor. This replacement also changes the J current into the K current and is equivalent to the passing from $\mathcal{L}^+$ into $\mathcal{L}^-$. This induces the astonishing idea that **two different energy–momentum tensors exist with the wave of the electron**. The existence of two Lagrangian densities and of two energy–momentum tensors was first encountered in de Broglie's theory of the photon [57][58].

We obtain the three other parts simply by replacing η^1 with ξ^1, η^8 and ξ^8, and by the replacement of the σ^μ with $\widehat{\sigma}^\mu$ whenever we replace η by ξ. It is thus enough to study T^1_L and then to apply this procedure to the others. What we carry out here is the generalization of the study in Chapter 1. And so we may then again use the same method of calculation which, with (1.310), gives:

$$\begin{aligned}\partial_\mu T^{1\mu}_L &= \partial_\mu T^{1\mu}_{L\lambda}\sigma^\lambda = \Re\big[\partial_\mu[\eta^{1\dagger}\sigma^\mu(-i\partial_\lambda + l^1_\lambda)\eta^1]\big]\sigma^\lambda \\ &= \Re\big[\partial_\mu[-i\eta^{1\dagger}\sigma^\mu\partial_\lambda\eta^1 + l^1_\lambda \mathrm{D}^{1\mu}_L]\big]\sigma^\lambda. \end{aligned} \tag{2.237}$$

Next we use the wave equation of η^1, and that gives:

$$\nabla\eta^1 = -il^1\eta^1;\ \partial_\mu \mathrm{D}^{1\mu}_L = 0, \tag{2.238}$$

$$\begin{aligned}\partial_\mu T^{1\mu}_L &= \Re\big[[-i(\nabla\eta^1)^\dagger\partial_\lambda\eta^1 - i\eta^{1\dagger}\partial_\lambda(\nabla\eta^1) - \mathrm{D}^{1\mu}_L\partial_\mu l^1_\lambda]\big]\sigma^\lambda \\ &= \Re[-i(i\eta^{1\dagger}l^1)\partial_\lambda\eta^1 - i\eta^{1\dagger}\partial_\lambda(-il^1\eta^1) + (\partial_\mu l^1_\lambda)\mathrm{D}^{1\mu}_L]\sigma^\lambda \\ &= (\partial_\mu l^1_\lambda - \partial_\lambda l^1_\mu)\mathrm{D}^{1\mu}_L\sigma^\lambda. \end{aligned} \tag{2.239}$$

Similarly, with the three other spinors, we obtain:

$$\partial_\mu T^{1\mu}_R = (\partial_\mu r^1_\lambda - \partial_\lambda r^1_\mu)\mathrm{D}^{1\mu}_R\sigma^\lambda, \tag{2.240}$$

$$\partial_\mu T^{8\mu}_L = (\partial_\mu l^8_\lambda - \partial_\lambda l^8_\mu)\mathrm{D}^{8\mu}_L\sigma^\lambda, \tag{2.241}$$

$$\partial_\mu T^{8\mu}_R = (\partial_\mu r^8_\lambda - \partial_\lambda r^8_\mu)\mathrm{D}^{8\mu}_R\sigma^\lambda. \tag{2.242}$$

The complete electromagnetic field F, which means with magnetic monopoles is the sum of a purely electric field (electric charge + magnetic dipole) that we denote as F^e, and of a purely magnetic field (magnetic charge + electric dipole) that we denote as F^m. They satisfy the following:

$$\partial_\mu \mathrm{A}^\mu = 0;\ \partial_\mu Z^{0\mu} = 0, \tag{2.243}$$

$$\begin{aligned} F = F^e + F^m;\ F^e = \nabla\widehat{\mathrm{A}} = \vec{E} + i\vec{H};\ \vec{E} = -\partial_0\vec{\mathrm{A}} - \vec{\partial}\mathrm{A}_0;\ \vec{H} = \vec{\partial}\times\vec{\mathrm{A}} \\ F^m = \nabla i\widehat{Z'^0} = \vec{E}^m + i\vec{H}^m;\ \vec{E}^m = \vec{\partial}\times\vec{Z'^0};\ \vec{H}^m = \partial_0\vec{Z'}^0 + \vec{\partial}Z'^0_0. \end{aligned} \tag{2.244}$$

We hence get:

$$F^e_{\mu\lambda} = \partial_\mu A_\lambda - \partial_\lambda A_\mu;\ iF^m_{\mu\lambda} = \partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu, \tag{2.245}$$

$$\begin{aligned}\partial_\mu T^\mu &= \frac{m}{k\mathbf{l}}\partial_\mu T^{1\mu}_L + \frac{m}{k\mathbf{r}}\partial_\mu T^{1\mu}_R + \frac{m}{km_l}\partial_\mu T^{8\mu}_L + \frac{m}{km_r}\partial_\mu T^{8\mu}_R = \partial_\mu T^\mu_\lambda\sigma^\lambda, \\ \partial_\mu T^\mu_\lambda &= \frac{m}{k\mathbf{l}}\partial_\mu T^{1\mu}_{L\lambda} + \frac{m}{k\mathbf{r}}\partial_\mu T^{1\mu}_{R\lambda} + \frac{m}{km_l}\partial_\mu T^{8\mu}_{L\lambda} + \frac{m}{km_r}\partial_\mu T^{8\mu}_{R\lambda}. \end{aligned} \tag{2.246}$$

And we have:

$$\begin{aligned} l^1 &= \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}^1\mathrm{v} = \frac{q}{2}(A - Z'^0) + \frac{q}{2}(A + Z'^0) + \mathbf{l}^1\mathrm{v} \\ &= qA + \mathbf{l}^1\mathrm{v}, && (2.247) \\ r^1 &= 2\mathrm{b} + \mathbf{r}^1\mathrm{v} = q(A - Z'^0) + \mathbf{r}^1\mathrm{v}, && (2.248) \\ l^8 &= \mathrm{b} - 3\mathrm{w}^3 + \mathbf{l}^8\mathrm{v} = \frac{q}{2}(A - Z'^0) - \frac{q}{2}(A + 3Z'^0) + \mathbf{l}^8\mathrm{v} \\ &= -2qZ'^0 + \mathbf{l}^8\mathrm{v}, && (2.249) \\ r^8 &= 2p\mathrm{b} + \mathbf{r}^8\mathrm{v} = pq(A - Z'^0) + \mathbf{r}^8\mathrm{v}. && (2.250) \end{aligned}$$

With the left wave of the electron we obtain:

$$\begin{aligned} \partial_\mu T^{1\mu}_{L\lambda} &= (\partial_\mu l^1_\lambda - \partial_\lambda l^1_\mu)\mathrm{D}^{1\mu}_L \\ &= \Big[q(\partial_\mu A_\lambda - \partial_\lambda A_\mu) + q(\partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu) + \mathbf{l}(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)\Big]\mathrm{D}^{1\mu}_L \\ &= (qF^e_{\mu\lambda} + iqF^m_{\mu\lambda} + \mathbf{l}^1 G_{\mu\lambda})\mathrm{D}^{1\mu}_L && (2.251) \\ G_{\mu\lambda} &:= \partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu, && (2.252) \end{aligned}$$

where G is a similar bivector, independent from the gauge interactions. With the right wave of the electron we obtain:

$$\begin{aligned} \partial_\mu T^{1\mu}_{R\lambda} &= (\partial_\mu r^1_\lambda - \partial_\lambda r^1_\mu)\mathrm{D}^{1\mu}_R \\ &= [q(\partial_\mu A_\lambda - \partial_\lambda A_\mu) - (\partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu) + \mathbf{r}(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)]\mathrm{D}^{1\mu}_R \\ &= (qF^e_{\mu\lambda} - iqF^m_{\mu\lambda} + \mathbf{r}^1 G_{\mu\lambda})\mathrm{D}^{1\mu}_R. && (2.253) \end{aligned}$$

With the left wave of the neutrino–monopole we obtain:

$$\begin{aligned} \partial_\mu T^{8\mu}_{L\lambda} &= (\partial_\mu l^8_\lambda - \partial_\lambda l^8_\mu)\mathrm{D}^{8\mu}_L \\ &= [0(\partial_\mu A_\lambda - \partial_\lambda A_\mu) - 2q(\partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu) + m_l(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)]\mathrm{D}^{8\mu}_L \\ &= \Big(0F^e_{\mu\lambda} - 2iqF^m_{\mu\lambda} + \mathbf{l}^8 G_{\mu\lambda}\Big)\mathrm{D}^{8\mu}_L. && (2.254) \end{aligned}$$

With the right wave of the neutrino–monopole we obtain:

$$\begin{aligned} \partial_\mu T^{8\mu}_{R\lambda} &= (\partial_\mu r^8_\lambda - \partial_\lambda r^8_\mu)\mathrm{D}^{8\mu}_R \\ &= [pq(\partial_\mu A_\lambda - \partial_\lambda A_\mu) - pq(\partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu) + m_r(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)]\mathrm{D}^{8\mu}_R \\ &= \Big(pqF^e_{\mu\lambda} - ipqF^m_{\mu\lambda} + \mathbf{r}^8 G_{\mu\lambda}\Big)\mathrm{D}^{8\mu}_R. && (2.255) \end{aligned}$$

Adding, we get:

$$\begin{aligned} \partial_\mu T^\mu_\lambda = &\frac{q}{k}F^e_{\mu\lambda}\Big(\frac{m}{\mathbf{l}^1}\mathrm{D}^{1\mu}_L + \frac{m}{\mathbf{r}^1}\mathrm{D}^{1\mu}_R + \frac{mp}{\mathbf{r}^8}\mathrm{D}^{8\mu}_R\Big) \\ &+ i\frac{q}{k}F^m_{\mu\lambda}\Big(\frac{m}{\mathbf{l}^1}\mathrm{D}^{1\mu}_L - \frac{m}{\mathbf{r}^1}\mathrm{D}^{1\mu}_R - 2\frac{m}{\mathbf{l}^8}\mathrm{D}^{8\mu}_L - p\frac{m}{\mathbf{r}^8}\mathrm{D}^{8\mu}_R\Big) && (2.256) \\ &+ \frac{m}{k}G_{\mu\lambda}(\mathrm{D}^{1\mu}_L + \mathrm{D}^{1\mu}_R + \mathrm{D}^{8\mu}_L + \mathrm{D}^{8\mu}_R). \end{aligned}$$

We hence obtain:

$$\partial_\mu T^\mu = \Big[qF^e_{\mu\lambda}(\underline{\mathrm{J}}^\mu + \frac{mp}{k\mathbf{r}^8}\mathrm{D}_R^{8\mu}) \tag{2.257}$$
$$+ iqF^m_{\mu\lambda}(\frac{m}{k\mathbf{l}^1}\mathrm{D}_L^{1\mu} - \frac{m}{k\mathbf{r}^1}\mathrm{D}_R^{1\mu} - 2\frac{m}{k\mathbf{l}^8}\mathrm{D}_L^{8\mu} - p\frac{m}{k\mathbf{r}^8}\mathrm{D}_R^{8\mu}) + \frac{m}{k}G_{\mu\lambda}\mathrm{J}_l^\mu)\Big]\sigma^\lambda.$$

We see that the $\mathrm{D}_L^{8\mu}$ term is missing in the first line: this results from the neutrality of the left wave of the neutrino which does not see the electric interaction (hence the name "neutrino"). When the electron is alone, when weak interactions are not at play, nor the G field, it remains:

$$\partial_\mu T^\mu = qF^e_{\mu\lambda}(\frac{m}{k\mathbf{l}^1}\mathrm{D}_L^{1\mu} + \frac{m}{k\mathbf{r}^1}\mathrm{D}_R^{1\mu})\sigma^\lambda, \tag{2.258}$$

This gives the Lorentz force (1.328) acting on the electric current $\mathrm{j}_e = e(\frac{m}{k\mathbf{l}^1}\mathrm{D}_R^1 + \frac{m}{k\mathbf{r}^1}\mathrm{D}_L^1)$ of the electron.

2.4.1 Probability density

The component T_0^0 of the energy–momentum tensor satisfies:

$$T_0^0 = \Re\Big[-i\Big(\frac{m}{k\mathbf{l}^1}\eta^{1\dagger}\mathrm{d}^1_{L0}\eta^1 + \frac{m}{k\mathbf{r}^1}\xi^{1\dagger}\mathrm{d}^1_{R0}\xi^1$$
$$+ \frac{m}{k\mathbf{l}^8}\eta^{8\dagger}\mathrm{d}^8_{L0}\eta^8 + \frac{m}{k\mathbf{r}^8}\xi^{8\dagger}\mathrm{d}^8_{R0}\xi^8\Big)\Big]. \tag{2.259}$$

For a solution to the wave equation with an energy E that is the same for the whole wave, we have:

$$-i\mathrm{d}^1_{R0}\xi^1 = \frac{E}{\hbar c}\xi^1(\vec{x});\ -i\mathrm{d}^8_{R0}\xi^8 = \frac{E}{\hbar c}\xi^8(\vec{x}),$$
$$-i\mathrm{d}^1_{L0}\eta^1 = \frac{E}{\hbar c}\eta^1(\vec{x});\ -i\mathrm{d}^8_{L0}\eta^8 = \frac{E}{\hbar c}\eta^8(\vec{x}). \tag{2.260}$$

That gives:

$$T_0^0 = \frac{E}{\hbar c}(\frac{m}{k\mathbf{l}^1}\eta^{1\dagger}\eta^1 + \frac{m}{k\mathbf{r}^1}\xi^{1\dagger}\xi^1 + \frac{m}{k\mathbf{l}^8}\eta^{8\dagger}\eta^8 + \frac{m}{k\mathbf{r}^8}\xi^{8\dagger}\xi^8)$$
$$= \frac{E}{\hbar c}(\frac{m}{k\mathbf{l}^1}D_L^{10} + \frac{m}{k\mathbf{r}^1}D_R^{10} + \frac{m}{k\mathbf{l}^8}D_L^{80} + \frac{m}{k\mathbf{r}^8}D_R^{80}) = \frac{E}{\hbar c}\underline{\mathrm{J}}_l^0, \tag{2.261}$$

It is the J_l current which is a generalization of the J current in Chapter 1. The reason of the existence of a probability current in physics is the same as for the electron alone: equivalence between inertial mass and gravitational mass, which implies:

$$E = \iiint dvT_0^0\ ;\quad \iiint dv\frac{\mathrm{J}_l^0}{\hbar c} = 1. \tag{2.262}$$

2.5 Quantization of the kinetic momentum

Noether's theorem derives the conservation of energy–momentum from the invariance of the Lagrangian density under translation. In the same way this same theorem derives the conservation of the kinetic momentum from the invariance of the Lagrangian density under space-time rotations. Relativistic mechanics replaced the group of spatial rotations with the Lorentz group, but quantum theory also replaced the $SO(3)$ group of rotations with the $SU(2)$ group. As well the orthochronous Lorentz group is replaced with the $SL(2,\mathbb{C})$ group. And endly we extended this invariance using the greater group $GL(2,\mathbb{C}) = Cl_3^*$ containing the $SL(2,\mathbb{C})$ group. Moreover we have not only one, but two invariant Lagrangian densities: $\mathcal{L}^+$ and $\mathcal{L}^-$. We will now start from the real Lagrangian density $\mathcal{L}^-$ and from the energy–momentum corresponding to this Lagrangian density, defined from:

$$\begin{aligned} V_\lambda^\mu := \Re\Big[-i\Big(&\frac{m}{k\mathbf{l}^1}\eta^{1\dagger}\sigma^\mu \mathrm{d}_{L\lambda}^1\eta^1 - \frac{m}{k\mathbf{r}^1}\xi^{1\dagger}\widehat{\sigma}^\mu \mathrm{d}_{R\lambda}^1\xi^1 \\ &+ \frac{m}{k\mathbf{l}^8}\eta^{8\dagger}\sigma^\mu \mathrm{d}_{L\lambda}^8\eta^8 - \frac{m}{k\mathbf{r}^8}\xi^{8\dagger}\widehat{\sigma}^\mu \mathrm{d}_{R\lambda}^8\xi^8\Big)\Big]. \end{aligned} \tag{2.263}$$

We note two possible methods for the demonstration of Noether's theorem. However that of Lasenby [89], using easy calculations of Clifford algebras, does not specify in detail what comes in the case of the spin half. Thus we will take the usual method in quantum field theory and we will follow Bailin [2]. We consider a general transformation satisfying:

$$M = 1 + \frac{1}{2}(\delta\omega^0 + \delta\omega^1\sigma_1 + \delta\omega^2\sigma_2 + \delta\omega^3\sigma_3 + \delta\omega^4 i\sigma_1 + \delta\omega^5 i\sigma_2 + \delta\omega^6 i\sigma_3 + \delta\omega^7 i) \tag{2.264}$$

where the eight $\delta\omega^n$ are infinitely small. We have:

$$\begin{aligned} M^\dagger = 1 + \frac{1}{2}(&\delta\omega^0 + \delta\omega^1\sigma_1 + \delta\omega^2\sigma_2 + \delta\omega^3\sigma_3 - \delta\omega^4 i\sigma_1 \\ &- \delta\omega^5 i\sigma_2 - \delta\omega^6 i\sigma_3 - \delta\omega^7 i) \\ \mathrm{x}' = \mathrm{x}'^\mu\sigma_\mu &= M\mathrm{x}M^\dagger = \mathrm{x} + \delta\mathrm{x}^\mu\sigma_\mu;\ \delta\mathrm{x}^\mu = X_i^\mu\delta\omega^i \end{aligned} \tag{2.265}$$

That gives:

$$\begin{aligned} \delta\mathrm{x}^0 &= \mathrm{x}^0\delta\omega^0 + \mathrm{x}^1\delta\omega^1 + \mathrm{x}^2\delta\omega^2 + \mathrm{x}^3\delta\omega^3, \\ \delta\mathrm{x}^1 &= \mathrm{x}^0\delta\omega^1 + \mathrm{x}^1\delta\omega^0 + \mathrm{x}^2\delta\omega^6 - \mathrm{x}^3\delta\omega^5, \\ \delta\mathrm{x}^2 &= \mathrm{x}^0\delta\omega^2 - \mathrm{x}^1\delta\omega^6 + \mathrm{x}^2\delta\omega^0 + \mathrm{x}^3\delta\omega^4, \\ \delta\mathrm{x}^3 &= \mathrm{x}^0\delta\omega^3 + \mathrm{x}^1\delta\omega^5 - \mathrm{x}^2\delta\omega^4 + \mathrm{x}^3\delta\omega^0. \end{aligned} \tag{2.266}$$

The only non zero X_i^μ terms are:

$$\begin{aligned}
&X_0^0 = \mathrm{x}^0;\ X_1^0 = \mathrm{x}^1;\ X_2^0 = \mathrm{x}^2;\ X_3^0 = \mathrm{x}^3,\\
&X_0^1 = \mathrm{x}^1;\ X_1^1 = \mathrm{x}^0;\ X_5^1 = -\mathrm{x}^3;\ X_6^1 = \mathrm{x}^2,\\
&X_0^2 = \mathrm{x}^2;\ X_2^2 = \mathrm{x}^0;\ X_6^2 = -\mathrm{x}^1;\ X_4^2 = \mathrm{x}^3,\\
&X_0^3 = \mathrm{x}^3;\ X_3^3 = \mathrm{x}^0;\ X_4^3 = -\mathrm{x}^2;\ X_5^3 = \mathrm{x}^1,
\end{aligned} \tag{2.267}$$

Bailin denotes the different fields φ_a, and their variations are denoted as:

$$\delta\varphi_a = \phi_i^a \delta\omega^i. \tag{2.268}$$

Since we may use the adjoint to obtain the real part, we can opt to consider only four spinor fields:

$$\varphi_1 = \eta^1;\ \varphi_2 = \xi^1;\ \varphi_3 = \eta^8;\ \varphi_4 = \xi^8. \tag{2.269}$$

And we have:

$$\begin{aligned}
&\eta^1 + \delta\eta^1 = \widehat{M}\eta^1;\ \xi^1 + \delta\xi^1 = M\xi^1;\ \eta^8 + \delta\eta^8 = \widehat{M}\eta^8;\ \xi^8 + \delta\xi^8 = M\xi^8,\\
&\widehat{M} = 1 + \frac{1}{2}(\delta\omega^0 - \delta\omega^1\sigma_1 - \delta\omega^2\sigma_2 - \delta\omega^3\sigma_3\\
&\qquad + \delta\omega^4 i\sigma_1 + \delta\omega^5 i\sigma_2 + \delta\omega^6 i\sigma_3 - \delta\omega^7 i).
\end{aligned} \tag{2.270}$$

That gives:

$$\begin{aligned}
2\delta\xi^1 &= \delta\omega^0\xi^1 + \delta\omega^1\sigma_1\xi^1 + \delta\omega^2\sigma_2\xi^1 + \delta\omega^3\sigma_3\xi^1 \qquad (2.271)\\
&\quad + \delta\omega^4 i\sigma_1\xi^1 + \delta\omega^5 i\sigma_2\xi^1 + \delta\omega^6 i\sigma_3\xi^1 + \delta\omega^7 i\xi^1,\\
2\delta\eta^1 &= \delta\omega^0\eta^1 - \delta\omega^1\sigma_1\eta^1 - \delta\omega^2\sigma_2\eta^1 - \delta\omega^3\sigma_3\eta^1 \qquad (2.272)\\
&\quad + \delta\omega^4 i\sigma_1\eta^1 + \delta\omega^5 i\sigma_2\eta^1 + \delta\omega^6 i\sigma_3\eta^1 - \delta\omega^7 i\eta^1.
\end{aligned}$$

And we obtain two similar formulas for ξ^8 and η^8. With the numbering in (2.269) we get:

$$\begin{aligned}
&\phi_0^1 = \frac{\eta^1}{2};\ \phi_1^1 = -\sigma_1\frac{\eta^1}{2};\ \phi_2^1 = -\sigma_2\frac{\eta^1}{2};\ \phi_3^1 = -\sigma_3\frac{\eta^1}{2},\\
&\phi_4^1 = i\sigma_1\frac{\eta^1}{2};\ \phi_5^1 = i\sigma_2\frac{\eta^1}{2};\ \phi_6^1 = i\sigma_3\frac{\eta^1}{2};\ \phi_7^1 = -i\frac{\eta^1}{2}, \qquad (2.273)\\
&\phi_0^2 = \frac{\xi^1}{2};\ \phi_1^2 = \sigma_1\frac{\xi^1}{2};\ \phi_2^2 = \sigma_2\frac{\xi^1}{2};\ \phi_3^2 = \sigma_3\frac{\xi^1}{2},\\
&\phi_4^2 = i\sigma_1\frac{\xi^1}{2};\ \phi_5^2 = i\sigma_2\frac{\xi^1}{2};\ \phi_6^2 = i\sigma_3\frac{\xi^1}{2};\ \phi_7^2 = i\frac{\xi^1}{2}, \qquad (2.274)
\end{aligned}$$

$$\phi_0^3 = \frac{\eta^8}{2};\ \phi_1^3 = -\sigma_1\frac{\eta^8}{2};\ \phi_2^3 = -\sigma_2\frac{\eta^8}{2};\ \phi_3^3 = -\sigma_3\frac{\eta^8}{2},$$
$$\phi_4^3 = i\sigma_1\frac{\eta^8}{2};\ \phi_5^3 = i\sigma_2\frac{\eta^8}{2};\ \phi_6^3 = i\sigma_3\frac{\eta^8}{2};\ \phi_7^3 = -i\frac{\eta^8}{2}, \quad (2.275)$$
$$\phi_0^4 = \frac{\xi^8}{2};\ \phi_1^4 = \sigma_1\frac{\xi^8}{2};\ \phi_2^4 = \sigma_2\frac{\xi^8}{2};\ \phi_3^4 = \sigma_3\frac{\xi^8}{2},$$
$$\phi_4^4 = i\sigma_1\frac{\xi^8}{2};\ \phi_5^4 = i\sigma_2\frac{\xi^8}{2};\ \phi_6^4 = i\sigma_3\frac{\xi^8}{2};\ \phi_7^4 = +i\frac{\xi^8}{2}. \quad (2.276)$$

Since the ξ'^n and η'^n are also solutions of the wave equations, the Lagrangian density always satisfies $0 = \mathcal{L}'^-$; thus Noether's theorem associates to each of the eight sub-groups with one parameter ω^i of the invariance group a conservative current:

$$j_i^\mu = \Big(\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}(\partial_\nu\varphi_a) - \mathcal{L}^-\delta_\nu^\mu\Big)X_i^\nu - \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_i^a. \quad (2.277)$$

In comparison with this general formula we now use a simplification because our equations are homogeneous, and this is associated to a Lagrangian density that is exactly null for each solution of the equation. Thus the currents satisfy:

$$j_i^\mu = \Big(\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}(\partial_\nu\varphi_a)\Big)X_i^\nu - \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_i^a. \quad (2.278)$$

With (2.160) to (2.163) the Lagrangian density (2.228) gives:

$$\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_1)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\eta^1)} = -\frac{im}{2k\mathbf{l}^1}\eta^{1\dagger}\sigma^\mu;\ \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_2)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\xi^1)} = +\frac{im}{2k\mathbf{r}^1}\xi^{1\dagger}\sigma^\mu, \quad (2.279)$$
$$\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_3)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\eta^8)} = -\frac{im}{2k\mathbf{l}^8}\eta^{8\dagger}\sigma^\mu;\ \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_4)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\xi^8)} = +\frac{im}{2k\mathbf{r}^8}\xi^{8\dagger}\sigma^\mu.$$

Before us, relativistic quantum theory only knew the 6 j_1^μ to j_6^μ vectors. These six space-time vectors are now joined with two other vectors, and it is precisely one of these new vectors, j_7, which is useful here. We have:

$$j_7^\mu = \Big(\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}(\partial_\nu\varphi_a)\Big)X_7^\nu - \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_7^a. \quad (2.280)$$

The only X_i^ν that are not null are listed in (2.267), and this list contains none X_7^ν. That comes from the commutative property: the generator i of the chiral gauge $U(1)$ belongs to the kernel of the homomorphism $f : M \mapsto R$ from Cl_3^* into the D^* group of similitudes (see 1.1.2 and 1.2). We thus have:

$$j_7^\mu = -\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_7^a. \quad (2.281)$$

With equations (2.273) to (2.276) and with (2.279), and since the adjoint of a real is likewise real, we get:

$$\begin{aligned} j_7^\mu &= 2\frac{im}{2k\mathbf{l}^1}\eta^{1\dagger}\sigma^\mu(-i)\frac{\eta^1}{2} - 2\frac{im}{2k\mathbf{r}^1}\xi^{1\dagger}\sigma^\mu(+i)\frac{\xi^1}{2} \\ &+ 2\frac{im}{2k\mathbf{l}^8}\eta^{8\dagger}\sigma^\mu(-i)\frac{\eta^8}{2} - 2\frac{im}{2k\mathbf{r}^8}\xi^{8\dagger}\sigma^\mu(+i)\frac{\xi^8}{2}, \end{aligned} \tag{2.282}$$

which means:

$$j_7 = \frac{1}{2}\Big(\frac{m}{k\mathbf{l}^1}\mathrm{D}_L^1 + \frac{m}{k\mathbf{r}^1}\mathrm{D}_R^1 + \frac{m}{k\mathbf{l}^8}\mathrm{D}_L^8 + \frac{m}{k\mathbf{r}^8}\mathrm{D}_R^8\Big) = \frac{1}{2}\underline{\mathrm{J}}_l^0. \tag{2.283}$$

With (2.262), the proper kinetic momentum, usually called spin, thus satisfies :

$$\iiint dv\frac{1}{c}j_7^0 = \frac{1}{2c}\iiint dv\underline{\mathrm{J}}_l^0 = \frac{\hbar}{2}. \tag{2.284}$$

We recall that this equality is obtained from the equivalence principle: the total energy of the electron (linked to the frequency, thus to space-time geometry) is equal to the sum over all space of the energy density T_0^0 (density linked to electromagnetic forces, thus inertial). We may then say that **both quantization and general relativity result from the same equivalence principle between inertial mass-energy and gravitational mass-energy**. This quantization of the spin from properties of the wave equation could not be obtained previously because two concepts were lacking: first, nobody suspected the presence of a form–invariance group more binding than the Lorentz group [12] . Second reason, nobody except the wise O. Costa de Beauregard [53], saw the existence of the strange V tensor in the wave of the electron.

The quantization of the spin concerns the complete lepton (electron + neutrino–monopole), the alone electron has a kinetic momentum $\hbar/2$, the alone neutrino–monopole has a kinetic momentum $\hbar/2$, the couple electron + neutrino–monopole has the same kinetic momentum $\hbar/2$. It is the temporal component of a space-time vector which is quantized, and this temporal component is obtained by summing a tensorial quantity over the whole space. **This is both very close to what we know from the experimental point of view, with the true value $\hbar/2$, and very far from previous quantum theory since that does not come from the proper value of a Hermitian operator**.

2.6 Dynamics of the neutrino–monopole

The magnetic monopole may be viewed from three different perspectives because it may be dotted of a right wave, or a left wave or both. The

12. This could be seen as early as 1928 but was obscured by the use of infinitesimal transformations which masked the great difference between $GL(2,\mathbb{C})$ and the Lorentz group.

magnetic monopole with a null right wave is called a neutrino or electron neutrino in today's physics. The $\mathrm{j}_m = g\mathrm{D}_R^8$ is thus null, and afterwards the k_m current is reduced to the left current. The force acting on the neutrino is:

$$\partial_\mu T^\mu = F^m_{\mu\lambda} i \mathrm{k}^\mu_m \sigma^\lambda;\ \mathrm{k}_m = -\frac{2q}{k}\mathrm{D}^8_L, \tag{2.285}$$

$$\partial_\mu T^\mu = \mathrm{f}^0 + \vec{\mathrm{f}};\ \frac{i}{2}F^m_{\mu\lambda}\mathrm{k}^\mu_m\sigma^\lambda = \vec{\mathrm{k}}_m \cdot \vec{H}^m + \mathrm{k}^0_m \vec{H}^m - \vec{\mathrm{k}}_m \times \vec{E}^m, \tag{2.286}$$

$$\frac{\hbar c}{2}\vec{\mathrm{f}} = \mathrm{k}^0_m \vec{H}^m - \vec{\mathrm{k}}_m \times \vec{E}^m;\ F^m = \vec{E}^m + i\vec{H}^m. \tag{2.287}$$

A second possibility which was not yet employed by the Standard Model is that the neutrino–monopole might only have a right wave. In this case it interacts both with particles with electric charge and with particles with magnetic charge, and (2.257) is reduced to:

$$\hbar c \partial_\mu T^\mu = F^e_{\mu\lambda}\mathrm{j}^\mu_m\sigma^\lambda + F^m_{\mu\lambda} i \mathrm{k}^\mu_m \sigma^\lambda;\ \mathrm{k}_m = -\frac{pm}{k\mathbf{r}^8}\mathrm{D}^8_R;\ \mathrm{j}_m = \frac{pm}{k\mathbf{r}^8}\mathrm{D}^8_R. \tag{2.288}$$

where g is the magnetic charge. We may remark that the two currents linked to this right wave are opposite. We must thus expect to get a different result for the force acting on this wave depending on whether an electric charge or a magnetic charge is at play.

The interaction between an electric charge and a magnetic charge (magnetic monopole) was previously described in a complicated way by writing the electromagnetic field of the monopole as if it were of an electric origin: $F = \nabla\widehat{W}$ instead $F = \nabla\widehat{iW}$. Similarly the interaction between a magnetic charge and an electric charge was described by Lochak [90, 91] using the electromagnetic field created by the electron as if it were of a magnetic origin: $F = \nabla\widehat{i\mathrm{A}}$ instead $F = \nabla\widehat{\mathrm{A}}$. But these calculations are correct because we indeed have:

$$F^e_{\mu\lambda}\mathrm{j}^\mu_m = \nabla\widehat{\mathrm{A}}\mathrm{j}^\mu_m = \nabla\widehat{i\mathrm{A}} i g \mathrm{D}^\mu_R. \tag{2.289}$$

We may then refer to these works [90, 91] for the demonstration of Dirac's formula $eg/\hbar c = 1/2$, a prediction that we shall correct in Chapter 4. Since we just explained how the quantization of the kinetic momentum follows from the equivalence principle, since the quantization of the electric charge and of the magnetic charge follows from Dirac's formula, we see how the quantization of charges also follows from the equivalence principle and from the extended invariance.

2.7 The soliton wave of the electron

The electron was first considered as an elementary particle, a point with a mass and an electric charge. Hence initial models of electron associated its mass-energy to the energy of the electric field $\vec{E}$ created by its charge. This energy is proportional to $\vec{E}^2$, thus with spherical symmetry. And we have, in 1.11, associated directly the electromagnetic field itself (not its square) with the energy-momentum tensor of the ϕ wave of the electron. We may thus think the electron as a soliton solution of its wave equation [45] [46], with a wave ruled by the same kind of partial differential equation as in the case of a bound state in a hydrogen atom. The electron being electrically charged creates around itself an electric field linked to the electric current of the electron wave. The electromagnetic potential A satisfies:

$$-\Box A = \Delta A = q\mathbf{J};\ A = \Delta^{-1}(q\mathbf{J}) = -\frac{u}{r}, \tag{2.290}$$

where u is a numeric constant to be determined (the charge of the bare electron). The wave equation then becomes:

$$0 = \nabla\widehat{\phi}\sigma_{21} - \frac{u}{r}\widehat{\phi} + m_g \mathrm{v}\widehat{\phi}. \tag{2.291}$$

Multiplying on the left side by $\overline{\phi}$ to obtain the invariant form of the wave equation, and with:

$$\overline{\phi}\phi = \rho e^{i\beta};\ \ \mathbf{lr} = m_g^2 = mm_a;\ \ m_g\overline{\phi}\mathrm{v}\widehat{\phi} = \rho\mathbf{m},\ \ \mathbf{m} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix}, \tag{2.292}$$

we get:

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} - \frac{u}{r}\overline{\phi}\widehat{\phi} + \rho\mathbf{m}. \tag{2.293}$$

This improved equation is solved in Chapter C, by separation of the spherical variables. Thanks to Krüger's identities [86], we can separate at one go the variables t and φ from the variables r and θ, and we search for a solution such as:

$$\begin{aligned} &S := e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2};\ \Omega = \widehat{\Omega} := r^{-1}(\sin\theta)^{-\frac{1}{2}}S;\ \ p := \lambda\varphi - E\mathrm{x}^0, \\ &\phi =: \Omega X e^{pi_3};\ \widehat{\phi} = \Omega\widehat{X}e^{pi_3}:\ \ \overline{\phi} = e^{-pi_3}\overline{X}\,\overline{\Omega}, \\ &X := \begin{pmatrix} AU & -\bar{B}V \\ CV & \bar{D}U \end{pmatrix};\ \widehat{X} = \begin{pmatrix} DU & -\bar{C}V \\ BV & \bar{A}U \end{pmatrix};\ \ \overline{X} = \begin{pmatrix} \bar{D}U & \bar{B}V \\ -CV & AU \end{pmatrix}, \end{aligned} \tag{2.294}$$

where A, B, C and D are functions with complex value of the radial variable r, while U and V are functions with real value of the angular variable θ. λ is a real constant named magnetic quantum number, which may be positive or negative. That gives the two up-down states of the electron. U and V are solution of the following system:

$$U' - \frac{\lambda U}{\sin\theta} = -\kappa V;\ V' + \frac{\lambda V}{\sin\theta} = \kappa U, \tag{2.295}$$

The κ constant is a positive non zero integer. The magnetic quantum number λ of the free electron is a half–odd number satisfying $|\lambda| < \kappa$. The double value of the free electron spin comes then from $\kappa = 1$ which gives $\lambda = \pm 1/2$. These values of κ and λ are the same as those of the $1\mathbf{s}1/2$ states, we thus may use the same calculation, replacing only the interaction term $-\alpha/r$ containing the fine structure constant $\alpha \approx \frac{1}{137}$ by the term of self-interaction $-u/r$. With the $\lambda = 1/2$ case (spin up) we have:

$$U = U_1\sqrt{\sin\theta}\cos\left(\frac{\theta}{2}\right);\quad V = V_1\sqrt{\sin\theta}\sin\left(\frac{\theta}{2}\right). \tag{2.296}$$

Since $\kappa = 1$ and $\lambda = 1/2$, (C.44) and (C.48) give:

$$U_1 = -1;\ V_1 = -1;\ U = -\sqrt{\sin\theta}\cos\left(\frac{\theta}{2}\right);\ V = -\sqrt{\sin\theta}\sin\left(\frac{\theta}{2}\right). \tag{2.297}$$

We let:

$$\mathbf{F}_1 := -\frac{A}{r};\ \mathbf{F}_2 := -\frac{\bar{D}}{r};\ \mathbf{F}_3 := -\frac{C}{r};\ \mathbf{F}_4 := -\frac{\bar{B}}{r}, \tag{2.298}$$

$$p := \frac{\varphi}{2} - E\mathrm{x}^0;\quad \underline{c} := \cos\left(\frac{\theta}{2}\right);\quad \underline{s} := \sin\left(\frac{\theta}{2}\right), \tag{2.299}$$

$$\theta_1 := \underline{c}^2 = \frac{1+\cos\theta}{2};\quad \theta_2 := \underline{s}^2 = \frac{1-\cos\theta}{2};\quad \theta_3 := \underline{c}\,\underline{s} = \frac{\sin\theta}{2}. \tag{2.300}$$

With (C.61) we then get:

$$\phi = S\frac{X}{r\sqrt{\sin\theta}}e^{pi_3} = S\begin{pmatrix}\underline{c}\mathbf{F}_1 & -\underline{s}\mathbf{F}_4\\ \underline{s}\mathbf{F}_3 & \underline{c}\mathbf{F}_2\end{pmatrix}e^{pi_3}, \tag{2.301}$$

$$\overline{X} = r\sqrt{\sin\theta}\begin{pmatrix}\underline{c}\mathbf{F}_2 & \underline{s}\mathbf{F}_4\\ -\underline{s}\mathbf{F}_3 & \underline{c}\mathbf{F}_1\end{pmatrix};\quad \overline{\phi} = e^{-pi_3}\begin{pmatrix}\underline{c}\mathbf{F}_2 & \underline{s}\mathbf{F}_4\\ -\underline{s}\mathbf{F}_3 & \underline{c}\mathbf{F}_1\end{pmatrix}S^{-1}, \tag{2.302}$$

We then have:

$$\phi\overline{\phi} = \overline{\phi}\phi = \det(\phi) = \rho e^{i\beta} = \underline{c}^2\mathbf{F}_1\mathbf{F}_2 + \underline{s}^2\mathbf{F}_3\mathbf{F}_4. \tag{2.303}$$

We also have:

$$\theta_1 + \theta_2 = 1;\quad \theta_1 - \theta_2 = \cos\theta:\quad \theta_3^2 = \frac{1-\cos(2\theta)}{8},$$

$$\theta_1^2 = \frac{3+4\cos\theta+\cos(2\theta)}{8};\quad \theta_1^2 = \frac{3-4\cos\theta+\cos(2\theta)}{8}, \tag{2.304}$$

$$\det(X) = \rho_X e^{i\beta} = X\overline{X} = r^2\sin\theta(\mathbf{F}_1\mathbf{F}_2\theta_1 + \mathbf{F}_3\mathbf{F}_4\theta_2), \tag{2.305}$$

$$\det(\phi) = \rho e^{i\beta} = \frac{\det(X)}{r^2\sin\theta} = \mathbf{F}_1\mathbf{F}_2\theta_1 + \mathbf{F}_3\mathbf{F}_4\theta_2, \tag{2.306}$$

$$\rho^2 = (\rho e^{i\beta})(\rho e^{-i\beta}) = |\mathbf{F}_1\mathbf{F}_2\theta_1 + \mathbf{F}_3\mathbf{F}_4\theta_2|^2, \tag{2.307}$$

$$\rho = |\mathbf{F}_1\mathbf{F}_2\theta_1 + \mathbf{F}_3\mathbf{F}_4\theta_2|;\quad e^{i\beta} = \frac{\mathbf{F}_1\mathbf{F}_2\theta_1 + \mathbf{F}_3\mathbf{F}_4\theta_2}{|\mathbf{F}_1\mathbf{F}_2\theta_1 + \mathbf{F}_3\mathbf{F}_4\theta_2|}. \tag{2.308}$$

Similarly for $\kappa = 1$ and $\lambda = -1/2$ (spin down case), we obtain:

$$U = -\sqrt{\sin\theta}\sin\Big(\frac{\theta}{2}\Big);\ V = \sqrt{\sin\theta}\cos\Big(\frac{\theta}{2}\Big), \tag{2.309}$$

$$U^2 = \sin\theta\frac{1-\cos\theta}{2};\quad V^2 = \sin\theta\frac{1+\cos\theta}{2};\quad UV = -\frac{\sin^2\theta}{2}. \tag{2.310}$$

We then have:

$$\det(X) = \rho_X e^{i\beta} = \sin\theta\Big[A\bar{D}\frac{1-\cos\theta}{2} + C\bar{B}\frac{1+\cos\theta}{2}\Big], \tag{2.311}$$

$$\frac{\det(X)}{r^2\sin\theta} = A\bar{D}\frac{1-\cos\theta}{2r^2} + C\bar{B}\frac{1+\cos\theta}{2r^2}, \tag{2.312}$$

$$\frac{\rho_X^2}{r^4\sin^2\theta} = \Big[A\bar{D}\frac{1-\cos\theta}{2r^2} + C\bar{B}\frac{1+\cos\theta}{2r^2}\Big]\Big[D\bar{A}\frac{1-\cos\theta}{2r^2} + B\bar{C}\frac{1+\cos\theta}{2r^2}\Big].$$

We may remark that the only changes coming from the sign of the spin are the replacement of $\cos\theta$ by $-\cos\theta$, which exchanges θ_1 and θ_2 and the change of sign for θ_3. We thus must detail the calculation only in the $\lambda = +1/2$ case. We now let:

$$k_1 := |\mathbf{F}_1\mathbf{F}_2|\theta_1;\quad k_2 := |\mathbf{F}_3\mathbf{F}_4|\theta_2, \tag{2.313}$$
$$k_3^2 = (\mathbf{F}_1\mathbf{F}_2\bar{\mathbf{F}}_3\bar{\mathbf{F}}_4 + \bar{\mathbf{F}}_1\bar{\mathbf{F}}_2\mathbf{F}_3\mathbf{F}_4)\theta_3^2.$$

We have:

$$\rho^2 = \frac{\rho_X^2}{r^4\sin^2\theta} = k_1^2 + k_2^2 + k_3^2;\quad \rho = \frac{\rho_X}{r^2\sin\theta} = \sqrt{k_1^2 + k_2^2 + k_3^2} \tag{2.314}$$

Accounting for the internal electromagnetic interaction, a $-\frac{u}{r}$ term is added and we then get, with (C.78):

$$\nabla'\big(\widehat{X}e^{pi_3}\big)\sigma_{21} - \frac{u}{r}\widehat{X}e^{pi_3} = -\sqrt{\sin\theta}\begin{pmatrix} \underline{c}\mathcal{D} & -\underline{s}\mathcal{C} \\ \underline{s}\mathcal{B} & \underline{c}\mathcal{A} \end{pmatrix}. \tag{2.315}$$

With:

$$\begin{aligned}
\mathcal{D} &:= -\Big(E + \frac{u}{r}\Big)D + iD' + \frac{i\kappa}{r}B. \\
\mathcal{B} &:= -\Big(E + \frac{u}{r}\Big)B - iB' - \frac{i\kappa}{r}D, \\
\mathcal{C} &:= -\Big(E + \frac{u}{r}\Big)\bar{C} - i\bar{C}' - \frac{i\kappa}{r}\bar{A}, \\
\mathcal{A} &:= -\Big(E + \frac{u}{r}\Big)\bar{A} + i\bar{A}' + \frac{i\kappa}{r}\bar{C},
\end{aligned} \tag{2.316}$$

The wave equation is equivalent to the (C.83)–(C.86) system:

$$\begin{aligned}
\bar{D}\mathcal{D}U^2 + \bar{B}\mathcal{B}V^2 &= -\rho_X\mathbf{l}, \\
\bar{D}\mathcal{C} &= \bar{B}\mathcal{A}, \\
C\mathcal{D} &= A\mathcal{B}, \\
A\mathcal{A}U^2 + C\mathcal{C}V^2 &= -\rho_X\mathbf{r}.
\end{aligned}$$

The separation of variables was made in C.1, we get the radial system:

$$\left(E+\frac{\alpha}{r}\right)D - iD' - i\frac{\kappa}{r}B + \mathbf{l}e^{il_1}A = 0, \tag{2.317}$$
$$\left(E+\frac{\alpha}{r}\right)B + iB' + i\frac{\kappa}{r}D + \mathbf{l}e^{il_1}C = 0, \tag{2.318}$$
$$\left(E+\frac{\alpha}{r}\right)A + iA' + i\frac{\kappa}{r}C - \mathbf{r}e^{-ir_1}D = 0, \tag{2.319}$$
$$\left(E+\frac{\alpha}{r}\right)C - iC' - i\frac{\kappa}{r}A - \mathbf{r}e^{-ir_1}B = 0, \tag{2.320}$$

where l_1 and r_1 are any real numbers.

2.7.1 Resolution of the radial system

The system of radial equations (2.317), 2.318 (2.319) and (2.320) is the same as the linear system resulting from the Dirac equation, when we suppose the cancellation of l_1 and r_1. The asymptotic behavior of the functions A, B, C and D is thus the same as with the Dirac equation. And this asymptotic behavior implies the use of the (C.107) functions:

$$\begin{aligned}
A &= (a_0 r^s + a_1 r^{s+1} + ... + a_n r^{s+n})e^{-\Lambda m r},\\
B &= (b_0 r^s + b_1 r^{s+1} + ... + b_n r^{s+n})e^{-\Lambda m r},\\
C &= (c_0 r^s + c_1 r^{s+1} + ... + c_n r^{s+n})e^{-\Lambda m r},\\
D &= (d_0 r^s + d_1 r^{s+1} + ... + d_n r^{s+n})e^{-\Lambda m r},
\end{aligned}$$

where s and Λ are two positive reals. We then get three kinds of numeric systems: with null index, with index between 0 and n and with index n. With the null index the system only depends on u, κ and s:

$$\begin{aligned}
0 &= (-u-is)a_0 - i\kappa c_0 \ ; \quad 0 = (-u-is)b_0 - i\kappa d_0,\\
0 &= i\kappa a_0 + (-u+is)c_0 \ ; \quad 0 = i\kappa b_0 + (-u+is)d_0.
\end{aligned} \tag{2.321}$$

These systems are exactly those of the linear Dirac equation. They are made of sub-systems. A non null solution is obtained only if the determinant of each sub-system is null, which means if s satisfies:

$$0 = (-u-is)(-u+is) - \kappa^2;\ \kappa^2 = s^2 + u^2;\ s = \sqrt{\kappa^2 - u^2}. \tag{2.322}$$

It is from there that the condition $\kappa \neq 0$ comes. Let:

$$s + iu := |\kappa|e^{i\gamma};\ s - iu = |\kappa|e^{-i\gamma}. \tag{2.323}$$

The (2.321) system gives:

$$c_0 = \frac{iu-s}{|\kappa|}a_0 = -e^{-i\gamma}a_0;\ d_0 = \frac{iu-s}{|\kappa|}b_0 = -e^{-i\gamma}b_0. \tag{2.324}$$

With the n index we have the following system:

$$0 = (-E + i\Lambda m)a_n + e^{-ir_1}\mathbf{r}d_n;\ 0 = (-E + i\Lambda m)b_n - e^{il_1}\mathbf{l}c_n,$$
$$0 = -(E + i\Lambda m)d_n - e^{il_1}\mathbf{l}a_n;\ 0 = -(E + i\Lambda m)c_n + e^{-ir_1}\mathbf{r}b_n. \quad (2.325)$$

That system is also made from two similar sub-systems, with the same determinant D. A non null solution exists only if this determinant cancels. That gives:

$$0 = D = \begin{vmatrix} -E + i\Lambda m & e^{-ir_1}\mathbf{r} \\ -e^{il_1}\mathbf{l} & -E - i\Lambda m \end{vmatrix} = E^2 + \Lambda^2 m^2 - e^{i(l_1 - r_1 + \pi)}\mathbf{lr}, \quad (2.326)$$

We obtain the awaited relation between mass and energy only if:

$$l_1 - r_1 = \pi \bmod 2\pi;\quad e^{ir_1} = -e^{il_1}. \quad (2.327)$$

And then we have:

$$0 = D \Leftrightarrow \mathbf{lr} = E^2 + \Lambda^2 m^2,$$
$$0 = D \Leftrightarrow E^2 + \Lambda^2 m^2 = m_g^2. \quad (2.328)$$

As with the electron states in the hydrogen atom, we suppose that the two masses are not equatl, thus that the harmonic mean m of the two masses, satisfying $\frac{2}{m} = \frac{1}{\mathbf{l}} + \frac{1}{\mathbf{r}}$, is less than the geometric mean $m_g = \sqrt{\mathbf{lr}}$, which allows us to define a δ angle such as:

$$E = m;\quad E + i\Lambda m =: m_g e^{i\delta};\quad E - i\Lambda m = m_g e^{-i\delta}, \quad (2.329)$$
$$\Lambda = \sqrt{\frac{m_g^2}{m^2} - 1} = \sqrt{\frac{(\mathbf{l} - \mathbf{r})^2}{4\mathbf{lr}}} = \frac{|d|}{m_g};\quad d = \frac{\mathbf{l} - \mathbf{r}}{2}. \quad (2.330)$$

The (2.325) system is then reduced to:

$$d_n = \frac{E - i\Lambda m}{e^{-ir_1}\mathbf{r}}a_n = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i(r_1 - \delta)}a_n, \quad (2.331)$$
$$b_n = \frac{E + i\Lambda m}{e^{-ir_1}\mathbf{r}}c_n = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i(r_1 + \delta)}c_n. \quad (2.332)$$

If $n = 0$, which means if the radial polynomials are constant, the radial system is reduced to (C.115), (C.121) and (C.122), and we hence have:

$$d_0 = \frac{E - i\Lambda m}{e^{-ir_1}\mathbf{r}}a_0 = \frac{iu - s}{|\kappa|}b_0, \quad (2.333)$$
$$c_0 = \frac{E - i\Lambda m}{e^{ir_1}\mathbf{l}}b_0 = \frac{iu - s}{|\kappa|}a_0. \quad (2.334)$$

We then get:

$$\frac{(E - i\Lambda m)^2}{\mathbf{lr}} a_0 b_0 = \frac{(iu - s)^2}{\kappa^2} b_0 a_0,$$
$$\frac{E - i\Lambda m}{m_g} = \mp \frac{s - iu}{|\kappa|}. \tag{2.335}$$

Since the kinetic momentum is $j = |\kappa| - 1/2$, we have $\kappa = 1$. And since E, s and m_g are positive, we have only one possibility:

$$\frac{E}{m_g} = s; \ \frac{\Lambda m}{m_g} = \frac{u}{1}; \quad uE = s\Lambda m,$$
$$u^2 E^2 = s^2(m_g^2 - E^2); \quad E^2 = \frac{s^2 m_g^2}{s^2 + u^2},$$
$$E = \frac{m_g}{\sqrt{1 + \frac{u^2}{(s+0)^2}}}, \tag{2.336}$$

With (2.294) the electron–wave becomes:

$$\phi = e^{-(\varphi/2)i_3} X(r,\theta) e^{(\varphi/2 - E\mathrm{x}^0)i_3} \tag{2.337}$$
$$= \begin{pmatrix} X_1(r,\theta)e^{-iE\mathrm{x}^0} & X_3(r,\theta)e^{-i\varphi}e^{iE\mathrm{x}^0} \\ X_2(r,\theta)e^{i\varphi}e^{-iE\mathrm{x}^0} & X_4(r,\theta)e^{iE\mathrm{x}^0} \end{pmatrix}; \ X_j = r^{s-1}e^{-\Lambda m r} f_j(\theta).$$

So the wave of the alone electron has a singularity at its center. The electron is a wave on the whole space **and** (not or) a point particle. That was exactly the point of view of Louis de Broglie. The decay rate at infinity of the amplitude is very fast, from the exponential factor. It is however much less fast than with a Gaussian probability. That allows us to understand why, when an electron go through an interference device like Young slits, even if the singularity goes through only one of the slits the other slit sees the passing of a wave which has indeed a very small amplitude, but which is not infinitely small. That wave, extended to the whole space may hence guide the movement of the singularity–particle.

2.7.2 Normalization of the wave

This necessary normalization of the wave is a consequence of the equality between gravitational mass–energy (proportional to the frequency of the wave) and inertial mass–energy (see 1.5.5). That gives $E = \iiint dv\, T_0^0$. This equality is indeed equivalent to:

$$\iiint dv\, \frac{\mathbf{J}^0}{\hbar c} = 1 : \ \mathbf{J} := \frac{m}{k\mathbf{l}} \mathrm{D}_L + \frac{m}{k\mathbf{r}} \mathrm{D}_R. \tag{2.338}$$

Again with $\kappa = 1$ and $\lambda = 1/2$ (spin up), we have:

$$\mathbf{J} = \frac{m}{k}\Big(\frac{\mathrm{D}_L}{\mathbf{l}} + \frac{\mathrm{D}_R}{\mathbf{r}}\Big) = \frac{1}{km_a}\phi\begin{pmatrix}\mathbf{l} & 0\\ 0 & \mathbf{r}\end{pmatrix}\phi^\dagger, \tag{2.339}$$

$$\mathbf{J}^0 = \frac{1}{km_a}\Big[\frac{\mathbf{l}+\mathbf{r}}{2}\mathrm{D}_0^0 + \frac{\mathbf{l}-\mathbf{r}}{2}\mathrm{D}_3^0\Big] \tag{2.340}$$

$$\mathrm{D}_0^0 = \frac{1}{2}(\underline{c}^2|\mathbf{F}_1|^2 + \underline{s}^2|\mathbf{F}_4|^2 + \underline{c}^2|\mathbf{F}_2|^2 + \underline{s}^2|\mathbf{F}_3|^2), \tag{2.341}$$

$$\mathrm{D}_3^0 = \frac{1}{2}(\underline{c}^2|\mathbf{F}_1|^2 - \underline{s}^2|\mathbf{F}_4|^2 - \underline{c}^2|\mathbf{F}_2|^2 + \underline{s}^2|\mathbf{F}_3|^2). \tag{2.342}$$

We then get:

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{\mathbf{l}}{l_P^3 m_a}\Big[\frac{a_1^2+a_3^2}{2} + \cos\theta\frac{a_1^2-a_3^2}{2}\Big] \tag{2.343}$$

Since $\int_0^\pi \sin\theta\cos\theta d\theta = 0$, since $\cos\delta = \frac{m}{m_g}$, we have:

$$a_1^2 + a_3^2 = 4ya_1^2\cos\delta = 4y\frac{m}{m_g}a_1^2, \tag{2.344}$$

$$1 = \frac{4a_1^2 ym\mathbf{l}}{l_P^3 m_a m_g}\int_0^{2\pi}d\varphi\int_0^\pi d\theta\int_0^\infty dr r^{2s+2}e^{-2\Lambda mr} = \frac{4a_1^2 ym\mathbf{l}(4\pi)\Gamma(2s+3)}{l_P^3 m_a m_g(2\Lambda m)^{2s+3}}.$$

We thus get:

$$a_1 = \sqrt{\frac{l_P^3 m_a m_g(2\Lambda m)^{2s+3}}{16\pi ym\mathbf{l}\Gamma(2s+3)}}. \tag{2.345}$$

In the proper reference frame of the electron, the probability density does not have spherical symmetry. That density is dynamic, not static, it turns around the third direction. That is how the electron is an object with spin and that allows the electron to be a magnet [56]. Important remark: the alone electron is always stationary in its proper reference frame. The wave $\phi' = M\phi$ seen by any observer moving relative to the electron is a progressive wave, as a result of the transformation $R : \mathrm{x} \mapsto \mathrm{x}' = M\mathrm{x}M^\dagger$ that acts on the time, hence on the phase of the wave, that propagates. The particle as a little clock imagined by de Broglie a century ago, may thus well be that **soliton** quantum wave, that was sought by so many physicists. It could even be possible to go back to Descartes' vortices. The soliton electron, normalized from the equivalence principle, has indeed a proper kinetic momentum with value $\hbar/2$. The necessity of the Lagrangian formalism is also among the properties of the soliton wave. That determines the energy–momentum tensors (two tensors!) and kinetic momentum tensors. We also may remark the disappearance of all problems linked to the punctual charge of the electron or with the first Lorentz's model of extended electron: at the center of the electron the ϕ field has indeed an infinite amplitude, but the sum on the whole space of the energy density is finite and equal to the mass-energy of the electron. Other remark: the electromagnetic field created by the electron comes from the existence and the components of the two tensors of energy–momentum.

Why this soliton solution was not previously found? First the value of the wave belongs to Cl_3^*, a Lie group, not to a Hilbert space. Secondly the soliton solution comes only from a complete wave equation, with its gauge interaction term. Thirdly the soliton solution is calculated from separation of variables in spherical coordinates, not from plane waves. Fourthly the wave equation does not follow a Hamiltonian formalism, but instead comes with a Lagrangian formalism: the improved equation cannot be a relativistic Schrödinger equation. Fifth and finally the soliton solution exists only with two different proper masses, one for the left wave, and a different proper mass for the right wave.

Chapter 3

Electroweak and strong interactions of quarks

We study the subspace of the $Cl_{3,3}$ algebra corresponding to the quark part of the fermion wave (first generation). We study in this algebra weak interactions of d and u quarks. We present in the same framework the $SU(3)$ group of chromodynamics. We generalize the mass term of the lepton wave and we get the wave equations of quarks with mass term. These wave equations are form-invariant and gauge-invariant precisely under a gauge group that is exactly the gauge group of the Standard Model. The wave equations come from Lagrangian equations that are only derived from algebraic properties of the geometric algebra. The dynamics of the quark waves gives the forces acting on the charged and coloured fluid. This dynamics implies the quantization of the kinetic momentum of the proton and the neutron and as well the confinement of the quarks. The inclusion of Cl_3^* into $\mathrm{End}(Cl_3)$ fixes the orientation of space. We explain the preference for the left waves. We study the Dirac equation in an algebra without complex numbers.

3.1 The quark sector

We now study the Ψ_q part of the fermion wave (2.3), included in the Ψ^D wave such as:

$$\Psi^D := \Psi^{3,3} = \begin{pmatrix} \mathcal{P}_1 - i\mathcal{I}_1 + \mathcal{P}_4 + i\mathcal{I}_4 & \mathcal{I}_2 - i\mathcal{P}_2 + \mathcal{I}_3 - i\mathcal{P}_3 \\ \mathcal{I}_2 - i\mathcal{P}_2 - \mathcal{I}_3 + i\mathcal{P}_3 & \mathcal{P}_1 - i\mathcal{I}_1 - \mathcal{P}_4 - i\mathcal{I}_4 \end{pmatrix}, \tag{3.1}$$

In any Clifford algebra, thus also in $Cl_{3,3}$ we may separate an even part Ψ^+ from an odd part Ψ^- with:

$$\Psi^+ := \Psi_0^{3,3} + \Psi_2^{3,3} + \Psi_4^{3,3} + \Psi_6^{3,3};\ \Psi^- := \Psi_1^{3,3} + \Psi_3^{3,3} + \Psi_5^{3,3} \tag{3.2}$$

We may also separate the parts containing, or not, a i factor in the 4×4 matrices:

$$\begin{aligned}\Psi^H &:= \begin{pmatrix}\mathcal{P}_1+\mathcal{P}_4 & \mathcal{I}_2+\mathcal{I}_3\\ \mathcal{I}_2-\mathcal{I}_3 & \mathcal{P}_1-\mathcal{P}_4\end{pmatrix}\\ &= \begin{pmatrix}\phi_e+\phi_{db} & 0 & 0 & \phi_{ur}+\phi_{ug}\\ 0 & \widehat{\phi}_e+\widehat{\phi}_{db} & \widehat{\phi}_{ur}+\widehat{\phi}_{ug} & 0\\ 0 & \phi_{ur}-\phi_{ug} & \phi_e-\phi_{db} & 0\\ \widehat{\phi}_{ur}-\widehat{\phi}_{ug} & 0 & 0 & \widehat{\phi}_e-\widehat{\phi}_{db}\end{pmatrix},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Psi^N &:= \begin{pmatrix}-i\mathcal{I}_1+i\mathcal{I}_4 & -i\mathcal{P}_2-i\mathcal{P}_3\\ -i\mathcal{P}_2+i\mathcal{P}_3 & -i\mathcal{I}_1-i\mathcal{I}_4\end{pmatrix} = -i\begin{pmatrix}\mathcal{I}_1-\mathcal{I}_4 & \mathcal{P}_2+\mathcal{P}_3\\ \mathcal{P}_2-\mathcal{P}_3 & \mathcal{I}_1+\mathcal{I}_4\end{pmatrix}\\ &= -i\begin{pmatrix}0 & \phi_n-\phi_{ub} & \phi_{dr}+\phi_{dg} & 0\\ \widehat{\phi}_n-\widehat{\phi}_{ub} & 0 & 0 & \widehat{\phi}_{dr}+\widehat{\phi}_{dg}\\ \phi_{dr}-\phi_{dg} & 0 & 0 & \phi_n+\phi_{ub}\\ 0 & \widehat{\phi}_{dr}-\widehat{\phi}_{dg} & \widehat{\phi}_n+\widehat{\phi}_{ub} & 0\end{pmatrix}.\end{aligned} \tag{3.4}$$

The part without a i factor is called Ψ^H because it contains all elements of a hydrogen atom: ϕ_e refers to the electron, while the wave of the proton is made with the three other terms (a d quark and two u quarks, with the three colors, which well fits to the three terms ϕ_{db}, ϕ_{ur} and ϕ_{ug}. The part with i factors is called Ψ^N because it contains all elements of a neutron: l'onde ϕ_n of the neutrino–monopole studied in the previous Chapter and the three quarks of a neutron, a u quark and two d quarks, whith here also the three colors, corresponding to the three terms ϕ_{ub}, ϕ_{dr} and ϕ_{dg}. And hence we called ϕ^D the complete wave, with value in End(Cl_3, because it includes all elements of a deuterium atom:

$$\Psi^D = \begin{pmatrix}\phi_e+\phi_{db} & -i(\phi_n-\phi_{ub}) & -i(\phi_{dr}+\phi_{dg}) & \phi_{ur}+\phi_{ug}\\ -i(\widehat{\phi}_n-\widehat{\phi}_{ub}) & \widehat{\phi}_e+\widehat{\phi}_{db} & \widehat{\phi}_{ur}+\widehat{\phi}_{ug} & -i(\widehat{\phi}_{dr}+\widehat{\phi}_{dg})\\ -i(\phi_{dr}-\phi_{dg}) & \phi_{ur}-\phi_{ug} & \phi_e-\phi_{db} & -i(\phi_n+\phi_{ub})\\ \widehat{\phi}_{ur}-\widehat{\phi}_{ug} & -i(\widehat{\phi}_{dr}-\widehat{\phi}_{dg}) & -i(\widehat{\phi}_n+\widehat{\phi}_{ub}) & \widehat{\phi}_e-\widehat{\phi}_{db}\end{pmatrix}. \tag{3.5}$$

We replace the index of color r, g, b by an upper numeric index:

$$\begin{aligned}&\Psi^1 := \Psi_l;\ \Psi^2 := \Psi_r;\ \Psi^3 := \Psi_g;\ \Psi^4 := \Psi_b,\\ &\Psi^D = \begin{pmatrix}\Psi^1+i\Psi^4 & \Psi^2+\Psi^3\\ \Psi^2-\Psi^3 & \Psi^1-i\Psi^4\end{pmatrix},\end{aligned} \tag{3.6}$$

$$\Psi_l = \Psi^1 = \mathcal{P}_1 - i\mathcal{I}_1 = \begin{pmatrix}\phi_e & -i\phi_n\\ -i\widehat{\phi}_n & \widehat{\phi}_e\end{pmatrix} =: \begin{pmatrix}\phi^1 & \phi^{8\dagger}\\ -\overline{\phi}^8 & \widehat{\phi}^1\end{pmatrix} \tag{3.7}$$

We use the identity in Cl_3 between the adjoint $\phi^\dagger$ and the reverse $\widetilde{\phi}$. Next the $P : \phi \mapsto \widehat{\phi}$ transformation is the main automorphism in Cl_3 (parity). We may identify $\mathcal{P}_n + \mathcal{I}_n$ with its first row:

$$\Psi^2 = \mathcal{I}_2 - i\mathcal{P}_2 = \begin{pmatrix} -i\phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{ur} & -i\widehat{\phi}_{dr} \end{pmatrix} = \begin{pmatrix} \phi^2 & \widetilde{\phi}^5 \\ \overline{\phi}^5 & -\widehat{\phi}^2 \end{pmatrix} =: \begin{pmatrix} \phi^2 & \widetilde{\phi}^5 \end{pmatrix}, \quad (3.8)$$

$$\Psi^3 = \mathcal{I}_3 - i\mathcal{P}_3 = \begin{pmatrix} -i\phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{ug} & -i\widehat{\phi}_{dg} \end{pmatrix} = \begin{pmatrix} \phi^3 & \widetilde{\phi}^6 \\ \overline{\phi}^6 & -\widehat{\phi}^3 \end{pmatrix} =: \begin{pmatrix} \phi^3 & \widetilde{\phi}^6 \end{pmatrix}, \quad (3.9)$$

$$\Psi^4 = \mathcal{I}_4 - i\mathcal{P}_4 = \begin{pmatrix} -i\phi_{db} & \phi_{ub} \\ \widehat{\phi}_{ub} & -i\widehat{\phi}_{db} \end{pmatrix} = \begin{pmatrix} \phi^4 & \widetilde{\phi}^7 \\ \overline{\phi}^7 & -\widehat{\phi}^4 \end{pmatrix} = \begin{pmatrix} \phi^4 & \widetilde{\phi}^7 \end{pmatrix}, \quad (3.10)$$

$$\Psi^n = \begin{pmatrix} \phi^n & \widetilde{\phi}^{3+n} \end{pmatrix}, \ n = 2, 3, 4, \quad (3.11)$$

easing calculations with the structure of modulus on the Cl_3 ring: $Cl_3 \times Cl_3$. The two supplementary dimensions of time that should allow us to pass from $Cl_{1,3}$ into $Cl_{3,3}$ do not have physical reality. This $Cl_{3,3}$ is interesting only as Lie algebra of the real Lie group $Cl^*_{3,3} = \mathrm{End}(Cl_3)$. The six R^n and the six L^n are the only mathematical objects that are really important in this chapter:

$$R^n = \phi^n \frac{1+\sigma_3}{2}; \ L^n = \phi^n \frac{1-\sigma_3}{2}; \ n = 2, 3, 4, \quad (3.12)$$

$$\widetilde{R}^{3+n} = \widetilde{\phi}^{3+n} \frac{1+\sigma_3}{2}; \ \widetilde{L}^{3+n} = \widetilde{\phi}^{3+n} \frac{1-\sigma_3}{2}. \quad (3.13)$$

As previously electroweak interactions (and further strong interactions) are obtained by replacing partial derivatives by gauge-invariant derivatives. We always use notations of B.2. We indicate in which algebra we are calculating as follows: The same vectors of space-time are underlined when we express them in $Cl_{3,3}$. They are in bold when we express them in $Cl_{1,3}$ and will be in ordinary characters or in Roman characters when we express them in Cl_3. In this chapter we will use the index 0 for the time component of space-time vectors, the 4 and 5 indexes being those of the two fictitious supplementary dimensions. We let:

$$\underline{W}^j = \Gamma^\mu W^j_\mu, \ j = 1, 2, 3 \ ; \ \underline{D} = \Gamma^\mu D_\mu \ ; \ \Gamma^0 = \Gamma_0 \ ; \ \Gamma^j = -\Gamma_j; \ \underline{\mathbf{i}} = \Gamma_{0123},$$

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}; \ \mathbf{W}^j = W^j_\mu \gamma^\mu = \begin{pmatrix} 0 & W^j \\ \widehat{W}^j & 0 \end{pmatrix}; \ W^j = W^j_\mu \sigma^\mu, \quad (3.14)$$

$$\underline{D} = \begin{pmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{D} \end{pmatrix}; \ \mathbf{D} = D_\mu \gamma^\mu = \begin{pmatrix} 0 & D \\ \widehat{D} & 0 \end{pmatrix}; \ D = D_\mu \sigma^\mu; \ \underline{\mathbf{i}} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}.$$

The partial derivatives become for the electroweak gauge:

$$\underline{D}(\Psi) = \underline{\partial}(\Psi) + \frac{g_1}{2} \underline{B} \ \underline{P}_0(\Psi) + \frac{g_2}{2} \underline{W}^j \underline{P}_j(\Psi), \quad (3.15)$$

$$\underline{\partial} = \Gamma^\mu \partial_\mu = \begin{pmatrix} \boldsymbol{\partial} & 0 \\ 0 & \boldsymbol{\partial} \end{pmatrix};\ \boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix};\ \nabla = \sigma^\mu \partial_\mu, \tag{3.16}$$

$$\underline{B} = \Gamma^\mu B_\mu = \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{pmatrix};\ \mathbf{B} = B_\mu \gamma^\mu = \begin{pmatrix} 0 & B \\ \widehat{B} & 0 \end{pmatrix};\ B = B_\mu \sigma^\mu.$$

We use two projectors $\underline{P}_\pm$ satisfying:

$$\underline{P}_\pm(\Psi_q) = \frac{1}{2}(\Psi_q \pm \underline{\mathbf{i}}\Psi_q \Gamma_{21});\ P_\pm(\Psi^n) = \frac{1}{2}(\Psi^n \pm \mathrm{i}\Psi^n \gamma_{21}), \tag{3.17}$$

$$P_+(\Psi^n) = \Psi^n_L;\ P_-(\Psi^n) = \Psi^n_R. \tag{3.18}$$

And we define $P_j(\Psi^n), j = 1, 2, 3,\ n = 1, 2, 3, 4$ (we recall that $\Psi_l = \Psi^1$) by:

$$\underline{P}_1(\Psi) = \Gamma_{0123} P_+(\Psi)\Gamma_{35}, \tag{3.19}$$

$$\underline{P}_2(\Psi) = \Gamma_{0123} P_+(\Psi)\Gamma_{5012}, \tag{3.20}$$

$$\underline{P}_3(\Psi) = P_+(\Psi)(-\Gamma_{0123}), \tag{3.21}$$

$$\underline{P}_j(\Psi) = \frac{1}{2}\begin{pmatrix} P_j(\Psi_l) + iP_j(\Psi^4) & P_j(\Psi^2) + P_j(\Psi^3) \\ P_j(\Psi^2) - P_j(\Psi^3) & P_j(\Psi_l) - iP_j(\Psi^4) \end{pmatrix}, j = 0, 1, 2, 3.$$

The three operators $\underline{P}_j,\ j = 1, 2, 3$ act on the quark sector as they do on the lepton sector:

$$P_1(\Psi^n) = \mathbf{i}P_+(\Psi^n)\gamma_3\gamma_5, \tag{3.22}$$

$$P_2(\Psi^n) = \mathbf{i}P_+(\Psi^n)(-i\gamma_3), \tag{3.23}$$

$$P_3(\Psi^n) = P_+(\Psi^n)(-\mathbf{i}). \tag{3.24}$$

On the contrary the fourth operator acts differently on the wave of leptons and on the quark sector (we will explain this difference at the end of this section). Here we again use the operator P_0 defined in (2.55). The operators acting on the waves of quarks have a similar yet nevertheless different form:

$$\begin{aligned} P_0(\Psi_l) &= \Psi_l\gamma_{21} + (1-p)P_-(\Psi_l)\mathbf{i} + p\mathbf{i}P_-(\Psi_l), \\ P_0(\Psi^n) &= -\frac{1}{3}\Psi^n\gamma_{21} + P_-(\Psi^n)\mathbf{i} \\ &= -\frac{1}{3}\Psi^n\gamma_{21} + \frac{1}{2}(\Psi^n\mathbf{i} - \mathbf{i}\Psi^n\gamma_{03}),\ n = 2, 3, 4. \end{aligned} \tag{3.25}$$

Even if p was null and thus there could not exist any magnetic monopole, an important difference should subsist between (2.55) and (3.25), since the coefficient of $\Psi_l\gamma_{21}$ is 1, while the coefficient of each of the three $\Psi^n\gamma_{21}$ is $-1/3$. We remark that since the quarks have color in triplicates, the sum of the coefficient is $1 + 3(-1/3) = 0$, which indeed is not at random [1]. Next

1. This cancellation is very useful in the Standard Model to suppress the “anomalies” linked to the chiral behavior of weak interactions. That played an important role in the discovery of the quarks and their three color charges.

for $n = 1, 2, 3, 4$ we let:

$$\frac{\phi^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & -\overline{\eta}_2^n \\ \xi_2^n & \overline{\eta}_1^n \end{pmatrix} ; \ \frac{R^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & 0 \\ \xi_2^n & 0 \end{pmatrix} ; \ \xi^n = \begin{pmatrix} \xi_1^n \\ \xi_2^n \end{pmatrix} ; \ \widehat{\eta}^n = \begin{pmatrix} -\overline{\eta}_2^n \\ \overline{\eta}_1^n \end{pmatrix} , \quad (3.26)$$

and for $n = 5, 6, 7, 8$ we let:

$$\frac{\widetilde{\phi}^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & -\overline{\eta}_2^n \\ \xi_2^n & \overline{\eta}_1^n \end{pmatrix} ; \ \frac{\widetilde{R}^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & 0 \\ \xi_2^n & 0 \end{pmatrix} ; \ \xi^n = \begin{pmatrix} \xi_1^n \\ \xi_2^n \end{pmatrix} ; \ \widehat{\eta}^n = \begin{pmatrix} -\overline{\eta}_2^n \\ \overline{\eta}_1^n \end{pmatrix} . \quad (3.27)$$

We then have for $n = 1, 2, 3, 4$:

$$\frac{\widehat{\phi}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & -\overline{\xi}_2^n \\ \eta_2^n & \overline{\xi}_1^n \end{pmatrix} ; \ \frac{\widehat{L}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & 0 \\ \eta_2^n & 0 \end{pmatrix} ; \ \eta^n = \begin{pmatrix} \eta_1^n \\ \eta_2^n \end{pmatrix} ; \ \widehat{\xi}^n = \begin{pmatrix} -\overline{\xi}_2^n \\ \overline{\xi}_1^n \end{pmatrix} , \quad (3.28)$$

and for $n = 5, 6, 7, 8$:

$$\frac{\overline{\phi}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & -\overline{\xi}_2^n \\ \eta_2^n & \overline{\xi}_1^n \end{pmatrix} ; \ \frac{\overline{L}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & 0 \\ \eta_2^n & 0 \end{pmatrix} ; \ \eta^n = \begin{pmatrix} \eta_1^n \\ \eta_2^n \end{pmatrix} ; \ \widehat{\xi}^n = \begin{pmatrix} -\overline{\xi}_2^n \\ \overline{\xi}_1^n \end{pmatrix} . \quad (3.29)$$

P_+ is the projector on left waves and P_- on right waves. For $n = 2, 3, 4$ we have:

$$P_-(\Psi^n) = \begin{pmatrix} R^n & \widetilde{R}^{3+n} \\ \overline{R}^{3+n} & -\widehat{R}^n \end{pmatrix} ; P_+(\Psi^n) = \begin{pmatrix} L^n & \widetilde{L}^{3+n} \\ \overline{L}^{3+n} & -\widehat{L}^n \end{pmatrix} , \quad (3.30)$$

where we recall that the waves numbered $2, 3, 4$ are the states of color r, g, b of the d quark while the waves numbered $5, 6, 7$ are the states of color r, g, b of the u quark. We have the same for the lepton part of the wave with yet the upper indexes 1 and 8 instead the n and $3 + n$. We then get for $n = 2, 3, 4$:

$$\begin{aligned} P_0(\Psi^n) &= -\frac{1}{3}\Psi^n\gamma_{21} + P_-(\Psi^n)\mathbf{i} \\ &= \frac{i}{3}\begin{pmatrix} 2R^n + L^n & -4\widetilde{R}^{3+n} + \widetilde{L}^{3+n} \\ 4\overline{R}^{3+n} - \overline{L}^{3+n} & 2\widehat{R}^n + \widehat{L}^n \end{pmatrix} , \end{aligned} \quad (3.31)$$

$$\frac{g_1}{2}\mathbf{B}P_0(\Psi^n) = \mathbf{b}P_0(\Psi^n) = \frac{i}{3}\begin{pmatrix} \mathrm{b}(4\overline{R}^{3+n} - \overline{L}^{3+n}) & \mathrm{b}(2\widehat{R}^n + \widehat{L}^n) \end{pmatrix} .$$

Since P_1, P_2 and P_3 remain unchanged when we move on to the quark sector, on the model of (2.68) and (2.70) we have:

$$P_1(\Psi^n) = i\begin{pmatrix} \widetilde{L}^{3+n} & L^n \\ -\widehat{L}^n & \overline{L}^{3+n} \end{pmatrix} ; \ P_2(\Psi^n) = i^2\begin{pmatrix} -\widetilde{L}^{3+n} & L^n \\ \widehat{L}^n & \overline{L}^{3+n} \end{pmatrix} . \quad (3.32)$$

For $j = 3$ we get:

$$P_3(\Psi^n) = i\begin{pmatrix} -L^n & \widetilde{L}^{3+n} \\ -\overline{L}^{3+n} & -\widehat{L}^n \end{pmatrix} . \quad (3.33)$$

We then have:

$$\mathbf{w}^j P_j(\Psi^n) = \left(-i[(\mathbf{w}^1 - i\mathbf{w}^2)\widehat{L}^n + \mathbf{w}^3\overline{L}^{3+n}] \quad i[(\mathbf{w}^1 + i\mathbf{w}^2)\overline{L}^{3+n} - \mathbf{w}^3\widehat{L}^n]\right). \tag{3.34}$$

Now (3.16) implies:

$$\begin{aligned}\mathbf{D}\Psi^n &= \boldsymbol{\partial}\Psi^n + \frac{g_1}{2}\mathbf{B}P_0(\Psi^n) + \frac{g_2}{2}\mathbf{W}^j P_j(\Psi^n) \\ &= \boldsymbol{\partial}\Psi^n + \mathbf{b}P_0(\Psi^n) + \mathbf{w}^j P_j(\Psi^n).\end{aligned} \tag{3.35}$$

That gives for the right waves:

$$D\widehat{R}^n = \nabla\widehat{R}^n - \frac{2i}{3}\mathrm{b}\widehat{R}^n;\ D\overline{R}^{3+n} = \nabla\overline{R}^{3+n} + \frac{4i}{3}\mathrm{b}\overline{R}^{3+n}. \tag{3.36}$$

And for the left waves we get:

$$D\widehat{L}^n = \nabla\widehat{L}^n - \frac{i}{3}\mathrm{b}\widehat{L}^n - i[(\mathrm{w}^1 + i\mathrm{w}^2)\overline{L}^{3+n} - \mathrm{w}^3\widehat{L}^n], \tag{3.37}$$

$$D\overline{L}^{3+n} = \nabla\overline{L}^{3+n} - \frac{i}{3}\mathrm{b}\overline{L}^{3+n} - i[(\mathrm{w}^1 - i\mathrm{w}^2)\widehat{L}^n + \mathrm{w}^3\overline{L}^{3+n}]. \tag{3.38}$$

Since the operators P_1, P_2 and P_3 act exactly in the same way in the sector of leptons as in the sector of quarks, the gauge invariance that we studied in 2.3 works similarly. This allows us to obtain the values of the gauge fields. And instead of (2.129) we have:

$$\begin{aligned}&\mathrm{D}_L^{n,\,3+n} - id_L^{n,\,3+n} = 2L^n L^{3+n};\ \mathrm{D}_L^n = L^n\widetilde{L}^n;\ \mathrm{D}_L^{3+n} = \widetilde{L}^{3+n}L^{3+n}, \\ &W_n^1 = \mathrm{D}_L^{n,\,3+n};\ W_n^2 = d_L^{n,\,3+n};\ W_n^3 = \mathrm{D}_L^{3+n} - \mathrm{D}_L^n.\end{aligned} \tag{3.39}$$

We added an index n to the W^j: even if they have the same properties, the W_n^j change with the color or when we pass from leptons to quarks. The gauge invariance is similar to that of the lepton wave. The result, as with the lepton part, is a simplification of the covariant derivatives which become:

$$\begin{aligned}D\widehat{R}^n &= \nabla\widehat{R}^n - \frac{2i}{3}\mathrm{b}\widehat{R}^n, \\ D\widehat{L}^n &= \nabla\widehat{L}^n - \frac{i}{3}\mathrm{b}\widehat{L}^n + 3i\mathrm{w}_3^3\widehat{L}^n;\ \mathrm{w}_n^j = \frac{g_2}{2}W_n^j,\end{aligned} \tag{3.40}$$

$$\begin{aligned}D\overline{R}^{3+n} &= \nabla\overline{R}^{3+n} + \frac{4i}{3}\mathrm{b}\overline{R}^{3+n}, \\ D\overline{L}^{3+n} &= \nabla\overline{L}^{3+n} - \frac{i}{3}\mathrm{b}\overline{L}^{3+n} - 3i\mathrm{w}_n^3\overline{L}^{3+n}.\end{aligned} \tag{3.41}$$

By using the Weinberg–Salam angle of 30° of the lepton case we have:

$$\mathrm{b} = \frac{q}{2}\mathrm{A} - \frac{q}{2\sqrt{3}}Z_n^0;\ 3\mathrm{w}_n^3 = \frac{q}{2}\mathrm{A} + \frac{q\sqrt{3}}{2}Z_n^0. \tag{3.42}$$

We thus have for the d quark:

$$\begin{aligned}
D\widehat{R}^n &= (\nabla - i\frac{q}{3}\mathrm{A} + i\frac{q}{3\sqrt{3}}Z_n^0)\widehat{R}^n, \\
D\widehat{L}^n &= (\nabla + i\frac{q}{3}\mathrm{A} + i\frac{5q}{3\sqrt{3}}Z_n^0)\widehat{L}^n, \\
D\widehat{\phi}^n &= \nabla\widehat{\phi}^n + \frac{q}{3}\mathrm{A}\widehat{\phi}^n\sigma_{12} + i\frac{q}{3\sqrt{3}}Z_n^0(\widehat{R}^n + 5\widehat{L}^n).
\end{aligned} \tag{3.43}$$

This matches well what we expect: The electric charge of the d quark is exactly one third of the charge of the electron (negative). As for the u quark we have:

$$\begin{aligned}
D\widehat{R}^{3+n} &= (\nabla + i\frac{2q}{3}\mathrm{A} - i\frac{2q}{3\sqrt{3}}Z_n^0)\widehat{R}^{3+n} \\
D\widehat{L}^{3+n} &= (\nabla - i\frac{2q}{3}\mathrm{A} - i\frac{4q}{3\sqrt{3}}Z_n^0)\widehat{L}^{3+n}, \\
D\widehat{\phi}^{3+n} &= \nabla\widehat{\phi}^{3+n} - \frac{2q}{3}\mathrm{A}\widehat{\phi}^{3+n}\sigma_{12} - i\frac{2q}{3\sqrt{3}}Z_n^0(\widehat{R}^{3+n} + 2\widehat{L}^{3+n}).
\end{aligned} \tag{3.44}$$

Here we also obtain the expected result since the charge of the u quark is positive and equal to -2 times the charge of the d quark. What we obtained in the first chapter for charge conjugation is indeed conserved: the antiquark of d seems to have a charge equal to half of that of the u quark, and the antiquark of u seems to have a charge double that of the d quark. We also recall that charge conjugation is not only an apparent change of sign of the electric charges: right and left waves are also exchanged. **Here we have an important result which reduces the too high number of free parameters in the Standard Model**: the simple replacement of the coefficient 1 of $\Psi\gamma_{21}$ by $-1/3$ in $\underline{P}_0$ is enough to obtain the two values of the electric charge of the two kinds of quarks. Thus we have only one free parameter instead two.

3.2 Chromodynamics

The Standard Model considers strong interactions as resulting also from a gauge invariance under a $SU(3)$ color group, from whence comes the word "chromodynamics." We transpose this group to Clifford algebra in a manner similar to that used for weak interactions. We now define Γ_k in a manner similar to $\underline{P}_j$ of the previous section. We know the $i\lambda_k$ generators of the $SU(3)$ group:

$$i\lambda_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ i\lambda_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ i\lambda_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.45}$$

$$i\lambda_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ i\lambda_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ i\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \tag{3.46}$$
$$i\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ i\lambda_8 = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

To simplify our notations, we use l, r, g, b instead of Ψ_l, $\Psi_r = \Psi^2$, $\Psi_g = \Psi^3$, $\Psi_b = \Psi^4$. So we have:

$$\Psi = \begin{pmatrix} l+ib & r+g \\ r-g & l-ib \end{pmatrix}. \tag{3.47}$$

The unique i of quantum mechanics must not be confused with the i of the above relation, which is responsible for the orientation of Cl_3. Thus in $Cl_3 \times Cl_3$ we must replace this commutative i by $\mathbf{i} = \gamma_{0123}$, responsible for the orientation of space-time, which does not commute. Therefore (3.45) and (3.46) give:

$$i\lambda_1\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{i}g \\ \mathbf{i}r \\ 0 \end{pmatrix},\ i\lambda_2\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} g \\ -r \\ 0 \end{pmatrix},\ i\lambda_3\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{i}r \\ -\mathbf{i}g \\ 0 \end{pmatrix},$$
$$i\lambda_4\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{i}b \\ 0 \\ \mathbf{i}r \end{pmatrix},\ i\lambda_5\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ -r \end{pmatrix},\ i\lambda_6\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{i}b \\ \mathbf{i}g \end{pmatrix}, \tag{3.48}$$
$$i\lambda_7\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ -g \end{pmatrix},\ i\lambda_8\begin{pmatrix} r \\ g \\ b \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} \mathbf{i}r \\ \mathbf{i}g \\ -2\mathbf{i}b \end{pmatrix}.$$

The $\mathbf{\Lambda}_k$ corresponding to the $i\lambda_k$ acting on Ψ are (see B.2):

$$\begin{aligned}
\mathbf{\Lambda}_1(\Psi) &= -\frac{1}{2}(\Gamma_{45}\Psi + \Gamma_{0123}\Psi S);\ S = \Gamma_{012345},\\
\mathbf{\Lambda}_2(\Psi) &= -\frac{1}{2}(\Gamma_4\Psi\Gamma_{01235} + \Gamma_{01235}\Psi\Gamma_4),\\
\mathbf{\Lambda}_3(\Psi) &= \frac{1}{2}(\Gamma_5\Psi\Gamma_{01235} - \Gamma_{01234}\Psi\Gamma_4),\\
\mathbf{\Lambda}_4(\Psi) &= \frac{1}{2}(\Gamma_{0123}\Psi\Gamma_4 - \Gamma_{01234}\Psi),\\
\mathbf{\Lambda}_5(\Psi) &= -\frac{1}{2}(S\Psi\Gamma_{01235} + \Gamma_{01235}\Psi S)\\
\mathbf{\Lambda}_6(\Psi) &= \frac{1}{2}(\Gamma_{01234}\Psi S - \Gamma_{45}\Psi\Gamma_4),\\
\mathbf{\Lambda}_7(\Psi) &= \frac{1}{2}(\Gamma_{01235}\Psi - \Psi\Gamma_{01235}),\\
\mathbf{\Lambda}_8(\Psi) &= -\frac{1}{2\sqrt{3}}(2\Gamma_{45}\Psi S + \Gamma_5\Psi\Gamma_{01235} + \Gamma_{01234}\Psi\Gamma_4).
\end{aligned} \tag{3.49}$$

The previous system is equivalent to:

$$\begin{aligned}
&\mathbf{\Lambda}_1(\Psi_l)=0;\ \mathbf{\Lambda}_1(\Psi_r)=\mathbf{i}\Psi_g;\ \mathbf{\Lambda}_1(\Psi_g)=\mathbf{i}\Psi_r;\ \mathbf{\Lambda}_1(\Psi_b)=0,\\
&\mathbf{\Lambda}_2(\Psi_l)=0;\ \mathbf{\Lambda}_2(\Psi_r)=\Psi_g;\ \mathbf{\Lambda}_2(\Psi_g)=-\Psi_r;\ \mathbf{\Lambda}_2(\Psi_b)=0,\\
&\mathbf{\Lambda}_3(\Psi_l)=0;\ \mathbf{\Lambda}_3(\Psi_r)=\mathbf{i}\Psi_5;\ \mathbf{\Lambda}_3(\Psi_g)=-\mathbf{i}\Psi_g;\ \mathbf{\Lambda}_3(\Psi_b)=0,\\
&\mathbf{\Lambda}_4(\Psi_l)=0;\ \mathbf{\Lambda}_4(\Psi_r)=\mathbf{i}\Psi_b;\ \mathbf{\Lambda}_1(\Psi_g)=0;\ \mathbf{\Lambda}_4(\Psi_b)=\mathbf{i}\Psi_r, \qquad (3.50)\\
&\mathbf{\Lambda}_5(\Psi_l)=0;\ \mathbf{\Lambda}_5(\Psi_r)=\Psi_b;\ \mathbf{\Lambda}_5(\Psi_g)=0;\ \mathbf{\Lambda}_5(\Psi_b)=-\Psi_r,\\
&\mathbf{\Lambda}_6(\Psi_l)=0;\ \mathbf{\Lambda}_6(\Psi_r)=0;\ \mathbf{\Lambda}_6(\Psi_g)=\mathbf{i}\Psi_b;\ \mathbf{\Lambda}_6(\Psi_b)=\mathbf{i}\Psi_g,\\
&\mathbf{\Lambda}_7(\Psi_l)=0;\ \mathbf{\Lambda}_7(\Psi_r)=0;\ \mathbf{\Lambda}_7(\Psi_g)=\Psi_b;\ \mathbf{\Lambda}_7(\Psi_b)=-\Psi_g,\\
&\mathbf{\Lambda}_8(\Psi_l)=0;\ \mathbf{\Lambda}_8(\Psi_r)=-\frac{\mathbf{i}}{\sqrt{3}}\Psi_g r;\ \mathbf{\Lambda}_8(\Psi_g)=\frac{\mathbf{i}}{\sqrt{3}}\Psi_g;\ \mathbf{\Lambda}_1(\Psi_b)=\frac{2\mathbf{i}}{\sqrt{3}}\Psi_b.
\end{aligned}$$

Each Λ_k projects the Ψ wave onto the quark sector Ψ_q. Thus **the lepton part of the wave does not see color forces.** That is well known experimentally, and we need no more to postulate this property. We extend to strong interactions the gauge-invariant derivative of the electroweak interactions (3.16) by letting:

$$\underline{D}(\Psi)=\underline{\partial}(\Psi)+\frac{g_1}{2}\underline{B}\ \underline{P}_0(\Psi)+\frac{g_2}{2}\underline{W}^j\underline{P}_j(\Psi)+\frac{g_3}{2}\underline{G}^k\mathbf{\Lambda}_k(\Psi), \qquad (3.51)$$

where g_3 is a third constant and the $\underline{G}^k$ are eight potential vectors called gluons. Since I_4 commutes with any element in $Cl_{1,3}$ and since we have $P_j(\mathbf{i}\Psi_{ind})=\mathbf{i}P_j(\Psi_{ind})$ for $j=0,1,2,3$ and $ind=l,r,g,b$, we obtain that each operator $\underline{\mathbf{i}}\Gamma_k$ commutes with each operator $\underline{P}_j$. Now we use twelve reals: a^0, a^j, $j=1,2,3$, b^k, $k=1,2,...,8$, and we let:

$$S_0=a^0\underline{P}_0;\ S_1=\sum_{j=1}^{j=3}a^j\underline{P}_j;\ S_2=\sum_{k=1}^{k=8}b^k\mathbf{\Lambda}_k;\ \Sigma=S_0+S_1+S_2, \qquad (3.52)$$

and by using the exponential function we get:

$$\begin{aligned}
\exp(\Sigma)&=\exp(S_0)\exp(S_1)\exp(S_2)=\exp(S_1)\exp(S_0)\exp(S_2)\\
&=\exp(S_0)\exp(S_2)\exp(S_1)=\dots \qquad (3.53)
\end{aligned}$$

in any order, thanks to the commutation of $\underline{P}_0$ with $\underline{P}_j$, $j=1,2,3$ as well as with the commutation of $\underline{P}_j$, $j=0,1,2,3$ with $\mathbf{\Lambda}_k$, $k=1,\dots,8$. The set of $\exp(S)$ operators is a $U(1)\times SU(2)\times SU(3)$ Lie group. The first difference from the Standard Model is: **we do not need to postulate this structure since it results from the calculation of commutators**. The invariance under Cl_3^* (and consequently the relativistic invariance) of this gauge-invariant derivation is similar to that obtained in 2.3. The gauge invariance [36][47][48] may be expressed as:

$$\Psi' = [\exp(a^0 \underline{P}_0 + S_1 + S_2)](\Psi) \ ; \ \underline{D} = \mathbf{\Lambda}^\mu \underline{D}_\mu \ ; \ \underline{D}' = \mathbf{\Lambda}^\mu \underline{D}'_\mu, \quad (3.54)$$

$$\underline{D}'_\mu \Psi' = \exp(a^0 \underline{P}_0 + S_1 + S_2) \underline{D}_\mu \Psi, \quad (3.55)$$

$$B'_\mu = B_\mu - \frac{2}{g_1} \partial_\mu a^0, \quad (3.56)$$

$$W'^j_\mu \underline{P}_j = \Big[\exp(S_1) W^j_\mu \underline{P}_j - \frac{2}{g_2} \partial_\mu [\exp(S_1)] \Big] \exp(-S_1), \quad (3.57)$$

$$\underline{G}'^k_\mu \mathbf{\Lambda}_k = \Big[\exp(S_2) \underline{G}^k_\mu \mathbf{\Lambda}_k - \frac{2}{g_3} \partial_\mu [\exp(S_2)] \Big] \exp(-S_2). \quad (3.58)$$

The $SU(3)$ group of chromodynamics generated by the $\mathbf{\Lambda}_k$ operators only acts on the quark sector. By letting:

$$\mathrm{diag}(\Psi) = \frac{1}{4}(\Psi + S\Psi S + \Gamma_4 \Psi \Gamma_4 - \Gamma_{01235} \Psi \Gamma_{01235}) = \begin{pmatrix} \Psi_l & 0 \\ 0 & \Psi_l \end{pmatrix}, \quad (3.59)$$

we have:

$$\mathrm{diag}\Big([\exp(b^k \mathbf{\Lambda}_k)](\Psi) \Big) = \mathrm{diag}(\Psi). \quad (3.60)$$

This comes from the fact that we begin with operators that do not at all act on Ψ_l. For the contrary case to be possible it would be necessary to consider some operators similar to $\mathbf{\Lambda}_k$ coupling the wave of Ψ_l with one of the three Ψ^n waves. That cannot exist because these operators project the right waves onto right waves and the left waves onto left waves, and because the right waves and the left waves of the lepton part, in weak interactions, transform differently compared to the waves of the colored part of the whole wave. We then get a $U(1) \times SU(2) \times SU(3)$ gauge group for a wave incorporating all fermions of the first generation. [2]. All that is certainly well established experimentally. The novelty here is simply the direct rise from the structure of the quantum wave.

Since that is independent of the scale of energies, we can understand why the grand unified theories (GUTs) had no experimental success: it is impossible to get a greater group. Thus it is impossible that a quark may decay into a lepton. This implies the conservation of a quantity that QFT calls the baryonic number. Moreover this conservation was experimentally supported by neutrino observatories like Kamioka. We may say that **our transposition of the Standard Model into Clifford algebra automatically satisfies this law of conservation**. This is a reinforcement of the Standard Model by concordance with experiment.

2. Later we will see how this group acts on the lepton sector only via the $U(1) \times SU(2)$ part. The physical interpretation is: the leptons are incapable of strong interactions. They interact only via electromagnetic and weak interactions.

3.2.1 Three generations, four neutrinos

The purpose of the physical theory is the understanding of experimental facts. Nowadays we must both justify why there are only three kinds of leptons and quarks, and also why there is a fourth neutrino, very different from the three others, which may explain black matter. Experiments show the existence of only three kinds of light leptons from studying the disintegration of Z^0, and experiments also suggest the possibility of the existence of a fourth neutrino. We justified the existence of three kinds of leptons in the previous chapters. This is easily generalized to the three generations of the Standard Model. The two other generations are obtained by replacing the σ_3 of the Dirac equation by σ_1 or σ_2 everywhere this direction is in use. Next, the passing from one generation to the other may be seen as a circular permutation of the indices $1 \mapsto 2 \mapsto 3 \mapsto 1$ or $1 \mapsto 3 \mapsto 2 \mapsto 1$ for the other generation. For instance, the σ_3 used for the projector defining the right wave and the left wave must be replaced by σ_1 or σ_2. And the σ_1 that links the wave of the particle and the antiparticle must be replaced by σ_2 or σ_3. These changes force us to treat each generation separately, and this explains the separate treatment of each generation in the Standard Model. Yet for a fourth generation a similar case is impossible, since the Cl_3 algebra is the algebra of the physical 3-dimensional space. There it is impossible to get a fourth set of operators similar to the P_μ.

But the existence of a fourth neutrino [27] is possible because Cl_3 contains four independent terms with square -1. The wave equation of the electron uses one of these four terms: $i\sigma_3 = \sigma_{12}$. Further, the equalities $i\sigma_1 = \sigma_{23}$ and $i\sigma_2 = \sigma_{31}$ explain why two other kinds of leptons exist. We can also build a form-invariant wave equation with the fourth generator $i = \sigma_{123}$:

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{123} + m\rho = 0. \tag{3.61}$$

Multiplying on the left side by $\overline{\phi}^{\,-1}$ we obtain using $\rho = e^{-i\beta}\overline{\phi}\phi$ the equivalent equation:

$$\nabla\widehat{\phi}i + me^{-i\beta}\phi = 0\ ;\quad \nabla\widehat{\phi} = ime^{-i\beta}\phi. \tag{3.62}$$

We may extend the gauge invariance to a local one:

$$0 = \nabla\widehat{\phi}i + g_1 B\widehat{\phi} + me^{-i\beta}\phi. \tag{3.63}$$

That is equivalent to:

$$0 = i\nabla\eta + g_1 B\eta + me^{-i\beta}\xi, \tag{3.64}$$

$$0 = i\nabla\widehat{\xi} + g_1 B\widehat{\xi} + me^{-i\beta}\widehat{\eta}. \tag{3.65}$$

Contrary to our improved wave equation for the electron which has the Dirac equation as its linear approximation, this wave equation cannot come from the linear quantum theory: no linear approximation exists because the angle β is no longer small. This angle is now the phase of the wave. We nevertheless may obtain the plane waves. We search for solutions satisfying

$$\phi = e^{-i\varphi}\phi_0 \ ; \quad \varphi = mv_\mu x^\mu \ ; \quad v = \sigma^\mu v_\mu, \tag{3.66}$$

where v is a fixed reduced velocity and ϕ_0 is also a fixed term. We get:

$$\nabla\widehat{\phi} = \sigma^\mu \partial_\mu(e^{i\varphi}\widehat{\phi}_0) = imve^{i\varphi}\widehat{\phi}_0. \tag{3.67}$$

And we have:

$$\phi\overline{\phi} = e^{-i\varphi}\phi_0 e^{-i\varphi}\overline{\phi}_0 = e^{-2i\varphi}\phi_0\overline{\phi}_0. \tag{3.68}$$

Then if we let:

$$\phi_0\overline{\phi}_0 = \rho_0 e^{i\beta_0}, \tag{3.69}$$

we finally have:

$$\beta = \beta_0 - 2\varphi \ ; \ e^{-i\beta}\phi = e^{-i(\beta_0 - 2\varphi)}e^{-i\varphi}\phi_0 = e^{-i(\beta_0-\varphi)}\phi_0. \tag{3.70}$$

Hence (3.67) is equivalent to:

$$imve^{i\varphi}\widehat{\phi}_0 = ime^{-i(\beta_0-\varphi)}\phi_0 \tag{3.71}$$

$$v\widehat{\phi}_0 = e^{-i\beta_0}\phi_0; \ e^{i\beta_0}v\widehat{\phi}_0 = \phi_0. \tag{3.72}$$

Using the parity conjugation we get:

$$e^{-i\beta_0}\widehat{v}\phi_0 = \widehat{\phi}_0. \tag{3.73}$$

Hence we have:

$$\phi_0 = e^{i\beta_0}v\widehat{\phi}_0 = e^{i\beta_0}v[e^{-i\beta_0}\widehat{v}\phi_0] = v\widehat{v}\phi_0. \tag{3.74}$$

Then if $\phi_0 \neq 0$ we get:

$$1 = v\widehat{v}, \tag{3.75}$$

which gives $v^0 = \sqrt{1+\vec{v}^2}$ or $v^0 = -\sqrt{1+\vec{v}^2}$ and since (3.72) implies:

$$\begin{aligned} &\phi_0 = ve^{i\beta_0}\widehat{\phi}_0; \ \mathrm{D}_0 = \phi_0\widetilde{\phi}_0 = ve^{i\beta_0}\widehat{\phi}_0\widetilde{\phi}_0 = ve^{i\beta_0}\rho_0 e^{-i\beta_0} = v\rho_0, \\ &\mathrm{D}_0^0 = v^0\rho_0; \ v^0 > 0; \ v^0 = \sqrt{1+\vec{v}^2}. \end{aligned} \tag{3.76}$$

Therefore no plane waves can exist with a sign of energy opposite to the sign of the mass. This wave equation may have a gauge term and may be expressed in an invariant manner of the form:

$$0 = \overline{\phi}(\nabla\widehat{\phi})i + \overline{\phi}qB\widehat{\phi} + m\rho. \tag{3.77}$$

Using the reversion we get

$$0 = -i(\overline{\phi}\nabla)\widehat{\phi} + \overline{\phi}qB\widehat{\phi} + m\rho. \tag{3.78}$$

With (1.122) to (1.125) we have:

$$\overline{\phi}(\nabla\widehat{\phi}) = \frac{1}{2}(\nabla \cdot \mathrm{D}_\mu)\sigma^\mu + iw_\mu\sigma^\mu, \quad (3.79)$$

$$\overline{\phi}B\widehat{\phi} = (B \cdot \mathrm{D}_\mu)\sigma^\mu. \quad (3.80)$$

Adding and subtracting (3.77) and (3.78) we get:

$$0 = \nabla \cdot \mathrm{D}_\mu,\ \mu = 0, 1, 2, 3, \quad (3.81)$$

$$0 = -w_0 + B \cdot \mathrm{D}_0 + m\rho, \quad (3.82)$$

$$0 = -w_j + B \cdot \mathrm{D}_j,\ j = 1, 2, 3. \quad (3.83)$$

The four equations (3.81) are the laws of conservation of the D_μ currents. Hence the probability current density is conserved. Multiplying by $\overline{\phi}^{-1}$ on the left side (3.77) is equivalent to (3.63). This is equivalent to the system:

$$0 = i\nabla\eta + qB\eta + m\mathrm{v}\eta, \quad (3.84)$$

$$0 = i\nabla\widehat{\xi} + qB\widehat{\xi} + m\mathrm{v}\widehat{\xi}. \quad (3.85)$$

Thus with v the wave equation (3.77) is:

$$0 = (\nabla\widehat{\phi})i + qB\widehat{\phi} + m\mathrm{v}\widehat{\phi}. \quad (3.86)$$

Since i commutes with σ_1, the multiplication by σ_1 on the right side of this equation changes nothing: **the fourth neutrino-monopole is its own antiparticle**.

3.3 Preserving the mass term

Similar to the previous section, and the previous chapter, a generalization of the mass term is possible for the improved equation. To see that we begin with the simplification of the part of the gauge-invariant derivative of chromodynamics: we consider equally the three $SU(2)$ subgroups of $SU(3)$ by using the potentials:

$$\mathrm{b} = \frac{g_1}{2}B;\ \mathrm{w}^j = \frac{g_2}{2}W^j,\ j = 1, 2, 3. \quad (3.87)$$

$$\begin{aligned}
\mathrm{h}_1^1 &= \frac{g_3}{2}G^1;\ \mathrm{h}_1^2 = \frac{g_3}{2}G^2;\ \mathrm{h}_1^3 - \mathrm{h}_3^3 = \frac{g_3}{2}\Big(-G^3 - \frac{G^8}{\sqrt{3}}\Big),\\
\mathrm{h}_2^1 &= \frac{g_3}{2}G^6;\ \mathrm{h}_2^2 = \frac{g_3}{2}G^7;\ \mathrm{h}_2^3 - \mathrm{h}_1^3 = \frac{g_3}{2}\Big(G^3 - \frac{G^8}{\sqrt{3}}\Big),\\
\mathrm{h}_3^1 &= \frac{g_3}{2}G^4;\ \mathrm{h}_3^2 = -\frac{g_3}{2}G^5;\ \mathrm{h}_3^3 - \mathrm{h}_2^3 = \frac{g_3}{2}\Big(2\frac{G^8}{\sqrt{3}}\Big).
\end{aligned} \quad (3.88)$$

These potentials introduce no supplementary dimension into the gauge group because the sum of $\mathrm{h}_1^3-\mathrm{h}_3^3$, $\mathrm{h}_2^3-\mathrm{h}_1^3$ and $\mathrm{h}_3^3-\mathrm{h}_2^3$ is null. Next we note:

$$\underline{n}=n \mod 3;\ \underline{3}=3;\ \underline{4}=1;\ \underline{5}=2. \tag{3.89}$$

For the gauge-invariant derivatives some supplementary terms appear containing the gauge potentials G^k or h_n^p:

$$\frac{g_3}{2}G^k\mathbf{\Lambda}_k(\Psi)=\begin{pmatrix}S(\Psi^2)-S(\Psi^3) & S(\Psi^1)-iS(\Psi^4)\\ S(\Psi^1)+iS(\Psi^4) & S(\Psi^2)+S(\Psi_3)\end{pmatrix}, \tag{3.90}$$

$$S(\Psi_1)=0, \tag{3.91}$$

$$\begin{aligned}S(\Psi^2)&=\frac{g_3}{2}\Big[(\mathbf{G}^1-\mathbf{G}^2\mathbf{i})\mathbf{i}\Psi^3-\mathbf{G}^3\mathbf{i}\Psi^2+(\mathbf{G}^4-\mathbf{G}^5\mathbf{i})\mathbf{i}\Psi^4-\frac{1}{\sqrt{3}}\mathbf{i}\mathbf{G}^8\Psi^2\Big]\\&=(\mathbf{h}_1^1-\mathbf{h}_1^2\mathbf{i})\mathbf{i}\Psi^3-\mathbf{h}_1^3\mathbf{i}\Psi^2+(\mathbf{h}_3^1+\mathbf{h}_3^2\mathbf{i})\mathbf{i}\Psi^4+\mathbf{h}_3^3\mathbf{i}\Psi^2,\end{aligned} \tag{3.92}$$

$$\begin{aligned}S(\Psi^3)&=\frac{g_3}{2}\Big[(\mathbf{G}^1+\mathbf{G}^2\mathbf{i})\mathbf{i}\Psi^2+\mathbf{G}^3\mathbf{i}\Psi^3+(\mathbf{G}^6-\mathbf{G}^7\mathbf{i})\mathbf{i}\Psi^4-\frac{1}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi^3\Big]\\&=(\mathbf{h}_2^1-\mathbf{h}_2^2\mathbf{i})\mathbf{i}\Psi^4-\mathbf{h}_2^3\mathbf{i}\Psi^3+(\mathbf{h}_1^1+\mathbf{h}_1^2\mathbf{i})\mathbf{i}\Psi^2+\mathbf{h}_1^3\mathbf{i}\Psi^3,\end{aligned} \tag{3.93}$$

$$\begin{aligned}S(\Psi^4)&=\frac{g_3}{2}\Big[(\mathbf{G}^4+\mathbf{G}^5\mathbf{i})\mathbf{i}\Psi^2+(\mathbf{G}^6+\mathbf{G}^7\mathbf{i})\mathbf{i}\Psi^3+\frac{2}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi^4\Big]\\&=(\mathbf{h}_3^1-\mathbf{h}_3^2\mathbf{i})\mathbf{i}\Psi^2+\mathbf{h}_3^3\mathbf{i}\Psi^4+(\mathbf{h}_2^1+\mathbf{h}_2^2\mathbf{i})\mathbf{i}\Psi^3+\mathbf{h}_2^3\mathbf{i}\Psi^4,\end{aligned} \tag{3.94}$$

The cancellation in (3.91) means that leptons do not experience strong interactions. Next the use of the modulo 3 indices allows us to express the general formula as:

$$S(\Psi^n)=(\mathbf{h}_{n-1}^1-\mathbf{h}_{n-1}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+1}}-\mathbf{h}_{n-1}^3\mathbf{i}\Psi^n+(\mathbf{h}_{\underline{n+1}}^1+\mathbf{h}_{\underline{n+1}}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+2}}+\mathbf{h}_{\underline{n+1}}^3\mathbf{i}\Psi^n. \tag{3.95}$$

We have with (3.51):

$$\begin{aligned}\underline{D}(\Psi)&=\begin{pmatrix}\mathbf{D}\Psi^2-\mathbf{D}\Psi^3 & \mathbf{D}\Psi^1-i\mathbf{D}\Psi^4\\ \mathbf{D}\Psi_l+i\mathbf{D}\Psi^4 & \mathbf{D}\Psi^2+\mathbf{D}\Psi^3\end{pmatrix},\\ \mathbf{D}\Psi_l&=\boldsymbol{\partial}\Psi_l+\frac{g_1}{2}\mathbf{B}P_0(\Psi_l)+\frac{g_2}{2}\mathbf{W}^jP_j(\Psi_l)=\boldsymbol{\partial}\Psi_l+\mathbf{b}P_0(\Psi_l)+\mathbf{w}^jP_j(\Psi_l),\\ \mathbf{D}\Psi^n&=\boldsymbol{\partial}\Psi^n+\frac{g_1}{2}\mathbf{B}P_0(\Psi^n)+\frac{g_2}{2}\mathbf{W}^jP_j(\Psi^n)+S(\Psi^n)\\&=\boldsymbol{\partial}\Psi^n+\mathbf{b}P_0(\Psi^n)+\mathbf{w}^jP_j(\Psi^n)+S(\Psi^n).\end{aligned} \tag{3.96}$$

With the notations in B.1.3 we also have:

$$\begin{aligned}(\mathbf{h}_{n-1}^1-\mathbf{h}_{n-1}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+1}}&=i\left(-(\mathrm{h}_{n-1}^1+i\mathrm{h}_{n-1}^2)\overline{\phi}^{3+\underline{n+1}}\quad -(\mathrm{h}_{n-1}^1+i\mathrm{h}_{n-1}^2)\widehat{\phi}^{\underline{n+1}}\right),\\ -\mathbf{h}_{n-1}^3\mathbf{i}\Psi^n&=i\left(\mathrm{h}_{n-1}^3\overline{\phi}^{3+n}\quad \mathrm{h}_{n-1}^3\widehat{\phi}^n\right).\end{aligned} \tag{3.97}$$

We next have:

$$\begin{aligned}(\mathbf{h}_{\underline{n+1}}^1+\mathbf{h}_{\underline{n+1}}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+2}}&=i\left(-(\mathrm{h}_{\underline{n+1}}^1-i\mathrm{h}_{\underline{n+1}}^2)\overline{\phi}^{3+\underline{n+2}}\quad -(\mathrm{h}_{\underline{n+1}}^1-i\mathrm{h}_{\underline{n+1}}^2)\widehat{\phi}^{\underline{n+2}}\right),\\ \mathbf{h}_{\underline{n+1}}^3\mathbf{i}\Psi^n&=i\left(-\mathrm{h}_{\underline{n+1}}^3\overline{\phi}^{3+n}\quad -\mathrm{h}_{\underline{n+1}}^3\widehat{\phi}^n\right).\end{aligned} \tag{3.98}$$

Then if we let:

$$S(\Psi^n) = \begin{pmatrix} S(\overline{\phi}^{3+n}) & S(\widehat{\phi}^n) \end{pmatrix}, \tag{3.99}$$

we get:

$$\begin{aligned} S(\overline{\phi}^{3+n}) &= -i(\mathrm{h}^1_{n-1} + i\mathrm{h}^2_{n-1})\overline{\phi}^{3+\underline{n+1}} + i\mathrm{h}^3_{n-1}\overline{\phi}^{3+n} \\ &\quad - i(\mathrm{h}^1_{\underline{n+1}} - i\mathrm{h}^2_{\underline{n+1}})\overline{\phi}^{3+\underline{n+2}} - i\mathrm{h}^3_{\underline{n+1}}\overline{\phi}^{3+n}, \end{aligned} \tag{3.100}$$

$$\begin{aligned} S(\widehat{\phi}^n) &= -i(\mathrm{h}^1_{n-1} + i\mathrm{h}^2_{n-1})\overline{\phi}^{3+\underline{n+1}} + i\mathrm{h}^3_{n-1}\widehat{\phi}^n \\ &\quad - i(\mathrm{h}^1_{\underline{n+1}} - i\mathrm{h}^2_{\underline{n+1}})\widehat{\phi}^{\underline{n+2}} - i\mathrm{h}^3_{\underline{n+1}}\widehat{\phi}^n. \end{aligned} \tag{3.101}$$

For the part containing the derivative and the electroweak interaction terms we use equations (3.36) to (3.38). And we use the conjugation $M \mapsto \widehat{M}$ on the right waves, which allows us to get for the gauge-invariant derivative:

$$\begin{aligned} -i\widehat{D}R^n = -i\widehat{\nabla}R^n + \frac{2}{3}\widehat{\mathrm{b}}R^n + (\widehat{\mathrm{h}}^1_{n-1} - i\widehat{\mathrm{h}}^2_{n-1})R^{\underline{n+1}} + (\widehat{\mathrm{h}}^1_{\underline{n+1}} + i\widehat{\mathrm{h}}^2_{\underline{n+1}})R^{\underline{n+2}} \\ - (\widehat{\mathrm{h}}^3_{n-1} - \widehat{\mathrm{h}}^3_{\underline{n+1}})R^n, \end{aligned} \tag{3.102}$$

$$\begin{aligned} -i\overline{D}\widetilde{R}^{3+n} &= -i\widehat{\nabla}\widetilde{R}^{3+n} - \frac{4}{3}\widehat{\mathrm{b}}\widetilde{R}^{3+n} + (\widehat{\mathrm{h}}^1_{n-1} - i\widehat{\mathrm{h}}^2_{n-1})\widetilde{R}^{3+\underline{n+1}} \quad (3.103) \\ &\quad + (\widehat{\mathrm{h}}^1_{\underline{n+1}} + i\widehat{\mathrm{h}}^2_{\underline{n+1}})\widetilde{R}^{3+\underline{n+2}} - (\widehat{\mathrm{h}}^3_{n-1} - \widehat{\mathrm{h}}^3_{\underline{n+1}})\widetilde{R}^{3+n}, \\ iD\widehat{L}^n &= i\nabla\widehat{L}^n + \frac{\mathrm{b}}{3}\widehat{L}^n + [(\mathrm{w}^1_n + i\mathrm{w}^2_n)\overline{L}^{3+n} - \mathrm{w}^3_n\widehat{L}^n] \quad (3.104) \\ &+(\mathrm{h}^1_{n-1} + i\mathrm{h}^2_{n-1})\widehat{L}^{\underline{n+1}} + (\mathrm{h}^1_{\underline{n+1}} - i\mathrm{h}^2_{\underline{n+1}})\widehat{L}^{\underline{n+2}} - (\mathrm{h}^3_{n-1} - \mathrm{h}^3_{\underline{n+1}})\widehat{L}^n, \\ i\widetilde{D}\overline{L}^{3+n} &= i\widetilde{\nabla}\overline{L}^{3+n} + \frac{\mathrm{b}}{3}\overline{L}^{3+n} + [(\mathrm{w}^1_n - i\mathrm{w}^2_n)\widehat{L}^n + \mathrm{w}^3_n\overline{L}^{3+n}] \quad (3.105) \\ &+(\mathrm{h}^1_{n-1} + i\mathrm{h}^2_{n-1})\overline{L}^{3+\underline{n+1}} + (\mathrm{h}^1_{\underline{n+1}} - i\mathrm{h}^2_{\underline{n+1}})\overline{L}^{3+\underline{n+2}} - (\mathrm{h}^3_{n-1} - \mathrm{h}^3_{\underline{n+1}})\overline{L}^{3+n}. \end{aligned}$$

For making this gauge-invariant derivative compatible with the mass term (and we recall that the mass term allows us a direct link with inertia and gravitation), we derive this in a manner completely analogous to that used in 2.2 for the lepton part of the wave. The $\mathrm{v}\widehat{\phi}\mathbf{m}\sigma_{12}$ form of the mass term of the improved equation of the electron is conserved. The only thing that changes is the definition of the unitary vector v. We now have indeed twelve chiral currents:

$$\mathrm{D}^n_R = R^n\widetilde{R}^n;\ \mathrm{D}^n_L = L^n\widetilde{L}^n;\ \mathrm{D}^{3+n}_R = \widetilde{R}^{3+n}R^{3+n};\ \mathrm{D}^{3+n}_L = \widetilde{L}^{3+n}L^{3+n}, \tag{3.106}$$

for $n = 2, 3, 4$. The J_q current that replaces J_l is the sum of these twelve currents:

$$\mathrm{J}_q = \sum_{n=2}^{n=4}[\mathrm{D}^n_R + \mathrm{D}^{3+n}_R + \mathrm{D}^n_L + \mathrm{D}^{3+n}_L];\ \rho_q^2 = (\mathrm{J}_q)^2 = \mathrm{J}_q\widehat{\mathrm{J}}_q;\ \mathrm{v}_q = \frac{\mathrm{J}_q}{\rho_q}. \tag{3.107}$$

Since the J_q current is the sum of twelve currents, the calculation of the squared scalar product of this vector has twelve squares, all null since each chiral current is on the light cone. And there are $66 = 12 \times 11/2$ scalar products of two distinct currents. Hence the ρ_q^2 term is the sum of 66 relativistic invariant terms:

$$\begin{aligned}\rho_q^2 &= \sum_{n=2}^{n=7} d_n d_n^* + \sum_{n,p,q} s_n^{pq}(s_n^{pq})^* \\ d_n &= R^n\overline{L}^n + L^n\overline{R}^n = 2\eta^{n\dagger}\xi^n = \mathrm{D}_R^n \cdot \mathrm{D}_L^n,\end{aligned} \tag{3.108}$$

where in the s_n^{pq}, $n = 2, 3, 4, 5$, and pq is one of the 15 possible pairs that may be formed with two different numbers taken among $2, 3, 4, 5, 6, 7$:

$$\begin{aligned}s_2^{pq} &= 2\eta^{p\dagger}\widehat{\eta}^q = -2\eta^{q\dagger}\widehat{\eta}^p = \mathrm{D}_L^p \cdot \mathrm{D}_L^q, \\ s_3^{pq} &= 2\eta^{q\dagger}\xi^p = \mathrm{D}_R^p \cdot \mathrm{D}_L^q;\ s_4^{pq} = 2\eta^{p\dagger}\xi^q = \mathrm{D}_L^p \cdot \mathrm{D}_R^q, \\ s_5^{pq} &= 2\widehat{\xi}^{p\dagger}\xi^q = -2\widehat{\xi}^{q\dagger}\xi^p = \mathrm{D}_R^p \cdot \mathrm{D}_R^q.\end{aligned} \tag{3.109}$$

The equations with mass term for the quarks are obtained exactly like the equations of the lepton part (2.87) [48] [47]:

$$\begin{aligned}&0 = D\widehat{L}^n + m_1 \mathrm{v}_q \widehat{L}^n \sigma_{12};\ 0 = \widehat{D}R^n + m_2 \widehat{\mathrm{v}}_q R^n \sigma_{12}, \\ &0 = D\overline{L}^{3+n} + m_3 \mathrm{v}_q \overline{L}^{3+n} \sigma_{12};\ 0 = \widehat{D}\widetilde{R}^{3+n} + m_4 \widehat{\mathrm{v}}_q \widetilde{R}^{3+n} \sigma_{12},\end{aligned} \tag{3.110}$$

and always for $n = 2, 3, 4$. Above, we have four equations in triplicates; thus we consider four proper masses $\hbar c m_j$, $j = 1, 2, 3, 4$. As in 2.2 the mass term accounts for the separation of the Ψ^n wave into four parts:

$$\begin{aligned}\Psi_{dL}^n &= \begin{pmatrix} L^n & 0 \end{pmatrix};\ \Psi_{uL}^n = \begin{pmatrix} 0 & \widetilde{L}^{3+n} \end{pmatrix}, \\ \Psi_{dR}^n &= \begin{pmatrix} R^n & 0 \end{pmatrix};\ \Psi_{uR}^n = \begin{pmatrix} 0 & \widetilde{R}^{3+n} \end{pmatrix}\end{aligned} \tag{3.111}$$

$$\begin{aligned}\mathbf{m}(\Psi^n) &= m_1\Psi_{dL}^n + m_2\Psi_{dR}^n + m_3\Psi_{uL}^n + m_4\Psi_{uR}^n \\ &= \begin{pmatrix} m_1 L^n + m_2 R^n & m_3\widetilde{L}^{3+n} + m_4\widetilde{R}^{3+n} \end{pmatrix}.\end{aligned} \tag{3.112}$$

And we gather these four equations (3.110) into:

$$0 = \mathbf{D}\Psi^n + \mathbf{v}_q\mathbf{m}(\Psi^n)\gamma_{21};\ \mathbf{v}_q = \begin{pmatrix} 0 & \mathrm{v}_q \end{pmatrix};\ \mathbf{v}_q^2 = 1. \tag{3.113}$$

Then letting:

$$\underline{M}(\Psi) = \begin{pmatrix} \mathbf{v}_q[\mathbf{m}(\Psi^2) - \mathbf{m}(\Psi^3)] & \mathbf{v}\mathbf{m}(\Psi^1) - i\mathbf{v}_q\mathbf{m}(\Psi^4) \\ \mathbf{v}\mathbf{m}(\Psi^1) + i\mathbf{v}_q\mathbf{m}(\Psi^4) & \mathbf{v}_q[\mathbf{m}(\Psi^2) + \mathbf{m}(\Psi_3)] \end{pmatrix}, \tag{3.114}$$

The wave equation that generalizes the improved equation of the electron is then expressed as:

$$0 = \underline{D}\Psi\Gamma_{012} + \underline{M}(\Psi)\Gamma_0, \tag{3.115}$$

while the equation invariant under Cl_3^* (we recall that this invariance automatically implies relativistic invariance) is obtained simply by multiplying on the left side by the reverse:

$$0 = \widetilde{\Psi}\underline{D}\Psi\Gamma_{012} + \widetilde{\Psi}\underline{M}(\Psi)\Gamma_0. \tag{3.116}$$

It is the close connection between the reversion in $Cl_{3,3}$ and the reversion in $Cl_{1,3}$ which enables the complete separation of the mass term of the lepton part from the quark part. This strict link is nontrivial and is established in B.2. Moreover except in a very particular case, $\Psi(\mathrm{x})$ is invertible. We may thus derive the invariant form (3.116) from (3.115) by multiplying on the left side by $\widetilde{\Psi}$. And multiplying the left side of (3.116) by $\widetilde{\Psi}^{-1}$, we get the usual (3.115) form of the wave equation. We recall that this justifies, for all lepton waves, that we are able to derive the wave equations from the Lagrangian density. Then the same behavior is observed for the quark waves.

3.4 Invariance

The invariance of the (3.110) equations is similar to that of the leptonic wave studied in 2.3. For the form invariance that includes relativistic invariance, it is enough to add to (2.92) and (2.93), and to the covariance of the b and w^j potentials that of the g_3G^k or h^m that are derived from (3.88):

$$\mathrm{h}_n^m = \overline{M}\mathrm{h}'^m_n\widehat{M};\ m_n = rm'_n;\ r = |\det(M)|. \tag{3.117}$$

We derive:

$$\begin{aligned}
&0 = D'\widehat{L'}^n + m'_1\mathrm{v}'_q\widehat{L'}^n\sigma_{12};\ 0 = \widehat{D}'R'^n + m'_2\widehat{\mathrm{v}}'_qR'^n\sigma_{12},\\
&0 = D'\overline{L}'^{3+n} + m'_3\mathrm{v}'_q\overline{L}'^{3+n}\sigma_{12};\ 0 = \widehat{D}'\widetilde{R}'^{3+n} + m'_4\widehat{\mathrm{v}}'_q\widetilde{R}'^{3+n}\sigma_{12},
\end{aligned} \tag{3.118}$$

which implies the form-invariance of the wave equations.

The gauge invariance under the $U(1)$ group generated by $\underline{P}_0$ results from the equalities (2.104)–(2.105) in which it is enough to replace P_0 by $\underline{P}_0$ with the Ψ in (3.1). What changes from the lepton case comes only with $\underline{P}_0$ which gives:

$$\begin{aligned}
\underline{P}_0(\Psi) &= \begin{pmatrix} P_0(\Psi^1) & P_0(\Psi^2) \\ P_0(\Psi^3) & P_0(\Psi^4) \end{pmatrix};\ P_0(\Psi^n) = -\frac{1}{3}\Psi^n\gamma_{21} + \frac{1}{2}(\Psi^n\mathbf{i} + \mathbf{i}\Psi^n\gamma_{30}),\\
P_0(\Psi^n) &= -\frac{i}{3}\begin{pmatrix} R^n - L^n & \widetilde{R}^{3+n} - \widetilde{L}^{3+n} \end{pmatrix} + i\begin{pmatrix} R^n & -\widetilde{R}^{3+n} \end{pmatrix}\\
&= \begin{pmatrix} \frac{2i}{3}R^n + \frac{i}{3}L^n & -\frac{4i}{3}\widetilde{R}^{3+n} + \frac{i}{3}\widetilde{L}^{3+n} \end{pmatrix},\ n = 2, 3, 4.
\end{aligned} \tag{3.119}$$

We then have:

$$\begin{aligned}\Psi'^n &= [\exp(a^0 P_0)](\Psi^n) = \begin{pmatrix} R'^n + L'^n & \widetilde{R}'^{3+n} + \widetilde{L}'^{3+n}\end{pmatrix},\\ R'^n &= e^{2ia^0/3}R^n;\ L'^n = e^{ia^0/3}L^n,\ n=2,3,4,\\ \widetilde{R}'^{3+n} &= e^{-4ia^0/3}\widetilde{R}^{3+n};\ \widetilde{L}'^{3+n} = e^{ia^0/3}\widetilde{L}^{3+n}.\end{aligned} \tag{3.120}$$

All left waves turn with the same angle $a^0/3$, and only the left waves have this property. This is how they come to be invariant under the $SU(2)$ gauge group mixing the different left waves. We get:

$$\mathrm{D}'^n_R = R'^n\widetilde{R}'^n = e^{2ia^0/3}R^n e^{-2ia^0/3}\widetilde{R}^n = R^n\widetilde{R}^n = \mathrm{D}^n_R. \tag{3.121}$$

And similarly we have:

$$\mathrm{D}'^n_L = \mathrm{D}^n_L;\ \mathrm{D}'^{3+n}_L = \mathrm{D}^{3+n}_L;\ \mathrm{D}'^{3+n}_R = \mathrm{D}^{3+n}_R;\ \mathrm{J}'_q = \mathrm{J}_q;\ \mathrm{v}'_q = \mathrm{v}_q, \tag{3.122}$$

and so the mass terms of the wave equations are invariant under the $U(1)$ gauge group. As in the case of the leptonic wave all left waves transform in the same manner: that is what is responsible for the commutation between the $\underline{P}_0$ operator and the three $\underline{P}_j$, $j = 1,2,3$. To study the other parts of the gauge group we start from (3.96), so that for $n = 2$ and with (3.92) we have:

$$\begin{aligned}\mathbf{D}\Psi^2 &= \boldsymbol{\partial}\Psi^2 + \mathbf{b}P_0(\Psi^2) + \mathbf{w}^j P_j(\Psi^2) + S(\Psi^2)\\ &= \boldsymbol{\partial}\Psi^2 + \mathbf{b}P_0(\Psi^2) + \mathbf{w}^j P_j(\Psi^2)\\ &+ (\mathbf{h}^1_2 - \mathbf{h}^2_2\mathbf{i})\mathbf{i}\Psi^4 - \mathbf{h}^3_2\mathbf{i}\Psi^3 + (\mathbf{h}^1_1 + \mathbf{h}^2_1\mathbf{i})\mathbf{i}\Psi^2 + \mathbf{h}^3_1\mathbf{i}\Psi^3.\end{aligned} \tag{3.123}$$

With (3.1) to (3.24) the previous equation is equivalent to the system:

$$\begin{aligned}iD\overline{\phi}^5 &= i\nabla\overline{\phi}^5 - \mathrm{b}(\frac{4}{3}\overline{R}^5 + \frac{1}{3}\overline{L}^5) + [(\mathrm{w}^1_2 - i\mathrm{w}^2_2)\widehat{L}^2 + \mathrm{w}^3_2\overline{L}^5]\\ &+ [(\mathrm{h}^1_1 + i\mathrm{h}^2_1)\overline{\phi}^6 - \mathrm{h}^3_1\overline{\phi}^5] + [(\mathrm{h}^1_3 - i\mathrm{h}^2_3)\overline{\phi}^7 + \mathrm{h}^3_3\overline{\phi}^5],\\ iD\widehat{\phi}^2 &= i\nabla\widehat{\phi}^2 + \mathrm{b}(\frac{2}{3}\widehat{R}^2 + \frac{1}{3}\widehat{L}^5) + [(\mathrm{w}^1_2 + i\mathrm{w}^2_2)\overline{L}^5 - \mathrm{w}^3_2\widehat{L}^2]\\ &+ [(\mathrm{h}^1_1 + i\mathrm{h}^2_1)\widehat{\phi}^3 - \mathrm{h}^3_1\widehat{\phi}^2] + [(\mathrm{h}^1_3 - i\mathrm{h}^2_3)\widehat{\phi}^4 + \mathrm{h}^3_3\widehat{\phi}^2],\end{aligned} \tag{3.124}$$

There are two other equations when using the matrix representation of $Cl_{1,3}$, which are equivalent to the two previous ones and result from the application to these equations of the main automorphism $P : M \mapsto \widehat{M}$. Next, by using this automorphism for the right waves we get the equivalent system:

$$\begin{aligned}
iD\eta^5 &= i\nabla\eta^5 + \frac{\mathrm{b}}{3}\eta^5 + [(\mathrm{w}_2^1 - i\mathrm{w}_2^2)\eta^2 + \mathrm{w}_2^3\eta^5] \\
&\quad + [(\mathrm{h}_1^1 + i\mathrm{h}_1^2)\eta^6 - \mathrm{h}_1^3\eta^5] + [(\mathrm{h}_3^1 - i\mathrm{h}_3^2)\eta^7 + \mathrm{h}_3^3\eta^5], \\
iD\eta^2 &= i\nabla\eta^2 + \frac{\mathrm{b}}{3}\eta^2 + [(\mathrm{w}_2^1 + i\mathrm{w}_2^2)\eta^5 - \mathrm{w}_2^3\eta^2] \\
&\quad + [(\mathrm{h}_1^1 + i\mathrm{h}_1^2)\eta^3 - \mathrm{h}_1^3\eta^2] + [(\mathrm{h}_3^1 - i\mathrm{h}_3^2)\eta^4 + \mathrm{h}_3^3\eta^2], \qquad (3.125) \\
-i\widehat{D}\xi^2 &= -i\widehat{\nabla}\xi^2 + \frac{2}{3}\widehat{\mathrm{b}}\xi^2 + [(\widehat{\mathrm{h}}_1^1 - i\widehat{\mathrm{h}}_1^2)\xi^3 - \widehat{\mathrm{h}}_1^3\xi^2] + [(\widehat{\mathrm{h}}_3^1 + i\widehat{\mathrm{h}}_3^2)\xi^4 + \widehat{\mathrm{h}}_3^3\xi^2], \\
-i\widehat{D}\xi^5 &= -i\widehat{\nabla}\xi^5 - \frac{4}{3}\widehat{\mathrm{b}}\xi^5 + [(\widehat{\mathrm{h}}_1^1 - i\widehat{\mathrm{h}}_1^2)\xi^6 - \widehat{\mathrm{h}}_1^3\xi^5] + [(\widehat{\mathrm{h}}_3^1 + i\widehat{\mathrm{h}}_3^2)\xi^7 + \widehat{\mathrm{h}}_3^3\xi^5].
\end{aligned}$$

Next we have two other systems with the same structure which are obtained by circularly permuting the indices $2, 3, 4$ and $5, 6, 7$ (corresponding to the r, g, b colors of the quarks) everywhere these indices are present:

$$\begin{aligned}
iD\eta^6 &= i\nabla\eta^6 + \frac{\mathrm{b}}{3}\eta^6 + [(\mathrm{w}_3^1 - i\mathrm{w}_3^2)\eta^3 + \mathrm{w}_3^3\eta^6] \\
&\quad + [(\mathrm{h}_2^1 + i\mathrm{h}_2^2)\eta^7 - \mathrm{h}_2^3\eta^6] + [(\mathrm{h}_1^1 - i\mathrm{h}_1^2)\eta^5 + \mathrm{h}_1^3\eta^6], \qquad (3.126)
\end{aligned}$$

$$\begin{aligned}
iD\eta^3 &= i\nabla\eta^3 + \frac{\mathrm{b}}{3}\eta^3 + [(\mathrm{w}_3^1 + i\mathrm{w}_3^2)\eta^6 - \mathrm{w}_3^3\eta^3] \\
&\quad + [(\mathrm{h}_2^1 + i\mathrm{h}_2^2)\eta^4 - \mathrm{h}_2^3\eta^3] + [(\mathrm{h}_1^1 - i\mathrm{h}_1^2)\eta^2 + \mathrm{h}_1^3\eta^3], \qquad (3.127) \\
-i\widehat{D}\xi^3 &= -i\widehat{\nabla}\xi^3 + \frac{2}{3}\widehat{\mathrm{b}}\xi^3 + [(\widehat{\mathrm{h}}_2^1 - i\widehat{\mathrm{h}}_2^2)\xi^4 - \widehat{\mathrm{h}}_2^3\xi^3] + [(\widehat{\mathrm{h}}_1^1 + i\widehat{\mathrm{h}}_1^2)\xi^2 + \widehat{\mathrm{h}}_1^3\xi^3], \\
-i\widehat{D}\xi^6 &= -i\widehat{\nabla}\xi^6 - \frac{4}{3}\widehat{\mathrm{b}}\xi^6 + [(\widehat{\mathrm{h}}_2^1 - i\widehat{\mathrm{h}}_2^2)\xi^7 - \widehat{\mathrm{h}}_2^3\xi^6] + [(\widehat{\mathrm{h}}_1^1 + i\widehat{\mathrm{h}}_1^2)\xi^5 + \widehat{\mathrm{h}}_1^3\xi^6]. \\
iD\eta^7 &= i\nabla\eta^7 + \frac{\mathrm{b}}{3}\eta^7 + [(\mathrm{w}_1^1 - i\mathrm{w}_1^2)\eta^4 + \mathrm{w}_1^3\eta^7] \\
&\quad + [(\mathrm{h}_3^1 + i\mathrm{h}_3^2)\eta^5 - \mathrm{h}_3^3\eta^7] + [(\mathrm{h}_2^1 - i\mathrm{h}_2^2)\eta^6 + \mathrm{h}_2^3\eta^7], \\
iD\eta^4 &= i\nabla\eta^4 + \frac{\mathrm{b}}{3}\eta^4 + [(\mathrm{w}_1^1 + i\mathrm{w}_1^2)\eta^7 - \mathrm{w}_1^3\eta^4] \\
&\quad + [(\mathrm{h}_3^1 + i\mathrm{h}_3^2)\eta^2 - \mathrm{h}_3^3\eta^4] + [(\mathrm{h}_2^1 - i\mathrm{h}_2^2)\eta^3 + \mathrm{h}_2^3\eta^4], \\
-i\widehat{D}\xi^4 &= -i\widehat{\nabla}\xi^4 + \frac{2}{3}\widehat{\mathrm{b}}\xi^4 + [(\widehat{\mathrm{h}}_3^1 - i\widehat{\mathrm{h}}_3^2)\xi^2 - \widehat{\mathrm{h}}_3^3\xi^4] + [(\widehat{\mathrm{h}}_2^1 + i\widehat{\mathrm{h}}_2^2)\xi^3 + \widehat{\mathrm{h}}_2^3\xi^4], \\
-i\widehat{D}\xi^7 &= -i\widehat{\nabla}\xi^7 - \frac{4}{3}\widehat{\mathrm{b}}\xi^7 + [(\widehat{\mathrm{h}}_3^1 - i\widehat{\mathrm{h}}_3^2)\xi^5 - \widehat{\mathrm{h}}_3^3\xi^7] + [(\widehat{\mathrm{h}}_2^1 + i\widehat{\mathrm{h}}_2^2)\xi^6 + \widehat{\mathrm{h}}_2^3\xi^7].
\end{aligned}$$

The invariance under the $SU(2)$ group is the same as what we saw for the leptonic wave. This invariance actually results from:

$$\begin{aligned}
&\mathrm{D}_L^{n,\,3+n} - i d_L^{n,\,3+n} = 2L^n L^{3+n};\ \mathrm{D}_L^n = L^n \widetilde{L}^n;\ \mathrm{D}_L^{3+n} = \widetilde{L}^{3+n} L^{3+n}, \\
&\quad \mathrm{w}_n^1 = \frac{g_2}{2}\mathrm{D}_L^{n,\,3+n};\ \mathrm{w}_n^2 = \frac{g_2}{2} d_L^{n,\,3+n};\ \mathrm{w}_n^3 = \frac{g_2}{2}(\mathrm{D}_L^{3+n} - \mathrm{D}_L^n), \qquad (3.128)
\end{aligned}$$

which are enough to obtain the gauge invariance, as we saw in 2.3.2. And this gives the $U(1)\times SU(2)$ structure of the electroweak gauge group. The only difference from the Standard Model is that we do not need to postulate this result: **we derive the structure of the gauge group from the operators themselves**.

For the $SU(2)$ part of the electroweak gauge group, and since the invariance has exactly the same form as in 2.3.2 , we obtain the following, using the same identities as (2.148) and (2.150) (a detailed calculation is in D.3):

$$\begin{aligned}(W_n^1+iW_n^2)\overline{L}^{3+n}-W_n^3\widehat{L}^n &= -3\widetilde{L}^{3+n}L^{3+n}\widehat{L}^n = -3\mathrm{D}_L^{3+n}\widehat{L}^n = -3W_n^3\widehat{L}^n,\\ (W_n^1-iW_n^2)\widehat{L}^n+W_n^3\overline{L}^{3+n} &= -3L^n\widetilde{L}^n\overline{L}^{3+n} = -3\mathrm{D}_L^n\overline{L}^{3+n} = 3W_n^3\overline{L}^{3+n}.\end{aligned} \tag{3.129}$$

In the case of the quarks we have moreover the same formula of transformation for $n=2,3,4$; this gives the commutation between the $\underline{P}_n$ and the Λ^k operators of the group of chromodynamics, which act only on the n index, thus giving rise to the $U(1)\times SU(2)\times SU(3)$ structure of the gauge group of the Standard Model. The gauge invariance under the $SU(3)$ group gives a simplification of the wave equations with proper mass (see D.3). Our improved equations have the form:

$$\begin{aligned}0 &= D\widehat{L}^n+im_1\mathrm{v}_q\widehat{L}^n;\ 0=\widehat{D}R^n+im_2\widehat{\mathrm{v}}_qR^n,\\ 0 &= \widetilde{D}\overline{L}^{3+n}+im_3\mathrm{v}_q\overline{L}^{3+n};\ 0=\overline{D}\widetilde{R}^{3+n}+im_4\widehat{\mathrm{v}}_q\widetilde{R}^{3+n},\end{aligned} \tag{3.130}$$

$$\begin{aligned}D\widehat{L}^n &= \sigma^\mu\Big[\partial_\mu+i\Big(-\frac{\mathrm{b}_\mu}{3}+3\mathrm{w}^3_{n\mu}-3\mathrm{h}^{d3}_{L\underline{n+1}\mu}+3\mathrm{h}^{d3}_{L\,n-1\,\mu}\Big)\Big]\widehat{L}^n,\\ \widehat{D}R^n &= \widehat{\sigma}^\mu\Big[\partial_\mu+i\Big(\frac{2\mathrm{b}_\mu}{3}+3\mathrm{h}^{d3}_{R\underline{n+1}\mu}-3\mathrm{h}^{d3}_{R\,n-1\,\mu}\Big)\Big]R^n,\\ \widetilde{D}\overline{L}^{3+n} &= \sigma^\mu\Big[\partial_\mu+i\Big(-\frac{\mathrm{b}_\mu}{3}-3\mathrm{w}^3_{n\mu}-3\mathrm{h}^{u3}_{L\underline{n+1}\mu}+3\mathrm{h}^{u3}_{L\,n-1\,\mu}\Big)\Big]\overline{L}^{3+n},\\ \overline{D}\widetilde{R}^{3+n} &= \widehat{\sigma}^\mu\Big[\partial_\mu+i\Big(-\frac{4\mathrm{b}_\mu}{3}+3\mathrm{h}^{u3}_{R\underline{n+1}\mu}-3\mathrm{h}^{u3}_{R\,n-1\,\mu}\Big)\Big]\widetilde{R}^{3+n}.\end{aligned} \tag{3.131}$$

Here the w potentials depend on color and moreover the h potentials have a double dependence: their two indices with value $2,3,4$ come from the generators of the $SU(3)$ group, while their indices L,R and d,u are linked with the spinors on which they act. Thus the wave equations that are used to obtain the Lagrangian density are the equations governing right and left waves.

3.5 Wave equation – Lagrangian density

We multiply the wave equations (3.130) of η^n by $-i\eta^{n\dagger}$, the wave equations of ξ^n by $-i\xi^{n\dagger}$, the wave equations of η^{3+n} by $-i\eta^{3+n\dagger}$ and finally the wave equations of ξ^{3+n} by $-i\xi^{3+n\dagger}$, always by the left side. For the lepton part we saw in Chapter 2 how the Lagrangian density of the electron wave is generalized for several Lagrangian densities coming from the different wave equations. Among these densities, $\mathcal{L}_q^+$ is obtained as the sum of the different real parts, and $\mathcal{L}_q^-$ is obtained as the difference between the real parts coming from the left waves and the right waves. Since the ρ_q density is calculated like ρ_l we will get similar results. We let:

$$\begin{aligned}
\mathcal{L}_q^+ &= \sum_{n=2}^{4}\begin{bmatrix} \frac{m}{km_1}\eta^{n\dagger}\sigma^\mu(-i\partial_\mu + d^1_{n\mu}) + \frac{m}{km_2}\xi^{n\dagger}\widehat{\sigma}^\mu(-i\partial_\mu + d^2_{n\mu}) \\ +\frac{m}{km_3}\eta^{3+n\dagger}\sigma^\mu(-i\partial_\mu + d^3_{n\mu}) + \frac{m}{km_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu(-i\partial_\mu + d^4_{n\mu})\end{bmatrix},\\
d^1_{n\mu} &:= -\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{n\mu} - 3\mathrm{h}^{d3}_{L\underline{n+1}\mu} + 3\mathrm{h}^{d3}_{L\,n-1\,\mu} + m_1\mathrm{v}_{q\mu},\\
d^2_{n\mu} &:= \frac{2\mathrm{b}_\mu}{3} + 3\mathrm{h}^{d3}_{R\underline{n+1}\mu} - 3\mathrm{h}^{d3}_{R\,n-1\,\mu} + m_2\mathrm{v}_{q\mu},\\
d^3_{n\mu} &:= -\frac{\mathrm{b}_\mu}{3} - 3\mathrm{w}^3_{n\mu} - 3\mathrm{h}^{u3}_{L\underline{n+1}\mu} + 3\mathrm{h}^{u3}_{L\,n-1\,\mu} + m_3\mathrm{v}_{q\mu},\\
d^4_{n\mu} &:= -\frac{4\mathrm{b}_\mu}{3} + 3\mathrm{h}^{u3}_{R\underline{n+1}\mu} - 3\mathrm{h}^{u3}_{R\,n-1\,\mu} + m_4\mathrm{v}_{q\mu}.
\end{aligned} \tag{3.132}$$

The Lagrangian densities $\mathcal{L}_q^+$ and $\mathcal{L}_q^-$ satisfy:

$$0 = \mathcal{L}_q^+ = \sum_{n=2}^{4}\begin{bmatrix} \frac{m}{km_1}(-i\eta^{n\dagger}\sigma^\mu\partial_\mu\eta^n + d^1_{n\mu}\mathrm{D}_L^{n\mu}) \\ +\frac{m}{km_2}(-i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^n + d^2_{n\mu}\mathrm{D}_R^{n\mu}) \\ +\frac{m}{km_3}(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n} + d^3_{n\mu}\mathrm{D}_L^{3+n\mu}) \\ +\frac{m}{km_4}(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n} + d^4_{n\mu}\mathrm{D}_R^{3+n\mu})\end{bmatrix}, \tag{3.133}$$

$$0 = \mathcal{L}_q^- = \sum_{n=2}^{4}\begin{bmatrix} -\frac{m}{km_1}(-i\eta^{n\dagger}\sigma^\mu\partial_\mu\eta^n + d^1_{n\mu}\mathrm{D}_L^{n\mu}) \\ +\frac{m}{km_2}(-i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^n + d^2_{n\mu}\mathrm{D}_R^{n\mu}) \\ -\frac{m}{km_3}(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n} + d^3_{n\mu}\mathrm{D}_L^{3+n\mu}) \\ +\frac{m}{km_4}(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n} + d^4_{n\mu}\mathrm{D}_R^{3+n\mu})\end{bmatrix}. \tag{3.134}$$

The fact that these Lagrangian densities are null at each point of space-time is due to their construction from (3.130) to (3.131). Moreover these tensor densities are real because their imaginary part is null. We see this for instance in (3.130) which gives for $n = 2$ next taking the adjoint:

$$\begin{aligned}
0 &= \eta^{2\dagger}\sigma^\mu(-i\partial_\mu - \frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{2\mu} - 3\mathrm{h}^{d3}_{L3\mu} + 3\mathrm{h}^{d3}_{L1\mu} + m_1\mathrm{v}_\mu)\eta^2, \\
0 &= i\partial_\mu\eta^{2\dagger}\sigma^\mu\eta^2 + \eta^{2\dagger}\sigma^\mu(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{2\mu} - 3\mathrm{h}^{d3}_{L3\mu} + 3\mathrm{h}^{d3}_{L1\mu} + m_1\mathrm{v}_\mu)\eta^2.
\end{aligned} \tag{3.135}$$

Then subtracting we get:

$$0 = -i(\eta^{2\dagger}\sigma^\mu\partial_\mu\eta^2 + \partial_\mu\eta^{2\dagger}\sigma^\mu\eta^2) = -i\partial_\mu(\eta^{2\dagger}\sigma^\mu\eta^2);\ 0 = \partial_\mu\mathrm{D}_L^{2\mu}. \tag{3.136}$$

The Lagrangian density $\mathcal{L}_q^+$ is a sum of twelve terms, with the same structure as the four terms that we have in Chapter 2 for the leptonic wave. We may thus replicate what we detailed in 2.3.4. Since we used there only the algebraic properties of multiplication in Cl_3, we can easily redo with η^n what we proved with η^1. Moreover, ξ^n acts like ξ^1, η^{3+n} acts like η^8 and ξ^{3+n} acts like ξ^8. Thus the wave equations allow us to arrive at $0 = \mathcal{L}_q^+$, and moreover the Lagrange equations, without any supplementary condition, allow us to obtain for each left or right spinor the numeric real equations equivalent to the wave equation expressed in Cl_3. When we vary the Lagrangian density in relation to the variables contained in η^2, the gauge potentials introduce no supplementary term. This comes from the mechanism described in 2.3.4 for the b potential, as well as for the other potentials, because w_3^3 acts on η^2 only by the term D_L^5. This is the same for the potentials of chromodynamics. The h_{L3}^3 potential acts on η^2 only by the D_L^3 term, and h_{L1}^3 acts on the η^2 term only by the D_L^4 term. About the antiparticles we may also use without any change what we said about the electron in Chapter 1 and about the lepton wave in Chapter 2. The only change in this passing to the "anti-world" is the replacement of the ∂_μ by $-\partial_\mu$ and the exchange of η and ξ. The double link between the wave equation and the Lagrangian density is totally conserved.

3.6 Energy-momentum tensors

Here also we again obtain what we learned from the Dirac equation and from its extension to the lepton wave: the existence of not only one Lagrangian density and its associated energy-momentum tensor, but two tensorial densities linked to two Lagrangian densities and associated with the invariance of those densities under space-time translations. The Lagrangian density $\mathcal{L}_q^+$ is the sum of 12 similar terms. It is invariant under the space-time translations; thus a conservative tensorial density of momentum-energy is associated, sum of twelve densities: A conservative tensor is associated to this Lagrangian density. This tensor is also a sum of 12 terms:

$$T = \sum_{n=2}^{4} \left(\frac{m}{km_1} T_L^n + \frac{m}{km_2} T_R^n + \frac{m}{km_3} T_L^{3+n} + \frac{m}{km_4} T_R^{3+n}\right), \tag{3.137}$$

$$T_{L\lambda}^{n\mu} = \Re(-i\eta^{n\dagger}\sigma^\mu\partial_\lambda\eta^n) + d_{n\lambda}^1\eta^{n\dagger}\sigma^\mu\eta^n, \tag{3.138}$$

$$T_{R\lambda}^{n\mu} = \Re(-i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\lambda\xi^n) + d_{n\lambda}^2\xi^{n\dagger}\widehat{\sigma}^\mu\xi^n, \tag{3.139}$$

$$T_{L\lambda}^{3+n\mu} = \Re(-i\eta^{3+n\dagger}\sigma^\mu\partial_\lambda\eta^{3+n}) + d_{n\lambda}^3\eta^{3+n\dagger}\sigma^\mu\eta^{3+n}, \tag{3.140}$$

$$T_{R\lambda}^{3+n\mu} = \Re(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\lambda\xi^{3+n}) + d_{n\lambda}^4\xi^{3+n\dagger}\widehat{\sigma}^\mu\xi^{3+n}. \tag{3.141}$$

In particular for the T_0^0 component we have:

$$T_0^0 = \sum_{n=2}^{4} \left(\frac{m}{km_1} T_{L0}^{n0} + \frac{m}{km_2} T_{R0}^{n0} + \frac{m}{km_3} T_{L0}^{3+n0} + \frac{m}{km_4} T_{R0}^{3+n0}\right), \tag{3.142}$$

$$T_{L0}^{n0} = \Re(-i\eta^{n\dagger}\partial_0\eta^n) + d_{n0}^1\eta^{n\dagger}\eta^n, \tag{3.143}$$
$$T_{R0}^{n0} = \Re(-i\xi^{n\dagger}\partial_0\xi^n) + d_{n0}^2\xi^{n\dagger}\xi^n, \tag{3.144}$$
$$T_{L0}^{3+n0} = \Re(-i\eta^{3+n\dagger}\partial_0\eta^{3+n}) + d_{n0}^3\eta^{3+n\dagger}\eta^{3+n}, \tag{3.145}$$
$$T_{R0}^{3+n0} = \Re(-i\xi^{3+n\dagger}\partial_\mu\xi^{3+n}) + d_{n0}^4\xi^{3+n\dagger}\xi^{3+n}. \tag{3.146}$$

For a solution of the wave equation with an energy E of the whole wave we have:

$$\begin{aligned}-T_0^0 &= \frac{E}{\hbar c}\sum_{n=2}^{4}[\frac{m}{km_1}\eta^{n\dagger}\eta^n + \frac{m}{km_2}\xi^{n\dagger}\xi^n + \frac{m}{km_3}\eta^{3+n\dagger}\eta^{3+n} + \frac{m}{km_4}\xi^{3+n\dagger}\xi^{3+n}]\\ &= \frac{E}{\hbar c}(\frac{m}{km_1}S_L^d + \frac{m}{km_2}S_R^d + \frac{m}{km_3}S_L^u + \frac{m}{km_4}S_R^u)^0 = \frac{E}{\hbar c}\underline{\mathrm{J}}_q^0, \qquad (3.147)\\ S_L^d &= \sum_{n=2}^{4}\eta^{n\dagger}\eta^n;\ S_R^d = \sum_{n=2}^{4}\xi^{n\dagger}\xi^n;\ S_L^u = \sum_{n=2}^{4}\eta^{3+n\dagger}\eta^{3+n},\\ S_R^u &= \sum_{n=2}^{4}\xi^{3+n\dagger}\xi^{3+n},\end{aligned}$$

naming $\underline{\mathbf{J}}$ the sum current. The reason for the existence of a probability current in quantum mechanics is again here the equality between inertial mass and gravitational mass, which implies:

$$E = \iiint dvT_0^0;\quad \iiint \frac{\mathrm{J}_q^0}{\hbar c}dv = 1. \tag{3.148}$$

As with the lone electron or the lepton wave we have two useful tensors of energy–momentum instead of only one. The second tensor is the V of Costa de Beauregard [53] that is conserved from the invariance under the translations of the Lagrangian density $\mathcal{L}_q^-$. This V reads:

$$V := \sum_{n=2}^{4}(\frac{m}{km_1}T_L^n - \frac{m}{km_2}T_R^n + \frac{m}{km_3}T_L^{3+n} - \frac{m}{km_4}T_R^{3+n}). \tag{3.149}$$

The dynamics of the quarks comes from the variations of the energy–momentum tensor. The calculation is similar to that for the lepton wave. We have:

$$\partial_\mu T^\mu = \sum_{n=2}^{4}(\frac{m}{km_1}\partial_\mu T_L^{n\mu} + \frac{m}{km_2}\partial_\mu T_R^{n\mu} + \frac{m}{km_3}\partial_\mu T_L^{3+n\mu} + \frac{m}{km_4}\partial_\mu T_R^{3+n\mu}). \tag{3.150}$$

With the first of these four terms we obtain:

$$\begin{aligned}\partial_\mu T_L^{n\mu} &= \partial_\mu T_{L\lambda}^{n\mu}\sigma^\lambda = \partial_\mu[-i\eta^{n\dagger}\sigma^\mu\partial_\lambda\eta^n + d_{n\lambda}^1\mathrm{D}_L^{n\mu}]\sigma^\lambda\\ &= [-i(\nabla\eta^n)^\dagger\partial_\lambda\eta^n - i\eta^{n\dagger}\partial_\lambda(\nabla\eta^n) + (\partial_\mu d_{n\lambda}^1)\mathrm{D}_L^{n\mu} + d_{n\lambda}^1\partial_\mu\mathrm{D}_L^{n\mu}]\sigma^\lambda. \qquad (3.151)\end{aligned}$$

And we have with (3.106) and with (B.94) and (B.95):

$$\nabla\eta^n = -id_n^1\eta^n, \tag{3.152}$$

$$\begin{aligned}\partial_\mu \mathrm{D}_L^{n\mu} &= (\partial_\mu\eta^{n\dagger})\sigma^\mu\eta^n + \eta^{n\dagger}\sigma^\mu(\partial_\mu\eta^n)\\ &= i\eta^{n\dagger}d_n^1\eta^n - i\eta^{n\dagger}d_n^1\eta^n = 0.\end{aligned} \tag{3.153}$$

That gives:

$$-i(\nabla\eta^n)^\dagger\partial_\lambda\eta^n - i\eta^{n\dagger}\partial_\lambda(\nabla\eta^n) \tag{3.154}$$

$$\begin{aligned}&= \eta^{n\dagger}d_n^1\partial_\lambda\eta^n - \eta^{n\dagger}(\partial_\lambda d_n^1)\eta^n - \eta^{n\dagger}d_n^1\partial_\lambda\eta^n)\\ &= -\partial_\lambda d_{n\mu}^1\mathrm{D}_L^{n\mu}.\end{aligned} \tag{3.155}$$

And we then get:

$$\partial_\mu T_L^{n\mu} = (\partial_\mu d_{n\lambda}^1 - \partial_\lambda d_{n\mu}^1)\mathrm{D}_L^{n\mu}\sigma^\lambda. \tag{3.156}$$

Similarly we next obtain:

$$\partial_\mu T_R^{n\mu} = (\partial_\mu d_{n\lambda}^2 - \partial_\lambda d_{n\mu}^2)\mathrm{D}_R^{n\mu}\sigma^\lambda, \tag{3.157}$$

$$\partial_\mu T_L^{3+n\mu} = (\partial_\mu d_{n\lambda}^3 - \partial_\lambda d_{n\mu}^3)\mathrm{D}_L^{3+n\mu}\sigma^\lambda, \tag{3.158}$$

$$\partial_\mu T_R^{3+n\mu} = (\partial_\mu d_{n\lambda}^4 - \partial_\lambda d_{n\mu}^4)\mathrm{D}_R^{3+n\mu}\sigma^\lambda. \tag{3.159}$$

With:

$$g^k = \sum_{n=2}^{4} d_n^k;\ \mathrm{J}_q = \sum_{n=2}^{7}(\mathrm{D}_R^n + \mathrm{D}_L^n), \tag{3.160}$$

we get:

$$\partial_\mu T^\mu = \begin{bmatrix} \frac{m}{km_1}(\partial_\mu g_\lambda^1 - \partial_\lambda g_\mu^1)S_L^{d\mu} + \frac{m}{km_2}(\partial_\mu g_\lambda^2 - \partial_\lambda g_\mu^2)S_R^{d\mu} \\ +\frac{m}{km_3}(\partial_\mu g_\lambda^3 - \partial_\lambda g_\mu^3)S_L^{u\mu} + \frac{m}{km_1}(\partial_\mu g_\lambda^4 - \partial_\lambda g_\mu^4)S_R^{u\mu}\end{bmatrix}\sigma^\lambda. \tag{3.161}$$

We are thus able to separate the forces acting on the whole wave into a part acting on the d quark and a part acting on the u quark. For the d quark we have:

$$\partial_\mu T_d^\mu = [\frac{m}{km_1}(\partial_\mu g_\lambda^1 - \partial_\lambda g_\mu^1)S_L^{d\mu} + \frac{m}{km_2}(\partial_\mu g_\lambda^2 - \partial_\lambda g_\mu^2)S_R^{d\mu}]\sigma^\lambda. \tag{3.162}$$

And similarly for the u quark:

$$\partial_\mu T_u^\mu = [\frac{m}{km_3}(\partial_\mu g_\lambda^3 - \partial_\lambda g_\mu^3)S_L^{u\mu} + \frac{m}{km_4}(\partial_\mu g_\lambda^4 - \partial_\lambda g_\mu^4)S_R^{u\mu}]\sigma^\lambda. \tag{3.163}$$

And we get:

$$\begin{aligned} g^1 = d_2^1 + d_3^1 + d_4^1 &= \begin{bmatrix} -\frac{\mathrm{b}}{3} + 3\mathrm{w}_2^3 - 3\mathrm{h}_{L3}^{d3} + 3\mathrm{h}_{L1}^{d3} \\ -\frac{\mathrm{b}}{3} + 3\mathrm{w}_3^3 - 3\mathrm{h}_{L1}^{d3} + 3\mathrm{h}_{L2}^{d3} \\ -\frac{\mathrm{b}}{3} + 3\mathrm{w}_4^3 - 3\mathrm{h}_{L2}^{d3} + 3\mathrm{h}_{L3}^{d3}\end{bmatrix} \\ &= \Big[-\frac{g_1}{2}B + \frac{3g_2}{2}(\mathrm{D}_L^5 - \mathrm{D}_L^2 + \mathrm{D}_L^6 - \mathrm{D}_L^3 + \mathrm{D}_L^7 - \mathrm{D}_L^4)\Big] \\ &= \Big[-\frac{g_1}{2}B + \frac{3g_2}{2}(S_L^u - S_L^d)\Big]. \end{aligned} \tag{3.164}$$

Between the first and the second line the potentials of chromodynamics completely disappear. This implies that there are no strong forces for a wave of quarks that equally contains the three color states. And this is well known in nuclear physics where there are no stable states formed by three d quarks or three u quarks with color r, g and b.

Above all this result means that the proton or the neutron is a single wave containing the three colored quarks. We will now see that it is this single wave of the proton or the neutron which has a quantized kinetic momentum, not a lone quark.

3.7 Quantization of the kinetic momentum

We may again use what we have shown in Chapter 2 for the lepton wave. First we have instead of (2.263):

$$V_\lambda^\mu = \sum_{n=2}^{4} \Re \begin{bmatrix} -i(\frac{m}{km_1}\eta^{n\dagger}\sigma^\mu d_{L\lambda}^n \eta^n - \frac{m}{km_2}\xi^{n\dagger}\widehat{\sigma}^\mu d_{R\lambda}^n \xi^n) \\ -i(\frac{m}{km_3}\eta^{3+n\dagger}\sigma^\mu d_{L\lambda}^{3+n}\eta^{3+n} - \frac{m}{km_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu d_{R\lambda}^{3+n}\xi^{3+n}) \end{bmatrix}. \tag{3.165}$$

We have twelve fields of spinors, six left and six right ones (and some of those spinors may be null) instead of the four of the lepton wave, but with very similar properties. Now we let:

$$\varphi_n = \eta^n;\ \varphi_{6+n} = \xi^n,\ n = 2, 3, \dots, 7. \tag{3.166}$$

Next, as in (2.268) and (2.270):

$$\begin{aligned} \delta\varphi_a &= \phi_i^a \delta\omega^i \\ \eta^n + \delta\eta^n &= \widehat{M}\eta^n;\ \xi^n + \delta\xi^n = M\xi^n. \end{aligned} \tag{3.167}$$

We then obtain as in (2.273) to (2.276):

$$\begin{aligned} \phi_0^n &= \frac{\eta^n}{2};\ \phi_1^n = -\sigma_1\frac{\eta^n}{2};\ \phi_2^n = -\sigma_2\frac{\eta^n}{2};\ \phi_3^n = -\sigma_3\frac{\eta^n}{2}, \\ \phi_4^n &= i\sigma_1\frac{\eta^n}{2};\ \phi_5^n = i\sigma_2\frac{\eta^n}{2};\ \phi_6^n = i\sigma_3\frac{\eta^n}{2};\ \phi_7^n = -i\frac{\eta^n}{2}, \end{aligned} \tag{3.168}$$

$$\begin{aligned} \phi_0^{6+n} &= \frac{\xi^n}{2};\ \phi_1^{6+n} = \sigma_1\frac{\xi^n}{2};\ \phi_2^{6+n} = \sigma_2\frac{\xi^n}{2};\ \phi_3^{6+n} = \sigma_3\frac{\xi^n}{2}, \\ \phi_4^{6+n} &= i\sigma_1\frac{\xi^n}{2};\ \phi_5^{6+n} = i\sigma_2\frac{\xi^n}{2};\ \phi_6^{6+n} = i\sigma_3\frac{\xi^{6+n}}{2};\ \phi_7^{6+n} = i\frac{\xi^n}{2}. \end{aligned} \tag{3.169}$$

Hence we always have (2.277) to (2.281), without other changes than the replacement of the Lagrangian densities from the lepton wave by those from the quark waves. We now get:

$$j_7^\mu = -\frac{\partial \mathcal{L}^-}{\partial(\partial_\mu \varphi_a)}\phi_7^a = \sum_{n=2}^{4}\Big[i\frac{m}{km_1}\eta^{n\dagger}\sigma^\mu\Big(\frac{-i}{2}\Big)\eta^n - i\frac{m}{km_2}\xi^{n\dagger}\widehat{\sigma}^\mu\Big(\frac{i}{2}\Big)\xi^n$$
$$+ i\frac{m}{km_3}\eta^{3+n\dagger}\sigma^\mu\Big(\frac{-i}{2}\Big)\eta^{2+n} - i\frac{m}{km_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu\Big(\frac{i}{2}\Big)\xi^{3+n}\Big]. \qquad (3.170)$$

With (3.147) we thus obtain:

$$j_7 = \frac{1}{2}\Big[\frac{m}{km_1}S_L^d + \frac{m}{km_2}S_R^d + \frac{m}{km_3}S_L^u + \frac{m}{km_4}S_R^u\Big] = \frac{1}{2}\underline{\mathrm{J}}_q. \qquad (3.171)$$

Thus (3.148) gives:

$$\iiint dv\underline{\mathrm{J}}_q^0 = \hbar c;\ \iiint dv\frac{j_7^0}{c} = \frac{1}{2c}\iiint dv\underline{\mathrm{J}}_q^0 = \frac{\hbar}{2}. \qquad (3.172)$$

This gives the quantization of kinetic momentum of the proton or the neutron. This satisfies all known properties of these two kinds of particles. The quantization of kinetic momentum is thus for the electrons and also for the protons and neutrons, a direct consequence of the wave equations of these particles and of the form-invariance under the extended group Cl_3^*.

This quantization of the kinetic momentum, not for each of the quarks separately but for the whole proton and the whole neutron with their three colored quarks, has a very well-known experimental consequence: it is impossible to move a lone quark outside a proton or a neutron. In spite of the fact that they are made of several quark waves, only the protons and the neutrons may have the individuality of a particle. The reason is that a proper kinetic momentum is always an integer multiple of de $\hbar/2$. Only objects that we can individually detect in any physical experiment are those which have a kinetic momentum multiple of $\hbar/2$. . That kinetic momentum is an even multiple of $\hbar/2$ in the boson case. We can get $3\hbar/2$ for any hadrons of for states with 5 quarks. But the kinetic momentum cannot be smaller than $\hbar/2$ and this minimal kinetic momentum needs three colored quarks, not only one.

We recall that the quantization of the kinetic momentum is the origin of Heisenberg's inequalities (see [61]). These inequalities thus apply to or neutron. The fact that quantum mechanics works similarly for an electron, a proton or a neutron gives the same "fundamental particle" character to these objects, and seems to question the quark model which is at the core of the Standard Model. Nevertheless it is the wave of a quark that is similar to the wave of an electron, not the wave of a proton or neutron. Once again, **all that supports the Standard Model**.

3.7.1 Case of the lone proton or the lone neutron

The proton is made of two u quarks and one d quark. Since the color of the different nucleons does not add, the Standard Model says the proton (or the neutron) is color neutral. We may then suppose, for instance, that a proton is at a given instant made of a u_r quark, a u_g quark and a d_b quark. From our previous calculations the proton is then composed of only six non-null spinor waves: L^4, L^5, L^6 and R^4, R^5, R^6, all other spinor waves being exactly null. The J_q and $\underline{J}$ currents are thus the sum of only six spinor currents instead of the twelve possible currents:

$$\mathrm{J}_q = \mathrm{D}_L^4 + \mathrm{D}_L^5 + \mathrm{D}_L^6 + \mathrm{D}_R^4 + \mathrm{D}_R^5 + \mathrm{D}_R^6. \tag{3.173}$$

The Lagrangian density of the proton hence comes from (3.133):

$$\begin{aligned} 0 = \mathcal{L}^+ = &\frac{m}{km_1}(-i\eta^{4\dagger}\sigma^\mu\partial_\mu\eta^4 + d^1_{4\mu}\mathrm{D}_L^{4\mu}) + \frac{m}{km_2}(-i\xi^{4\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^4 + d^2_{4\mu}\mathrm{D}_R^{4\mu}) \\ &+\sum_{n=2}^{3}\begin{bmatrix} \frac{m}{km_3}(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n} + d^3_{n\mu}\mathrm{D}_L^{3+n\mu}) \\ +\frac{m}{km_4}(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n} + d^4_{n\mu}\mathrm{D}_R^{3+n\mu}) \end{bmatrix}, \end{aligned} \tag{3.174}$$

where d^k is taken from (**??**), in which we replace n by 4 for the calculation of d^1 and d^2, and we consider only $n = 2$ and $n = 3$ for the calculation of d^3 and d^4. The same restrictions on the indices must be done in the calculation of the energy–momentum tensors and the kinetic momentum tensors, as well as in the calculation of the forces of inertia. The only part of the calculation that changes is in (3.164) where the square brackets include only one line instead of three. And the u quark uses two of the three lines of the square brackets. And it is the sum of the three lines that disappears, this comes from dimension 8 and not 9 of the group of chromodynamics. Strong interactions do not then disappear for a lone proton.

The case is similar for a neutron made of a u quark and two d quarks. There are also six non-null spinor waves, for instance L^2, L^3, L^7 and R^2, R^3, R^7. The quantization of kinetic momentum applies to both proton and neutron – this is in accordance with particle physics. Protons and neutrons were discovered many years before the hypothesis that they are made of three quarks linked together by the forces of chromodynamics. The main problem of this hypothesis is the confinement of the quarks, the practical impossibility to bring a quark out of the bags that are mesons and baryons.

The previous calculation explains this confinement: the quantum of kinetic momentum exists for a proton made of three quarks or for a neutron, not for an isolated quark. This quantum of kinetic momentum exists also for a lone electron, a lone neutrino, or for an electron–neutrino pair. If a lone quark was able to get a quantum of kinetic momentum it should be possible to push this quark out of the bag, but there is no kinetic momentum lower than $\hbar/2$. Particles accelerators use electrons and protons which each have one quantum of kinetic momentum. The only objects that the collisions can produce also have a kinetic momentum $n\hbar/2$, where n is an integer that may be null if the object (a meson) contains two opposite $\hbar/2$ spins. Actually this quantum of kinetic momentum also explains another restriction: we never observe a left neutrino alone just as we never see a quark alone. We see a left neutrino only with a charged particle when and where they interact, or with a right neutrino, or with both a right neutrino and an electron or another particle with an electric charge. It is also detectable when its left wave is joined by a right wave or both a neutrino right wave and the wave of a particle with an electric charge. But the complete neutrino, which we may also call magnetic monopole, has a quantum of kinetic momentum and can then be considered as an observable particle. And there is already some evidence of its being observed [36, 51, 52, 88].

3.8 Preference for left waves

Since the P transformation $P : M \mapsto \widehat{M}$ is an automorphism of Cl_3, the ring $\mathrm{End}(Cl_3) = Cl_{3,3}$ contains the subring of all terms such as:

$$\mathbf{\Psi} := \begin{pmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_1 \end{pmatrix} ; \ \mathcal{P}_1 = \begin{pmatrix} \phi_e & 0 \\ 0 & \widehat{\phi}_e \end{pmatrix} . \tag{3.175}$$

Since the transformation $P : M \mapsto \widehat{M}$ is an automorphism in Cl_3, the ring $\mathrm{End}(Cl_3) = Cl_{3,3}$ contains the sub-ring of all the ϕ_e. This subring may be considered as Cl_3, thus Cl_3 is a subring of $\mathrm{End}(Cl_3)$ and the operations of the Cl_3 ring are a particular case of the operations on $\mathrm{End}(Cl_3)$. The result is an identification between the wave of first quantization ϕ_e and the wave $\mathbf{\Psi}$ of second quantization (an operator). Now Cl_3 is also isomorphic to the even subalgebra $Cl^+_{1,3}$ of $Cl_{1,3}$ which is the set of all $\mathcal{P}_1 = \alpha + \mathbf{A} + \zeta\mathbf{i}$ of (B.95). We detailed this isomorphism in B.1.1 . With $\mathbf{A} = \vec{\mathbf{a}} + \mathbf{i}\vec{\mathbf{b}}$ we have $\mathcal{P}_1 = \alpha + \vec{\mathbf{a}} + \mathbf{i}\vec{\mathbf{b}} + \zeta\mathbf{i}$ whose self-adjoint part is $\alpha + \vec{\mathbf{a}}$. Quantum mechanics included space-time in this framework by setting $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu$ (see (1.33), which gives

$$\det(\mathrm{x}) = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \tag{3.176}$$

This automatically introduces the $+\ -\ -\ -$ signature for space-time. It is the main reason for preferring $Cl_{1,3}$ to $Cl_{3,1}$. This other algebra could be still more important since $Cl_{3,1} = M_4(\mathbb{R})$ is the Majorana algebra. And $\mathrm{End}(Cl_3) = M_8(\mathbb{R})$, each 8×8 real matrix comprising four 4×4 real matrices. Starting from the four γ_μ of (1.4) which generate $Cl_{1,3}$, the four $i\gamma_\mu$ generate $Cl_{3,1}$. The even subalgebra $Cl^+_{3,1}$ is thus the set of all

$$\widetilde{\mathcal{P}}_1 = \alpha - \mathbf{A} + \zeta\mathbf{i} = \alpha - \vec{\mathbf{a}} - \mathbf{i}\vec{\mathbf{b}} + \zeta\mathbf{i}. \tag{3.177}$$

Hence for space-time as the self-adjoint part of Cl_3, the passing from the $Cl^+_{1,3}$ version of Cl_3 to the $Cl^+_{3,1}$ version of the same Cl_3 induces a transformation from $\alpha + \vec{\mathbf{a}}$ to $\alpha - \vec{\mathbf{a}}$, which is the P (parity) transformation. It is the use of $Cl^+_{1,3}$ and the non-use of $Cl^+_{3,1}$ in Chapter 1 and Chapter 2 that fixes the preference for one of the two possible orientations of space, by the identification:

$$\phi_e \in Cl_3 = \psi_e \in Cl_{1,3}^{+} = \mathbf{\Psi} \in \mathrm{End}(Cl_3)]. \tag{3.178}$$

This identification explains why second quantization may use all the results of first quantization in the electron case. The use of $Cl_{1,3}$ both for the electron-neutrino wave and for space-time, as required by the determinant, leads to the use of $\nabla\widehat{\phi}_e$. And the left wave is the left column of $\widehat{\phi}_e$. Next there are two gauge invariances, the electric gauge generated by the 2-vector $\sigma_2\sigma_1 = -i\sigma_3$ and the chiral gauge generated by the 3-vector i. Under the electric gauge the left wave L_e rotates like $\widehat{\phi}_e$, the right wave R_e rotates like ϕ_e, and since $\widehat{i\sigma_3} = i\sigma_3$ thus R_e rotates like L_e. Since $L(-i\sigma_3) = -iL$, L_e rotates with opposite angles under the electric gauge and the chiral gauge. This results in the equalities of the coefficients of B for the left waves, seen in 2.2 in the lepton case and in 3.4 in the quark case. Next the value $\sin(\theta_W) = 1/2$ in (2.221) comes from the fact that A is the electric gauge potential. And this implies in (2.224) the suppression of the only term $qA\overline{L}^8$. The term $qA\overline{R}^8$ is not suppressed. This is the origin of the "maximal parity violation" in weak interactions.

3.9 Wave equation with real values

Until the invention, a century ago by E. Schrödinger, of his non relativistic wave equation of the electron, the complex number played, in physics, only the role of simple computational convenience. The end of the calculation ended always by a return to real quantities. It is with much surprise that Schrödinger understood that it was impossible to do without the i that he had introduced into his wave equation. We started in

We previously wrote the Dirac equation (1.2) with the Pauli and Dirac matrices (1.4). The i present in σ_2 and in the components ξ_j and η_j may be replaced by a real matrix. We yet used that in 2.3.4:

$$i := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\ \xi_j =: a_j + ib_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix};\ \eta_j =: c_j + id_j = \begin{pmatrix} c_j & -d_j \\ d_j & c_j \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \mapsto \boldsymbol{\sigma}_2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \gamma_{0135}, \tag{3.179}$$

$$\gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \mapsto \Lambda_2 = \begin{pmatrix} 0 & -\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\gamma_{0135} \\ \gamma_{0135} & 0 \end{pmatrix}. \tag{3.180}$$

We remark a change of sign in Λ_2 compared to what should be awaited from γ_μ matrices, corresponding to a change of the space orientation when we use real matrices. The Λ_2 matrix and the five other $\Lambda_a,\ a = 0, 1, 3, 4, 5$

matrices are generators of the $M_8(\mathbf{R})$ algebra studied in B.2.1. The Dirac equation (1.2) becomes [16]:

$$0 = [\Lambda^\mu(\partial_\mu + qA_\mu\Lambda_{45}) + m\Lambda_{45}]\mathbf{\Psi};\ \Lambda_{45} = \begin{pmatrix} \gamma_{31} & 0 \\ 0 & \gamma_{31} \end{pmatrix}, \quad (3.181)$$

$$\mathbf{\Psi} := \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix};\ \boldsymbol{\xi} := \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix};\ \boldsymbol{\eta} := \begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix}. \quad (3.182)$$

Multiplying on the left side by Λ_{54}, and using the commutation of this term with each of the four Λ^μ, the Dirac equation reads:

$$0 = [\Lambda^\mu(\Lambda_{54}\partial_\mu + qA_\mu) + m]\mathbf{\Psi}. \quad (3.183)$$

Instead Λ_{54} there are two terms, only two, which can play the same role: Λ_{01234} and Λ_{01235}, and that gives two similar wave equations:

$$0 = [\Lambda^\mu(\Lambda_{01234}\partial_\mu + qA_\mu) + m]\mathbf{\Psi}, \quad (3.184)$$
$$0 = [\Lambda^\mu(\Lambda_{01235}\partial_\mu + qA_\mu) + m]\mathbf{\Psi}. \quad (3.185)$$

That matches what we observe in particle physics: two objects, similar and nevertheless different from the electron, which are the muon and the tau. Notice the difference with the electron, for which the number of indices in Λ_{54} is even, while for the two other ones the number of indices in Λ_{01234} and Λ_{01235} is odd. The tensorial densities to the wave become:

$$d_{\text{ind}} := \overline{\mathbf{\Psi}}\Lambda_{\text{ind}}\mathbf{\Psi},\ \overline{\mathbf{\Psi}} := (\boldsymbol{\eta}^t\ \ \boldsymbol{\xi}^t). \quad (3.186)$$

And since the rewriting of the $n \times p$ complex matrices into $2n \times 2p$ real matrices conserves the operations and transform the complex adjoint matrix into the transposed real matrix, we get with the notations of A.4:

$$\overline{\mathbf{\Psi}}\mathbf{\Psi} = 2(a_1c_1 + b_1d_1 + a_2c_2 + b_2b_2) = \Omega_1, \quad (3.187)$$
$$\overline{\mathbf{\Psi}}\Lambda^\mu\mathbf{\Psi} = \overline{\psi}\gamma^\mu\psi = \mathrm{D}_0^\mu, \quad (3.188)$$
$$\overline{\mathbf{\Psi}}\Lambda^\mu\Lambda^\nu\Lambda_{45}\mathbf{\Psi} = \overline{\psi}i\gamma^\mu\gamma^\nu\psi = S_3^{\mu\nu}, \quad (3.189)$$
$$\overline{\mathbf{\Psi}}\Lambda^\mu\Lambda_{012345}\mathbf{\Psi} = \mathrm{K}^\mu = \mathrm{D}_3^\mu, \quad (3.190)$$
$$\overline{\mathbf{\Psi}}\Lambda_{0123}\mathbf{\Psi} = \Omega_2;\ \Omega_1 + i\Omega_2 = \rho e^{i\beta}, \quad (3.191)$$

The above $16 = 1+4+6+4+1$ tensorial densities, as we saw in 1.3.2, are among the $36 = 9 \times 8/2$ possible densities which exactly correspond to the 36 generators of $Cl_{3,3} = M_8(\mathbb{R})$ with a I_8 square. Besides D_0 and D_3, two other vectors exist, defined in 1.3.2, which satisfy:

$$\mathrm{D}_1 = \phi\sigma_1\phi^\dagger = \mathrm{D}_1^\mu\sigma_\mu;\ \mathrm{D}_2 = \phi\sigma_2\phi^\dagger = \mathrm{D}_2^\mu\sigma_\mu,$$
$$\mathrm{D}_1^\mu = \overline{\mathbf{\Psi}}\Lambda^\mu\Lambda_5\mathbf{\Psi};\ \mathrm{D}_2^\mu = \overline{\mathbf{\Psi}}\Lambda^\mu\Lambda_4\mathbf{\Psi}. \quad (3.192)$$

Similarly the 6 components of the bivector S are among the $18 = 3 \times 6$ densities, defining three similar bivectors:

$$\begin{aligned} S_3 &= \phi\sigma_3\overline{\phi} = S_3^{23}\sigma_1 + S_3^{31}\sigma_2 + S_3^{12}\sigma_3 + S_3^{10}i\sigma_1 + S_3^{20}i\sigma_2 + S_3^{30}i\sigma_3, \\ S_3^{\mu\nu} &= \overline{\mathbf{\Psi}}\Lambda^\mu\Lambda^\nu\Lambda_{45}\mathbf{\Psi}, \end{aligned} \tag{3.193}$$

$$\begin{aligned} S_1 &= \phi\sigma_1\overline{\phi} = S_1^{23}\sigma_1 + S_1^{31}\sigma_2 + S_1^{12}\sigma_3 + S_1^{10}i\sigma_1 + S_1^{20}i\sigma_2 + S_1^{30}i\sigma_3, \\ S_1^{\mu\nu} &= \overline{\mathbf{\Psi}}\Lambda^\mu\Lambda^\nu\Lambda_5\mathbf{\Psi}, \end{aligned} \tag{3.194}$$

$$\begin{aligned} S_2 &= \phi\sigma_2\overline{\phi} = S_2^{23}\sigma_1 + S_2^{31}\sigma_2 + S_2^{12}\sigma_3 + S_2^{10}i\sigma_1 + S_2^{20}i\sigma_2 + S_2^{30}i\sigma_3, \\ S_2^{\mu\nu} &= \overline{\mathbf{\Psi}}\Lambda^\mu\Lambda^\nu\Lambda_4\mathbf{\Psi}. \end{aligned} \tag{3.195}$$

The relativistic invariance has here a slightly different form in comparison with that using complex matrices. We just associate to each M in Cl_3^* the 4×4 real matrix $\mathbf{M}$, and to the 4×4 complex matrix N the 8×8 real matrix $\mathbf{N}$ such as:

$$N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix};\ \phi' = M\phi;\ \psi' = N\psi, \tag{3.196}$$

$$M = \begin{pmatrix} a_1 + ib_1 & -c_2 + id_2 \\ a_2 + ib_2 & c_1 - id_1 \end{pmatrix}:\quad \mathbf{M} = \begin{pmatrix} a_1 & -b_1 & -c_2 & -d_2 \\ b_1 & a_1 & d_2 & -c_2 \\ a_2 & -b_2 & c_1 & d_1 \\ b_2 & a_2 & -d_1 & c_1 \end{pmatrix}, \tag{3.197}$$

$$\mathbf{\Psi}' = \mathbf{N}\mathbf{\Psi};\ \mathbf{N} = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \widehat{\mathbf{M}} \end{pmatrix};\ \mathbf{\Psi}' = \mathbf{N}\mathbf{\Psi} \tag{3.198}$$

We must ensure here to distinguish the 4×4 real matrix $\mathbf{M}$ above to the 4×4 real matrix R in (1.36), which is quadratic compared to $\mathbf{M}$. We recall the form that takes the transformation of space-time variables:

$$\begin{aligned} &\mathrm{x}' = M\mathrm{x}M^\dagger;\ \mathrm{x} = \mathrm{x}^\mu\sigma_\mu;\ \mathrm{x}' = \mathrm{x}'^\mu\sigma_\mu;\ \mathrm{x}'^\mu = R^\mu_\nu\mathrm{x}^\nu, \\ &\nabla = \overline{M}\nabla'\widehat{M};\ \nabla = \sigma^\mu\partial_\mu;\ \nabla' = \sigma^\mu\partial'_\mu;\ R^\mu_\nu\partial'_\mu = \partial_\nu. \end{aligned} \tag{3.199}$$

That gives, with space-time algebra and with the real matrices:

$$R^\mu_\nu\Lambda^\nu = \overline{\mathbf{N}}\Lambda^\mu\mathbf{N};\ \overline{\mathbf{N}} = \begin{pmatrix} \widehat{\mathbf{M}}^t & 0 \\ & \mathbf{M}^t \end{pmatrix}:\ \Lambda^\mu\partial_\mu = \overline{\mathbf{N}}\Lambda^\mu\partial'_\mu\mathbf{N}, \tag{3.200}$$

$$\mathbf{x} := \mathrm{x}^\mu\Lambda_\mu;\ \mathbf{x}' := \mathrm{x}'^\mu\Lambda_\mu;\ \mathbf{x}' = \mathbf{N}\mathbf{x}\mathbf{N}^t. \tag{3.201}$$

We may easily go from the Dirac equation to our improved equation by using the Yvon-Takabayasi angle in the mass term and by replacing the proper mass by the following mass matrix:

$$0 = [\Lambda^\mu(\Lambda_{54}\partial_\mu + qA_\mu) + e^{-\beta\Lambda_{0123}}\mathbf{m}]\mathbf{\Psi};\ \mathbf{m} = \begin{pmatrix} \mathbf{l}I_4 & 0 \\ 0 & \mathbf{r}I_4 \end{pmatrix}. \tag{3.202}$$

Chapter 4

Gravitation

The space-time manifold of general relativity is a submanifold in Cl_3^*. The connection of this manifold is calculated both from the quantum wave (gravitation) and from the invariance group (inertia). Equations of the gravitational field are equalities between two affine connections. The i which defines the orientation of space belongs both to the invariance group and to the gauge group. We generalize the form-invariant derivative. This derivative simplifies the weak interactions part for quark waves. We study the double link between wave equations in the usual form and form-invariant equations, its consequences on the conservation of currents, and the homothety ratio. We show the compatibility between gravitation and our results for the energy–momentum and kinetic momentum tensor densities. Instead of propagating in a linear configuration space, the fermion wave propagates in the space-time manifold. The duality between Lie groups and Lie algebras, applied to space-time in its entirety, explains the arrow of time, gives a beginning of explanation for the resolution of the EPR problem and justifies an expansion of the universe with a recent acceleration of the expansion.

The space-time manifold of general relativity was naturally thought of as a pseudo-Riemannian manifold based on properties of the space-time pseudo-metric. With our choice (1.4) of matrices we have:

$$\mathbf{x} := \mathrm{x}^\mu \gamma_\mu = \begin{pmatrix} 0 & \mathrm{x} \\ \widehat{\mathrm{x}} & 0 \end{pmatrix};\ \mathrm{x} = \mathrm{x}^\mu \sigma_\mu. \tag{4.1}$$

With (1.36) The R similitude defined by the dilator M satisfies:

$$\mathrm{x}' = M\mathrm{x}M^\dagger;\ \ \widehat{\mathrm{x}}' = \widehat{M}\widehat{\mathrm{x}}\overline{M};\ \ M^\dagger = \widetilde{M};\ \ \overline{M} = \widehat{M}^\dagger, \tag{4.2}$$

$$\mathbf{x}' = \begin{pmatrix} 0 & \mathrm{x}' \\ \widehat{\mathrm{x}}' & 0 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{x} \\ \widehat{\mathrm{x}} & 0 \end{pmatrix} \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}, \tag{4.3}$$

$$\mathbf{x}' = N\mathbf{x}\widetilde{N};\ N := \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix};\ \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}. \tag{4.4}$$

The electron, in the tangent space-time at each point of the space-time manifold, has a wave following the improved Dirac equation described in Chapter 1. The set of N is the even subalgebra $Cl_{1,3}^+$, isomorphic to Cl_3. The Dirac theory restricted M to $SL(2,\mathbb{C})$ (wrongly, as we saw in 1.1.2), which implied:

$$\det(M) = 1;\ M^{-1} = \overline{M}, \tag{4.5}$$

$$\nabla\widehat{\phi} = \sigma^\nu\partial_\nu\widehat{\phi} = \overline{M}\nabla'\widehat{M}\widehat{\phi} = M^{-1}\sigma^\mu M^{\dagger-1}\partial'_\mu\widehat{\phi} = M^{-1}\sigma^\mu M^{\dagger-1}R^\nu_\mu\partial_\nu\widehat{\phi}$$

$$\sigma^\nu = M^{-1}\sigma^\mu R^\nu_\mu (M^\dagger)^{-1};\ \sigma^\mu R^\nu_\mu = M\sigma^\nu M^\dagger;\ \gamma^\mu R^\nu_\mu = N\gamma^\nu\widetilde{N}. \tag{4.6}$$

This calculation is valid as long as M is a fixed term. But the last relations (one relation for each value of ν), are carelessly used even in the case of a variable M. Moreover the N matrix is generally supposed to satisfy:

$$N = \exp\Big[\sum_{a<b}\theta^{ab}\boldsymbol{\sigma}_{ab}\Big];\ \boldsymbol{\sigma}_{ab} := \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a),\ \theta^{ab}\in\mathbb{R}. \tag{4.7}$$

But in the previous calculations the similitude R is completely different from the dilator M. And even if M belongs to $SL(2,\mathbb{C})$ the Lorentz transformation R is not the Pauli matrix M. Next (4.7) may unhappily be false: with $M = -1 + \sigma_1 + i\sigma_2$ we obtain:

$$M = -\exp[-(\sigma_1 + i\sigma_2)] = e^{i\pi - \sigma_0\sigma_1 + \sigma_1\sigma_3}, \tag{4.8}$$

$$N = e^{\pi\boldsymbol{\sigma}_{01}\boldsymbol{\sigma}_{23} + \boldsymbol{\sigma}_{01} + \boldsymbol{\sigma}_{13}} = e^{\pi(1+2k)\boldsymbol{\sigma}_{01}\boldsymbol{\sigma}_{23} + \boldsymbol{\sigma}_{01} + \boldsymbol{\sigma}_{13}},\ k\in\mathbb{Z}. \tag{4.9}$$

Thus any calculation based on (4.7) forgets the periodicity of the complex exponential function, mathematical reason of any wave motions. We hence must adopt a more serious approach, which we now develop.

The quantum wave studied in the three previous chapters introduces a major change with the inclusion of space and next of space-time in Cl_3. The first inclusion, that of space, was implicitly realized by Pauli nearly a century ago. The inclusion of time, with the Dirac equation, was also very much implicit. The pseudo-metric of space-time reads:

$$||\mathrm{x}||^2 = \det(\mathrm{x}) = \mathrm{x}\widehat{\mathrm{x}} = \mathrm{x}P(\mathrm{x}). \tag{4.10}$$

where P is the parity transformation. And that is very different in comparison with the approach of Riemann geometry: this geometry generalizes the study of curves and surfaces of the Euclidean geometry of the physical space. $||\mathrm{x}||$ gives only a pseudo-norm since $\det(\mathrm{x})$ may be positive, zero or negative. Moreover a determinant is not a symmetric bilinear form but an antisymmetric one. And the parity transformation is directly associated to geometry by (4.10). We encountered in the first chapter another important relation (1.304):

$$D_{\mathrm{x}} : X \mapsto \mathrm{x} = \phi X \phi^{\dagger}$$

defining the general element X of what we termed "invariant space-time". Since $\det(AB) = \det(A)\det(B)$ and since the M_{ϕ} element of $GL(2, \mathbb{C})$ was defined in (1.160) as $\phi = \sqrt{\rho}e^{i\beta/2}M_{\phi}$, we have:

$$\det(\mathrm{x}) = \rho \det(X). \tag{4.11}$$

We saw that for each important solution used in the Dirac theory, $\rho = \rho(\mathrm{x})$ is nonzero everywhere. And since no observer can travel on the light cone, x satisfies $\det(\mathrm{x}) \neq 0$, we will then make the hypothesis that $\det(X) \neq 0$. This means that X belongs to the Lie group Cl_3^* and the set of X is the self-adjoint part of Cl_3^*, satisfying $X = X^{\dagger}$. The determinant of a product is the product of determinants; thus the relation between X and x implies that x also has a nonzero determinant: this means that the set of x, which is the space-time manifold, is itself the self-adjoint part of the Lie group Cl_3^*. Whitney's theorem indicates that any 4-dimensional manifold, whatever its differential structure, may be included in $\mathbb{R}^8$, and Cl_3 itself, locally isomorphic to $\mathbb{R}^8$, is a Lie group 8-dimensional on $\mathbb{R}$. Consequently to $\det(\mathrm{x}) \neq 0$, space-time geometry does not need Cl_3 itself (isomorphic to $\mathbb{R}^8$ as a manifold), geometry instead needs the Lie group Cl_3^* itself, which is an 8-dimensional separate manifold having Cl_3 as its Lie algebra. [1]Since it is a Lie group, each point is locally identical to the unity point $\mathrm{x} = 1$. This unity point, as any one, means an event: "I am here, now". At any point-event, a tangent space-time exists, where this event "I am here, now" becomes the zero point of the Cl_3 Lie algebra of the Cl_3^* Lie group. Since distances are given by the determinant and since this determinant is not null, by definition of Cl_3^*, the light cone of each point-event is a subset only of the tangent space-time at the considered point-event. This tangent space-time must be distinguished from space-time itself, because the tangent space-time is itself a flat space.

We saw in 1.1.2 and 1.2 that the invariance group $\mathcal{G}$ generalizing to the relativistic case the invariance group $SU(2)$ of nonrelativistic quantum physics for the particles with spin 1/2, may be, without any difficulty, extended to the $GL(2, \mathbb{C})$ group, which is the group of all 2×2 complex invertible matrices. This group is isomorphic to the Cl_3^* group, consisting of all invertible elements in the Cl_3 algebra. This algebra contains the group of

1. The Riemannian geometry uses a metric. That automatically induces the separability of the topology of the manifold. But the pseudo-metric of space-time geometry induces just as automatically the non-separability of distinct events at a null space-time distance. The inclusion of the whole space-time ensures the condition of separability, essential to may use the general results of differential geometry.

its invertible elements, and moreover it is the Lie algebra of this Lie group. The difference between the dilator M and the similitude R generated by this dilator is the same as in the particular case of a Lorentz transformation: the dilator group is a 8-dimensional Lie group while the similitude group is only a 7-dimensional group. The kernel of the homomorphism $f : M \mapsto R$ is the 1-dimensional group made of $M = e^{ix}$, where x is any real number. The f function cannot be invertible: no way exists to define M from R. It is the true reason explaining why Cl_3 is the most important linear space that we use here and why Cl_3^* is the most important Lie group used in this Chapter. It is so because it is impossible otherwise: the isomorphism between $GL(n, \mathbb{C})$ and Cl_p^*, as Lie groups, has properties which exist only in the very particular case $n = 2$ and $p = 3$.

4.1 Differential geometry

4.1.1 Gravitation from the quantum wave

All differential operators that we used in previous chapters are built from the $\nabla = \sigma^\mu \partial_\mu$ operator of the Cl_3 algebra. The operator $\boldsymbol{\partial} = \gamma^\mu \partial_\mu$ used by Hestenes depends on ∇ (see B.1.1) since, as space-time algebra is a modulus on the Cl_3 ring, we have:

$$\boldsymbol{\partial} = \begin{pmatrix} 0 & \nabla \end{pmatrix}. \tag{4.12}$$

The Hestenes' space-time algebra is thus only a possible tool for calculation, but the simpler use of Cl_3 is always available. We hence use here only the Cl_3 algebra. We go from the operator ∇ that operates in a neighborhood of the event $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu$ to the operator ∇' that operates in a neighborhood of the event $\mathrm{x}' = \mathrm{x}'^\mu \sigma_\mu$ by:

$$\mathrm{x}' = R(\mathrm{x}) = M \mathrm{x} M^\dagger;\ \nabla = \overline{M} \nabla' \widehat{M};\ \nabla' = \sigma^\mu \partial'_\mu. \tag{4.13}$$

We recall that $\sigma_\mu,\ \mu = 0, 1, 2, 3$ are exactly the same when writing either x or x′, and likewise for ∇ and ∇'. [2] We also explained in 1.8 how the ϕ wave of the electron defines at each point-event a similitude:

$$D_\mathrm{x} : X = X^\mu \sigma_\mu \mapsto \mathrm{x} = \phi X \phi^\dagger = \phi X^\nu \sigma_\nu \phi^\dagger = X^\nu \mathrm{D}_\nu, \tag{4.14}$$

$$\mathrm{D}_\nu = \mathrm{D}_\nu^\mu \sigma_\mu = \phi \sigma_\nu \phi^\dagger. \tag{4.15}$$

So the ϕ function describes the inclusion of the space-time manifold into the 8-dimensional manifold, seen from our included manifold. The function ϕ is all that may be perceived from our space-time manifold.

We saw in 1.3.2 and in 3.9 that the four $\mathrm{D}_\nu,\ \nu = 0, 1, 2, 3$ vectors form an orthogonal basis of space-time (but not an orthonormal basis), at each

2. The fixedness of the σ^μ comes from the fact that the four matrices $(1 \pm \sigma_3)/2$ and $(\sigma_1 \pm i\sigma_2)/2$ constitute the canonical basis of $M_2(\mathbb{C})$ and are thus intrinsic in the $GL(2, \mathbb{C})$ group.

point-event. To the Cartan's mobile basis $(\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$ is thus associated an affine connection. These vectors are calculated in A.4.2. We recall that we have:

$$\rho e^{i\beta} = \phi\overline{\phi} = \det(\phi);\ \rho e^{-i\beta} = \widehat{\phi}\widetilde{\phi}, \tag{4.16}$$

$$\mathrm{D}_\mu \cdot \mathrm{D}_\nu = 0,\ \mu \neq \nu, \tag{4.17}$$

$$\rho^2 = \mathrm{D}_0 \cdot \mathrm{D}_0 = -\mathrm{D}_1 \cdot \mathrm{D}_1 = -\mathrm{D}_2 \cdot \mathrm{D}_2 = -\mathrm{D}_3 \cdot \mathrm{D}_3. \tag{4.18}$$

This connection calculated in D.4 was first studied in [21]. We let:

$$\begin{aligned} \boldsymbol{\partial}_\nu &= \frac{\partial}{\partial X^\nu} = \mathrm{D}^\mu_\nu \partial_\mu;\ d\mathrm{x} = dX^\nu \mathrm{D}_\nu, \\ d\mathrm{D}_\mu &= \Gamma^\beta_{\mu\nu} dX^\nu \mathrm{D}_\beta. \end{aligned} \tag{4.19}$$

If $\rho \neq 0$ we have:

$$\begin{aligned} d\mathrm{x} &= d\mathrm{x}^\mu \sigma_\mu = \mathrm{D}^\mu_\nu \sigma_\mu dX^\nu = \mathrm{D}_\nu dX^\nu, \\ \mathrm{D}_\nu &= \phi\sigma_\nu\phi^\dagger = \mathrm{D}^\mu_\nu \sigma_\mu\ ;\quad \sigma_\mu = (\mathrm{D}^{-1})^\beta_\mu \mathrm{D}_\beta. \end{aligned} \tag{4.20}$$

Now we use the similitude $\overline{\mathrm{D}}$ such that:

$$\overline{\mathrm{D}}(\mathrm{x}) = \overline{\phi}\mathrm{x}\widehat{\phi}. \tag{4.21}$$

We have:

$$\begin{aligned} \mathrm{D} \circ \overline{\mathrm{D}}(\mathrm{x}) &= \mathrm{D}[\overline{\mathrm{D}}(\mathrm{x})] = \phi\overline{\phi}\mathrm{x}\widehat{\phi}\phi^\dagger = \rho e^{i\beta}\mathrm{x}\rho e^{-i\beta} = \rho^2\mathrm{x}, \\ \mathrm{D} \circ (\rho^{-2}\overline{\mathrm{D}})(\mathrm{x}) &= \mathrm{x}, \\ \mathrm{D}^{-1}(\mathrm{x}) &= \rho^{-2}\overline{\mathrm{D}}(\mathrm{x}). \end{aligned} \tag{4.22}$$

And we get:

$$\begin{aligned} d\mathrm{D}_\mu &= \boldsymbol{\partial}_\nu(\mathrm{D}_\mu)dX^\nu = \boldsymbol{\partial}_\nu(\mathrm{D}^\xi_\mu \sigma_\xi)dX^\nu = \boldsymbol{\partial}_\nu(\mathrm{D}^\xi_\mu)\sigma_\xi dX^\nu \\ &= \boldsymbol{\partial}_\nu(\mathrm{D}^\xi_\mu)(\mathrm{D}^{-1})^\beta_\xi \mathrm{D}_\beta dX^\nu = \Gamma^\beta_{\mu\nu}\mathrm{D}_\beta dX^\nu. \end{aligned} \tag{4.23}$$

Therefore the connection coefficients are:

$$\Gamma^\beta_{\mu\nu} = \boldsymbol{\partial}_\nu(\mathrm{D}^\xi_\mu)(\mathrm{D}^{-1})^\beta_\xi\ ;\quad \boldsymbol{\partial}_\nu = \mathrm{D}^\tau_\nu \partial_\tau. \tag{4.24}$$

By using the $\overline{\mathrm{D}}$ similitude we get:

$$\Gamma^\beta_{\mu\nu} = \rho^{-2}\boldsymbol{\partial}_\nu(\mathrm{D}^\xi_\mu)\overline{\mathrm{D}}^\beta_\xi\ ;\quad \boldsymbol{\partial}_\nu = \mathrm{D}^\tau_\nu \partial_\tau. \tag{4.25}$$

Since $\overline{\mathrm{D}}^0_0 = \mathrm{D}^0_0$ and $\overline{\mathrm{D}}^0_j = -\mathrm{D}^j_0$ we have:

$$\Gamma^0_{0\nu} = \Gamma^1_{1\nu} = \Gamma^2_{2\nu} = \Gamma^3_{3\nu} = \boldsymbol{\partial}_\nu[\ln(\rho)] = \mathrm{D}^\mu_\nu \partial_\mu[\ln(\rho)]. \tag{4.26}$$

Since $\overline{\mathrm{D}}_0^j = -\mathrm{D}_j^0$ and $\overline{\mathrm{D}}_j^k = \mathrm{D}_k^j$ we have:

$$\Gamma^j_{0\nu} = \Gamma^0_{j\nu} \ , \quad j = 1,\ 2,\ 3, \tag{4.27}$$

$$\Gamma^j_{k\nu} = -\Gamma^k_{j\nu} \ , \quad j = 1,\ 2,\ 3 \ , \quad k = 1,\ 2,\ 3, \quad k \neq j. \tag{4.28}$$

A complete calculation of the connection needs the following quantities:

$$S_k = \phi\sigma_k\overline{\phi}, \tag{4.29}$$

$$\mathcal{S}_{(k)} + i\mathcal{S}'_{(k)} = \frac{\nabla S_k^\dagger}{\det(\phi)^\dagger}, \tag{4.30}$$

$$\mathcal{A}_{(k)} + i\mathcal{A}'_{(k)} = \frac{AS_k^\dagger}{\det(\phi)^\dagger}, \tag{4.31}$$

$$\tau = \frac{1}{2}[(\nabla\widehat{\phi})\phi^\dagger - \sigma^\mu\widehat{\phi}\partial_\mu\phi^\dagger], \tag{4.32}$$

$$\mathcal{T} + i\mathcal{T}' = \frac{\tau}{\det(\phi)^\dagger}. \tag{4.33}$$

The tensor τ is Durand's spin density [15, 53]. Using our improved wave equation of the electron, we obtain in D.4:

$$\Gamma^0_{1\mu} = \mathrm{D}_\mu \cdot [\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}] + 2m\rho\delta^2_\mu, \tag{4.34}$$

$$\Gamma^0_{2\mu} = \mathrm{D}_\mu \cdot [\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}] - 2m\rho\delta^1_\mu, \tag{4.35}$$

$$\Gamma^0_{3\mu} = \mathrm{D}_\mu \cdot \mathcal{S}_{(3)}, \tag{4.36}$$

$$\Gamma^2_{3\mu} = -\mathrm{D}_\mu \cdot [\mathcal{S}'_{(1)} + 2q\mathcal{A}'_{(2)}] - 2d\rho\delta^1_\mu, \tag{4.37}$$

$$\Gamma^3_{1\mu} = -\mathrm{D}_\mu \cdot [\mathcal{S}'_{(2)} + 2q\mathcal{A}'_{(1)}] + 2d\rho\delta^2_\mu, \tag{4.38}$$

$$\Gamma^1_{2\mu} = -\mathrm{D}_\mu \cdot [\mathcal{S}'_{(3)} + 2qA] - 2m\rho\delta^0_\mu + 2d\rho\delta^3_\mu, \tag{4.39}$$

$$\Gamma^0_{0\mu} = \mathrm{D}_\mu \cdot [-2\mathcal{T} + 2q\mathcal{A}'_{(3)}], \tag{4.40}$$

where $\delta^0_0 = 1$, $\delta^j_j = -1, j = 1, 2, 3$ and $\delta^\nu_\mu = 0, \mu \neq \nu$. The very particular role of the index 3 in the Dirac equation of the electron is still very visible in these relations. For the second or the third generation it is enough to make a circular permutation of indices. So a particular index, 1 or 2, is thus also visible. The connection is not torsion-free, and the proper mass is linked to this torsion: this is the reason to think of this connection as yielding gravitation. Moreover, the mass term $m\rho$, and thus also Christoffel's symbols, have the physical dimension L^{-1} of a radius of curvature. We may thus consider that **the link between mass-energy and geometry is not made with the curvature tensor, but directly with the affine connection and the torsion of the space-time manifold**. This is a generalization of Einstein's attempt at a space-time manifold without curvature and with torsion to account for both gravitation and electromagnetism [68].

With the plane wave we obtained in 1.5.3:

$$\phi = \phi_0 e^{-\varphi\sigma_{12}} \ ; \quad \varphi = m_g \mathrm{v}_\mu x^\mu \ ; \quad \mathrm{v} = \sigma^\mu \mathrm{v}_\mu, \ m_g = \sqrt{\mathbf{lr}}. \tag{4.41}$$

where the reduced velocity v and ϕ_0 are fixed terms. We obtain:

$$\phi = \sigma^\mu \partial_\mu(\widehat{\phi}_0 e^{-\varphi\sigma_{12}}) = -m_g \mathrm{v}\widehat{\phi}\sigma_{12}. \tag{4.42}$$

That gives:

$$\phi = e^{i\beta}\mathrm{v}\widehat{\phi}\frac{\widehat{\mathbf{m}}}{m_g}, \tag{4.43}$$

which implies that:

$$\phi = e^{i\beta}\mathrm{v}(e^{-i\beta}\widehat{\mathrm{v}}\phi\frac{\mathbf{m}}{m_g})\frac{\widehat{\mathbf{m}}}{m_g} = \mathrm{v}\widehat{\mathrm{v}}\phi. \tag{4.44}$$

Thus if ϕ_0 is invertible we must take:

$$1 = \mathrm{v}\widehat{\mathrm{v}} = \mathrm{v}\cdot\mathrm{v} = \mathrm{v}_0^2 - \vec{\mathrm{v}}^{\,2}, \tag{4.45}$$

$$\mathrm{v}_0^2 = 1 + \vec{\mathrm{v}}^{\,2}\ ;\quad \mathrm{v}_0 = \sqrt{1+\vec{\mathrm{v}}^{\,2}}. \tag{4.46}$$

which is the relativistic relation for the reduced velocity of a particle. We also get:

$$\rho e^{i\beta} = \det(\phi) = \det(\phi_0)\det(e^{i\beta}) = \det(\phi_0). \tag{4.47}$$

Therefore ρ and β are fixed. It is the same for:

$$\mathrm{D}_0 = \phi_0\phi_0^\dagger\ ;\quad \mathrm{D}_3 = \phi_0\sigma_3\phi_0^\dagger. \tag{4.48}$$

The D_1 and D_2 vectors, on the contrary, are variable. We let:

$$d_1 = \phi_0\sigma_1\phi_0^\dagger\ ;\quad d_2 = \phi_0\sigma_2\phi_0^\dagger, \tag{4.49}$$

which gives:

$$\begin{aligned}\mathrm{D}_1 &= \cos(2\varphi)d_1 + \sin(2\varphi)d_2,\\ \mathrm{D}_2 &= -\sin(2\varphi)d_1 + \cos(2\varphi)d_2.\end{aligned} \tag{4.50}$$

As D_0 and D_3 are fixed we get:

$$\boldsymbol{\partial}_\nu(\mathrm{D}_0^\xi) = \boldsymbol{\partial}_\nu(\mathrm{D}_3^\xi) = 0 \tag{4.51}$$

$$\Gamma_{0\nu}^\beta = \Gamma_{3\nu}^\beta = 0. \tag{4.52}$$

With D_1 and D_2 we obtain:

$$\partial_\tau(\mathrm{D}_1^\xi) = \partial_\tau[\cos(2\varphi)d_1^\xi + \sin(2\varphi)d_2^\xi] = 2m_g\mathrm{v}_\tau\mathrm{D}_2^\xi,$$

$$\partial_\tau(\mathrm{D}_2^\xi) = \partial_\tau[-\sin(2\varphi)d_1^\xi + \cos(2\varphi)d_2^\xi] = -2m_g\mathrm{v}_\tau\mathrm{D}_1^\xi,$$

$$\boldsymbol{\partial}_\nu(\mathrm{D}_1^\xi) = \mathrm{D}_\nu^\tau\partial_\tau(\mathrm{D}_1^\xi) = 2m_g\mathrm{D}_\nu^\tau\mathrm{v}_\tau\mathrm{D}_2^\xi = 2m_g(\mathrm{D}_\nu\cdot\mathrm{v})\mathrm{D}_2^\xi, \tag{4.53}$$

$$\boldsymbol{\partial}_\nu(\mathrm{D}_2^\xi) = \mathrm{D}_\nu^\tau\partial_\tau(\mathrm{D}_2^\xi) = -2m - g\mathrm{D}_\nu^\tau\mathrm{v}_\tau\mathrm{D}_1^\xi = -2m_g(\mathrm{D}_\nu\cdot\mathrm{v})\mathrm{D}_1^\xi. \tag{4.54}$$

Then we get:

$$\mathrm{D}_\nu \cdot \mathrm{v} = \frac{1}{\rho}\mathrm{D}_\nu \cdot \mathrm{D}_0 = \rho\delta^0_\nu. \tag{4.55}$$

Thus we have:

$$\Gamma^\beta_{11} = \Gamma^\beta_{12} = \Gamma^\beta_{13} = \Gamma^\beta_{21} = \Gamma^\beta_{22} = \Gamma^\beta_{23} = 0. \tag{4.56}$$

And we get:

$$\Gamma^\beta_{10} = \frac{2m_g}{\rho}\mathrm{D}^\xi_2\overline{\mathrm{D}}^\beta_\xi \ ; \ \ \Gamma^\beta_{20} = -\frac{2m_g}{\rho}\mathrm{D}^\xi_1\overline{\mathrm{D}}^\beta_\xi, \tag{4.57}$$

which gives:

$$\begin{aligned}\Gamma^2_{10} &= \frac{2m_g}{\rho}\mathrm{D}^\xi_2\overline{\mathrm{D}}^2_\xi = \frac{2m_g}{\rho}(\mathrm{D}^0_2\overline{\mathrm{D}}^2_0 + \mathrm{D}^1_2\overline{\mathrm{D}}^2_1 + \mathrm{D}^2_2\overline{\mathrm{D}}^2_2 + \mathrm{D}^3_2\overline{\mathrm{D}}^2_3)\\ &= \frac{2m_g}{\rho}(-\mathrm{D}^0_2\mathrm{D}^0_2 + \mathrm{D}^1_2\mathrm{D}^1_2 + \mathrm{D}^2_2\mathrm{D}^2_2 + \mathrm{D}^3_2\mathrm{D}^3_2)\\ &= \frac{2m_g}{\rho}(-\mathrm{D}_2 \cdot \mathrm{D}_2) = 2m_g\rho. \end{aligned} \tag{4.58}$$

We also have:

$$\begin{aligned}\Gamma^0_{10} &= \frac{2m_g}{\rho}(\mathrm{D}_2 \cdot \mathrm{D}_0) = 0,\\ \Gamma^3_{10} &= \frac{2m_g}{\rho}(-\mathrm{D}_2 \cdot \mathrm{D}_3) = 0,\\ \Gamma^1_{10} &= \frac{2m_g}{\rho}(-\mathrm{D}_2 \cdot \mathrm{D}_1) = 0. \end{aligned} \tag{4.59}$$

Similarly for the Γ^β_{20} we get:

$$\Gamma^1_{20} = -2m_g\rho \ ; \ \ \Gamma^0_{20} = \Gamma^2_{20} = \Gamma^3_{20} = 0. \tag{4.60}$$

To resume, among the 64 $\Gamma^\beta_{\mu\nu}$ terms, 62 terms are zero. Only two terms are not zero:

$$\Gamma^2_{10} = -\Gamma^1_{20} = 2m_g\rho. \tag{4.61}$$

Therefore the torsion has two fixed components:

$$\frac{1}{2}(\Gamma^2_{10} - \Gamma^2_{01}) = m_g\rho, \tag{4.62}$$

$$\frac{1}{2}(\Gamma^1_{20} - \Gamma^1_{02}) = -m_g\rho. \tag{4.63}$$

As the nonvanishing $\Gamma^\beta_{\mu\nu}$ terms are all fixed, the curvature tensor cancels out. We thus see that, for the improved equation, the manifold linked to a plane wave is without curvature but with a fixed torsion, and the mass term is proportional to this torsion.

4.1.2 Connection linked to the invariance group

We must now consider four kinds of spinors which vary with four different manners in the similitudes induced by the extended invariance group: the $Cl_3^* = GL(2,\mathbb{C})$ group has four kinds of nonequivalent representations. So in addition to the invariance of what do not change (for instance the the Lagrangian density), we get no less than six kinds of variance: the contravariance of vectors transforming like x, the covariance of vectors transforming like ∇, and four kinds of spinors that we encountered in previous chapters:

$$\mathrm{x}' = M\mathrm{x}M^\dagger;\ \nabla = \overline{M}\nabla'\widehat{M} = \overline{M}\sigma^\mu\widehat{M}\partial'_\mu, \tag{4.64}$$

$$R'^n = MR^n;\ \widehat{L}'^n = \widehat{M}\widehat{L}^n,\ n = 1,2,3,4, \tag{4.65}$$

$$R'^{4+n} = R^{4+n}\widetilde{M};\ \widehat{L}'^{4+n} = \widehat{L}^{4+n}\overline{M},\ n = 1,2,3,4. \tag{4.66}$$

Differential geometry studies what happens in the neighborhood of a given point-event. That induces to consider in the neighborhood of x a dilator M which differs from unity only by an infinitesimal. We thus let:

$$M = 1+\mathrm{dx}^\mu(a_\mu^0+a_\mu^1\sigma_1+a_\mu^2\sigma_2+a_\mu^3\sigma_3+a_\mu^4 i\sigma_1+a_\mu^5 i\sigma_2+a_\mu^6 i\sigma_3+a_\mu^7 i), \tag{4.67}$$

where the a_μ^n, for $\mu = 1,2,3,4$ and $n = 0,1,\dots,7$ are 32 smooth enough real functions of x, and dx^μ are increments of x at this point-event in the relevant local basis. This gives:

$$\begin{aligned}
M^\dagger &= 1 + \mathrm{dx}^\mu(a_\mu^0 + a_\mu^1\sigma_1 + a_\mu^2\sigma_2 + a_\mu^3\sigma_3 - a_\mu^4 i\sigma_1 - a_\mu^5 i\sigma_2 - a_\mu^6 i\sigma_3 - a_\mu^7 i),\\
\widehat{M} &= 1 + \mathrm{dx}^\mu(a_\mu^0 - a_\mu^1\sigma_1 - a_\mu^2\sigma_2 - a_\mu^3\sigma_3 + a_\mu^4 i\sigma_1 + a_\mu^5 i\sigma_2 + a_\mu^6 i\sigma_3 - a_\mu^7 i),\\
\overline{M} &= 1 + \mathrm{dx}^\mu(a_\mu^0 - a_\mu^1\sigma_1 - a_\mu^2\sigma_2 - a_\mu^3\sigma_3 - a_\mu^4 i\sigma_1 - a_\mu^5 i\sigma_2 - a_\mu^6 i\sigma_3 + a_\mu^7 i).
\end{aligned} \tag{4.68}$$

We also have:

$$M\overline{M} = \det(M) = 1 + 2\mathrm{dx}^\mu(a_\mu^0 + ia_\mu^7), \tag{4.69}$$

$$\det(M^{-1}) = 1 - 2\mathrm{dx}^\mu(a_\mu^0 + ia_\mu^7), \tag{4.70}$$

$$\overline{M}^{-1} = M\det(M^{-1}), \tag{4.71}$$

$$= 1 + \mathrm{dx}^\mu(-a_\mu^0 + a_\mu^1\sigma_1 + a_\mu^2\sigma_2 + a_\mu^3\sigma_3 + a_\mu^4 i\sigma_1 + a_\mu^5 i\sigma_2 + a_\mu^6 i\sigma_3 - a_\mu^7 i),$$

$$\widehat{M}^{-1} = (\overline{M}^{-1})^\dagger \tag{4.72}$$

$$= 1 + \mathrm{dx}^\mu(-a_\mu^0 + a_\mu^1\sigma_1 + a_\mu^2\sigma_2 + a_\mu^3\sigma_3 - a_\mu^4 i\sigma_1 - a_\mu^5 i\sigma_2 - a_\mu^6 i\sigma_3 + a_\mu^7 i).$$

The similitude R defined par M changes x into x$'$, such as:

$$\mathrm{x}' = R(\mathrm{x}) + \mathrm{dx} = M\mathrm{x}M^\dagger + \mathrm{dx}, \tag{4.73}$$

$$\mathrm{x}'^0 = \mathrm{x}^0 + \mathrm{dx}^0 + 2(a_\mu^0\mathrm{x}^0 + a_\mu^1\mathrm{x}^1 + a_\mu^2\mathrm{x}^2 + a_\mu^3\mathrm{x}^3)\mathrm{dx}^\mu, \tag{4.74}$$

$$x'^1 = x^1 + dx^1 + 2(a^1_\mu x^0 + a^0_\mu x^1 + a^6_\mu x^2 - a^5_\mu x^3)dx^\mu, \quad (4.75)$$
$$x'^2 = x^2 + dx^2 + 2(a^2_\mu x^0 - a^6_\mu x^1 + a^0_\mu x^2 + a^4_\mu x^3)dx^\mu, \quad (4.76)$$
$$x'^3 = x^3 + dx^3 + 2(a^3_\mu x^0 + a^5_\mu x^1 - a^4_\mu x^2 + a^0_\mu x^3)dx^\mu. \quad (4.77)$$

Since Christoffel symbols $\mathbf{\Gamma}^\alpha_{\beta\gamma}$ are defined as:

$$x'^\alpha = x^\alpha + dx^\alpha + \mathbf{\Gamma}^\alpha_{\beta\gamma} x^\beta dx^\gamma, \quad (4.78)$$

we thus have:

$$\mathbf{\Gamma}^0_{0\mu} = \mathbf{\Gamma}^1_{1\mu} = \mathbf{\Gamma}^2_{2\mu} = \mathbf{\Gamma}^3_{3\mu} = 2a^0_\mu, \quad (4.79)$$
$$\mathbf{\Gamma}^1_{0\mu} = \mathbf{\Gamma}^0_{1\mu} = 2a^1_\mu \ ; \ \mathbf{\Gamma}^2_{0\mu} = \mathbf{\Gamma}^0_{2\mu} = 2a^2_\mu \ ; \ \mathbf{\Gamma}^3_{0\mu} = \mathbf{\Gamma}^0_{3\mu} = 2a^3_\mu, \quad (4.80)$$
$$\mathbf{\Gamma}^2_{3\mu} = -\mathbf{\Gamma}^3_{2\mu} = 2a^4_\mu \ ; \ \mathbf{\Gamma}^3_{1\mu} = -\mathbf{\Gamma}^1_{3\mu} = 2a^5_\mu \ ; \ \mathbf{\Gamma}^1_{2\mu} = -\mathbf{\Gamma}^2_{1\mu} = 2a^6_\mu. \quad (4.81)$$

So the connection defined by the $\mathbf{\Gamma}^\alpha_{\beta\gamma}$ symbols depends only on 28 of the 32 real functions contained in the dilator M in (4.67). The four a^7_μ are necessarily absent in the connection, because they are factors of the i generator of the chiral gauge [21] [23] which belongs to the center of the group, which comes from the commutation of this i with any element in Cl_3^*. Differential geometry cannot perceive these a^7_μ! Einstein thought that something was lacking in the physical theory for the integration of quantum physics into classical physics. The four parameters that are lacking in the geometric part of the connection are not lacking in the spinor part of differential geometry. Thus something was actually lacking, only not where it was expected. [3] The equalities (4.79) to (4.81) have the same structure as the equalities between $\Gamma^\alpha_{\beta\mu}$ in (4.26) to (4.28). The construction of General Relativity by Einstein stated from an equality of structure similar, between gravitation and inertia. Identifying the curvature tensor of the space-time manifold to the density of energy–momentum, Einstein equated that identity of structure. His construction did not satisfy his aesthetic research of physical laws: That was putting together a very nice geometrical object, the curvature tensor, and a very badly assembled object, the energy–impulse density, where it was possible to put anything and everything. He thus searched for a theory in which the two sides of his equations should have the same geometric value. He especially studied a manifold with torsion [113], our approach is thus in the continuity of his research.

The curves and surfaces studied since Euler with the tools of differential geometry are the objects from which were defined the manifolds. Those manifolds are orientable (or not), they have a boundary (or not). We may think of the space-time manifold as a hyper-surface (with $1+3$ dimensions) in that 8-dimensional group, presenting a boundary (the light cone) and

3. This number, $28 = 8\times 7/2$, is also the dimension of the $SO(8)$ group of the rotations in Cl_3. And $36 = 64 - 28 = 8 \times 9/2$ is the number of densities that can be constructed from the electron wave [16].

two faces: we may see the hyper–surface above (that orientes the manifold) or below (with the other orientation). That explains the difference of orientation between the calculations of Chapter 1, based on the systematic use of quantum numbers in the geometry of space-time and the ϕ wave, and the calculations in 3.9 using only real quantities. But if we may distinguish two faces, the space-time manifold remains single, with only one connection. That is why we may only identify our two connections, with the 64 identities:

$$\mathbf{\Gamma}^{\alpha}_{\beta\mu} = \Gamma^{\alpha}_{\beta\mu}. \tag{4.82}$$

We may group those equations in $7 = 28/4$ equations:

$$2a^1_\mu = \Gamma^0_{1\mu} = D_\mu \cdot [\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}] + 2m_g\rho\delta^2_\mu, \tag{4.83}$$
$$2a^2_\mu = \Gamma^0_{2\mu} = D_\mu \cdot [\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}] - 2m_g\rho\delta^1_\mu, \tag{4.84}$$
$$2a^3_\mu = \Gamma^0_{3\mu} = D_\mu \cdot \mathcal{S}_{(3)}, \tag{4.85}$$
$$2a^4_\mu = \Gamma^2_{3\mu} = -D_\mu \cdot [\mathcal{S}'_{(1)} - 2q\mathcal{A}'_{(2)}], \tag{4.86}$$
$$2a^5_\mu = \Gamma^3_{1\mu} = -D_\mu \cdot [\mathcal{S}'_{(2)} + 2q\mathcal{A}'_{(1)}], \tag{4.87}$$
$$2a^6_\mu = \Gamma^1_{2\mu} = -D_\mu \cdot [\mathcal{S}'_{(3)} + 2qA] - 2m_g\rho\delta^0_\mu, \tag{4.88}$$
$$2a^0_\mu = \Gamma^0_{0\mu} = D_\mu \cdot [-2\mathcal{T} + 2q\mathcal{A}'_{(3)}]. \tag{4.89}$$

We again see clearly seven a^n, $n = 0, 1, \dots, 6$ vectors, with a^7 lacking. These vectors will be called inertial potentials. These 7 equations may be considered the gravitational field equations.

Vectors transforming like (4.78) are called contravariant. On the other hand, covariant vectors transform like ∇:

$$\nabla = \sigma^\mu\partial_\mu = \overline{M}\sigma^\mu\widehat{M}\partial'_\mu, \tag{4.90}$$

with always the same σ^μ. These relations, demonstrated in A.4.5, do not place the ∂'_μ operators behind $\widehat{M}$ but before, because M is taken to be constant. Nevertheless for a variable M, it is (4.90) that is proved in A.4.5, because the proof only uses algebraic properties of partial derivatives. That gives:

$$\begin{aligned}\nabla\widehat{\phi} = \sigma^\mu\partial_\mu\widehat{\phi} &= \overline{M}\sigma^\mu\widehat{M}\partial'_\mu\widehat{\phi}\\ &= \overline{M}\sigma^\mu[\partial'_\mu(\widehat{M}\widehat{\phi}) - (\partial'_\mu\widehat{M})\widehat{\phi}].\end{aligned} \tag{4.91}$$

And we have:

$$\begin{aligned}(\partial'_\mu\widehat{M})\widehat{M}^{-1} + \widehat{M}(\partial'_\mu\widehat{M}^{-1}) = \partial'_\mu(\widehat{M}\widehat{M}^{-1}) = \partial'_\mu(1) = 0,\\ \partial'_\mu\widehat{M} = -\widehat{M}(\partial'_\mu\widehat{M}^{-1})\widehat{M}.\end{aligned} \tag{4.92}$$

If we defined the $\mathbf{D}$ derivation as

$$\mathbf{D}\widehat{\phi} := [\nabla - \frac{1}{2}(\nabla\widehat{M}^{-1})\widehat{M}]\widehat{\phi}, \tag{4.93}$$

necessary we have:

$$\overline{\phi}'\mathbf{D}'\widehat{\phi}' = \overline{\phi}'\Big[\nabla' - \frac{1}{2}(\nabla'\widehat{M}'^{-1})\widehat{M}'\Big]\widehat{\phi}', \tag{4.94}$$

$$M' = M^{-1}, \tag{4.95}$$

$$\overline{\phi}'\mathbf{D}'\widehat{\phi}' = \overline{\phi}'\Big[\nabla' - \frac{1}{2}(\nabla'\widehat{M})\widehat{M}^{-1}\Big]\widehat{\phi}'. \tag{4.96}$$

Therefore we get:

$$\begin{aligned}
&\overline{\phi}'(\mathbf{D}'\widehat{\phi}') = \overline{\phi}'\Big[\nabla' - \frac{1}{2}(\nabla'\widehat{M})\widehat{M}^{-1}\Big]\widehat{\phi}' = \overline{\phi}\,\overline{M}\sigma^\mu\Big[\partial'_\mu(\widehat{M}\widehat{\phi}) - \frac{1}{2}(\partial'_\mu\widehat{M})\widehat{\phi}\Big]\\
&= \overline{\phi}\,\overline{M}\sigma^\mu\Big[(\partial'_\mu\widehat{M})\widehat{\phi} + \widehat{M}(\partial'_\mu\widehat{\phi}) - \frac{1}{2}(\partial'_\mu\widehat{M})\widehat{\phi}\Big] \qquad (4.97)\\
&= \overline{\phi}\,\overline{M}\sigma^\mu\Big[\widehat{M}\partial'_\mu\widehat{\phi} + \frac{1}{2}(\partial'_\mu\widehat{M})\widehat{\phi}\Big] = \overline{\phi}(\overline{M}\sigma^\mu\widehat{M}\partial'_\mu\widehat{\phi}) - \frac{1}{2}\overline{\phi}(\overline{M}\sigma^\mu\widehat{M}\partial'_\mu\widehat{M}^{-1})\widehat{M}\widehat{\phi}\\
&= \overline{\phi}[\nabla - \frac{1}{2}(\nabla\widehat{M}^{-1})\widehat{M}]\widehat{\phi} = \overline{\phi}(\mathbf{D}\widehat{\phi}).
\end{aligned}$$

We may then say that $\mathbf{D}$ is **form-invariant**. In a shortened form we name $\mathbf{D}$ the invariant derivation. Using the reversion and the $M \mapsto \widehat{M}$ conjugation we have:

$$\widehat{\mathbf{D}}\phi = [\widehat{\nabla} - \frac{1}{2}(\widehat{\nabla}M^{-1})M]\phi, \tag{4.98}$$

$$\overline{\phi}\widetilde{\mathbf{D}} = (\overline{\phi}\nabla) - \frac{1}{2}\overline{\phi}\,\overline{M}(\overline{M}^{-1}\nabla), \tag{4.99}$$

$$\widetilde{\phi}\overline{\mathbf{D}} = (\widetilde{\phi}\widehat{\nabla}) - \frac{1}{2}\widetilde{\phi}\widetilde{M}(\widetilde{M}^{-1}\widehat{\nabla}). \tag{4.100}$$

The quantum wave in a non-null gravitational field follows exactly the same invariant wave equations as in a null field. The only difference: the differential operator ∇ is replaced by the invariant $\mathbf{D}$. That uses the 8 (not 7) space-time vectors a^n in (4.67):

$$a^n = \sigma^\mu a^n_\mu, \tag{4.101}$$

$$\begin{aligned}
\mathbf{D}\widehat{\phi} &= \Big[\nabla - \frac{1}{2}(\nabla\widehat{M}^{-1})\widehat{M}\Big]\widehat{\phi} \qquad (4.102)\\
&= \Big[\nabla - \frac{1}{2}(a^0 - a^1\sigma_1 - a^2\sigma_2 - a^3\sigma_3 + a^4 i\sigma_1 + a^5 i\sigma_2 + a^6 i\sigma_3 - a^7 i)\Big]\widehat{\phi}.
\end{aligned}$$

Here we must consider all 32 functions, including the four that are not implied in the calculation of the tensors of torsion and curvature.

Under the similitude $\mathrm{x} \mapsto \underline{\mathrm{x}}$ that comes from a fixed dilator N satisfying:

$$\underline{\mathrm{x}} = N\mathrm{x}N^\dagger;\ \nabla = \overline{N}\underline{\nabla}\widehat{N}, \tag{4.103}$$

we must have, with the covariance of ∇ and linked gauge terms:

$$\mathbf{D} = \overline{N}\underline{\mathbf{D}}\widehat{N};\ \nabla\widehat{M}^{-1} = \overline{N}\underline{\nabla}(\widehat{M}\widehat{N}^{-1})^{-1}. \tag{4.104}$$

Hence with:

$$\underline{M} = MN^{-1}; \ \widehat{\underline{M}} = \widehat{M}\widehat{N}^{-1}; \ \widehat{M} = \widehat{\underline{M}}\widehat{N}, \tag{4.105}$$

We have:

$$(\nabla\widehat{M}^{-1})\widehat{M} = \overline{N}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\widehat{M} = \overline{N}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\,\widehat{\underline{M}}\widehat{N}, \tag{4.106}$$

$$\mathbf{D} = \overline{N}\underline{\nabla}\widehat{N} - \frac{1}{2}[\overline{N}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\widehat{\underline{M}}\widehat{N}] = \overline{N}\underline{\mathbf{D}}\widehat{N}, \tag{4.107}$$

$$\underline{\mathbf{D}} = \underline{\nabla} - \frac{1}{2}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\widehat{\underline{M}}. \tag{4.108}$$

We also recall that with the M which transforms x into $\mathrm{x}' = M\mathrm{x}M^\dagger$ we have $\phi' = M\phi$, and with the X of the invariant space-time in (1.304) we get:

$$\mathrm{x}' = M\mathrm{x}M^\dagger = M(\phi X\phi^\dagger)M^\dagger = (M\phi)X(M\phi)^\dagger = \phi' X\phi'^\dagger. \tag{4.109}$$

Thus the general element X is independent of ϕ, and thus the set of X may still be called the invariant space-time.

4.2 Invariant wave equations

In the invariant derivation there are two terms containing the same generator i: b and a^7, because this i which governs the orientation of space commutes with any element of the Cl_3 algebra (mathematical reason: physical space has an odd number of dimensions). Thus we have no reason to distinguish a gauge transformation acting by multiplication on the right side, from a transformation acting by multiplication on the left side. We must therefore identify these two transformations with each other and use a single gauge potential. We suppose:

$$b = -a^7; \ 0 = a^7 + \mathrm{b}, \tag{4.110}$$

and not $b = a_7$, since the two sides of the space-time hyper-surface have opposite orientation. The sum $a^7 + \mathrm{b}$ uses the incorporation of charges into potentials advocated by one of us [109]. We may also say: when these constants are integrated into potentials, gravitation is completely at equality with gauge forces. We will obtain wave equations in twisted space-time or with gravitation (it is the same mathematical object) via replacing partial derivatives by derivations comprising gauge potentials and connection symbols. Since the wave equations studied in previous chapters may be considered as approximations of the complete equations when the gravitational field is negligible, we must obtain these equations simply by suppressing $a^n, \ n = 0, 1, \ldots, 6$ which are connection terms. For the lepton wave (2.152) and (2.153) become:

$$0 = [\nabla + X + iY + i(\mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})]\widehat{L}^1, \tag{4.111}$$

$$0 = [\widehat{\nabla} + \widehat{X} - i\widehat{Y} - i(2\widehat{\mathrm{b}} + \mathbf{r}\widehat{\mathrm{v}})]R^1, \tag{4.112}$$

$$0 = [\widetilde{\nabla} + X - iY + i(\mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v})\Big]\overline{L}^8, \tag{4.113}$$

$$0 = [\overline{\nabla} + \widehat{X} + i\widehat{Y} - i(2p\widehat{\mathrm{b}} + m_r\widehat{\mathrm{v}})]\widetilde{R}^8. \tag{4.114}$$

Hence that comes from:

$$0 = \Big[\nabla + X + iY - i\frac{\mathrm{b}}{2} + i\Big(\frac{3}{2}\mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v}\Big)\Big]\widehat{L}^1, \tag{4.115}$$

$$0 = \Big[\widehat{\nabla} + \widehat{X} - i\widehat{Y} + i\frac{\widehat{\mathrm{b}}}{2} - i\Big(\frac{5}{2}\widehat{\mathrm{b}} + \mathbf{r}\widehat{\mathrm{v}}\Big)\Big]R^1, \tag{4.116}$$

$$0 = \Big[\widetilde{\nabla} + X - iY + i\frac{\mathrm{b}}{2} + i\Big(\frac{1}{2}\mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v}\Big)\Big]\overline{L}^8, \tag{4.117}$$

$$0 = \Big[\overline{\nabla} + \widehat{X} + i\widehat{Y} - i\frac{\widehat{\mathrm{b}}}{2} - i\Big(\frac{4p-1}{2}\widehat{\mathrm{b}} + m_r\widehat{\mathrm{v}}\Big)\Big]\widetilde{R}^8. \tag{4.118}$$

For the wave equation of quarks we obtain the following in place of (3.130) and for $n = 2, 3, 4$:

$$\begin{aligned}
0 &= \Big[\nabla + X + iY + i\Big(-\frac{\mathrm{b}}{3} + 3\mathrm{w}^3_n - 3\mathrm{h}^{d3}_{L\underline{n+1}} + 3\mathrm{h}^{d3}_{L\,n-1} + m_1\mathrm{v}_q\Big)\Big]\widehat{L}^n,\\
0 &= \Big[\widehat{\nabla} + \widehat{X} - i\widehat{Y} + i\Big(\frac{2\widehat{\mathrm{b}}}{3} + 3\widehat{\mathrm{h}}^{d3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{d3}_{R\,n-1} + m_2\widehat{\mathrm{v}}_q\Big)\Big]R^n,\\
0 &= \Big[\widetilde{\nabla} + X - iY + i\Big(-\frac{\mathrm{b}}{3} - 3\mathrm{w}^3_n - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1} + m_3\mathrm{v}_q\Big)\Big]\overline{L}^{3+n},\\
0 &= \Big[\overline{\nabla} + \widehat{X} + i\widehat{Y} + i\Big(-\frac{4\widehat{\mathrm{b}}}{3} + 3\widehat{\mathrm{h}}^{u3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{u3}_{L\,n-1} + m_4\widehat{\mathrm{v}}_q\Big)\Big]\widetilde{R}^{3+n}.
\end{aligned} \tag{4.119}$$

That is equivalent to:

$$\mathbf{D} := \nabla + X + iY - i\frac{\mathrm{b}}{2}, \tag{4.120}$$

$$0 = \Big[\mathbf{D} + i\Big(\frac{\mathrm{b}}{6} + 3\mathrm{w}^3_n - 3\mathrm{h}^{d3}_{L\underline{n+1}} + 3\mathrm{h}^{d3}_{L\,n-1} + m_1\mathrm{v}_q\Big)\Big]\widehat{L}^n, \tag{4.121}$$

$$0 = \Big[\widehat{\mathbf{D}} + i\Big(\frac{\widehat{\mathrm{b}}}{6} + 3\widehat{\mathrm{h}}^{d3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{d3}_{R\,n-1} + m_2\widehat{\mathrm{v}}_q\Big)\Big]R^n, \tag{4.122}$$

$$0 = \Big[\widetilde{\mathbf{D}} + i\Big(-\frac{5\mathrm{b}}{6} - 3\mathrm{w}^3_n - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1} + m_3\mathrm{v}_q\Big)\Big]\overline{L}^{3+n}, \tag{4.123}$$

$$0 = \Big[\overline{\mathbf{D}} + i\Big(-\frac{5\widehat{\mathrm{b}}}{6} + 3\widehat{\mathrm{h}}^{u3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{u3}_{L\,n-1} + m_4\widehat{\mathrm{v}}_q\Big)\Big]\widetilde{R}^{3+n}. \tag{4.124}$$

We can notice some similarities and differences in comparison with the lepton wave equations: terms coming from inertial potentials are the same, and gravitation works in the same manner on any material wave: **gravitation is universal**. The quark sector has more gauge terms: this comes from the fact that chromodynamics acts only on quarks, as leptons are not sensitive to strong interactions. **The quark sector seems more simple and more regular than the lepton sector**: mass terms all have the same sign whereas signs are different for right or left waves in the lepton case. The gauge terms of the $U(1)$ group are also more simplified by the invariant differential term linked to gravitation: in the quark sector, only two coefficients remain as factors of the chiral potential b: $1/6$ and $-5/6$.

4.2.1 Quantization of charges

The Standard Model employs the renormalization before you can compare between theoretical calculations and experimental values. In the case of weak interactions the success of this process requires the cancellation of "anomalies" related to chirality. This cancellation is obtained only if the sum of all the different charges of particles in each generation is zero. Since these charges come from weak charges we will obtain this suppression of anomalies by imposing, as it is done in the Standard Model, that the sum of all coefficients of the gauge potential b is null:

$$\begin{aligned}&0=\frac{3}{2}-\frac{5}{2}+\frac{1}{2}-\frac{4p-1}{2}+3\Big(\frac{1}{6}+\frac{1}{6}-\frac{5}{6}-\frac{5}{6}\Big)=-2p-4\\&p=-2.\end{aligned}\tag{4.125}$$

We remark that (4.125) should imply, in the absence of quarks, that the sum should be null only if $p=0$, which means only in the case without magnetic monopoles. **Thus the magnetic monopole exists only because quarks exist**. We now have in the lepton sector:

$$0=\Big[\mathbf{D}+i\Big(\frac{3}{2}\mathrm{b}+3\mathrm{w}^3+\mathbf{l}\mathrm{v}\Big)\Big]\widehat{L}^1,\tag{4.126}$$

$$0=\Big[\mathbf{D}+i\Big(\frac{5}{2}\mathrm{b}+\mathbf{r}\mathrm{v}\Big)\Big]\widehat{R}^1,\tag{4.127}$$

$$0=\Big[\widetilde{\mathbf{D}}+i\Big(\frac{1}{2}\mathrm{b}-3\mathrm{w}^3+m_l\mathrm{v}\Big)\Big]\overline{L}^8,\tag{4.128}$$

$$0=\Big[\widetilde{\mathbf{D}}-i\Big(-\frac{9}{2}\mathrm{b}+m_r\mathrm{v}\Big)\Big]\overline{R}^8.\tag{4.129}$$

The sum of coefficients of the chiral potential is:

$$\frac{3}{2}+\frac{5}{2}+\frac{1}{2}-\frac{9}{2}=0.\tag{4.130}$$

Then (4.125) and (4.130) imply the value $e/3$ for the sum of charges of the u and d quarks, and thus only with the choice of the coefficient $-1/3$ in the definition of the projector P_0, which implies that the choice made there is not arbitrary, but constrained by its consequences. We may also remark in (4.121) to (4.124) that the coefficients of the potential b are the same for both left and right waves of each quark. This is also a consequence of the choice $-1/3$ in the projector P_0.

From the interaction between electron and magnetic monopole Dirac found a formula relating electric charges e and magnetic charges g to the Planck constant:

$$\frac{eg}{\hbar c} = \frac{1}{2}. \tag{4.131}$$

We then have:

$$\frac{e}{2} = \frac{e^2 g}{\hbar c} = \alpha g;\ g = \frac{e}{2\alpha}, \tag{4.132}$$

where α is the fine structure constant. This formula, the only one provided to explain the quantification of charges, was obtained via many ways [70][92]. The smartest, from the point of view of quantum physics, was Lochak's way. He used the property that, with an $1/r$ electric potential, a supplementary symmetry exists aside from rotation invariance, which transforms this invariance into $SO(4)$ invariance [1]. The continuity of the magnetic monopole wave under the $SO(4)$ group then allowed Lochak to obtain the formula (4.131) [90] [91] [92].

The various ways [70] of obtaining the Dirac formula, including Lochak's, are all based on the same presupposition: they suppose that an electric central charge acts by a potential like B on a magnetic monopole, or that a magnetic central charge acts by a potential like A on an electric charge. The problem is: potential terms are not pure tools for calculation, they are embedded in the electromagnetic quantum field. And Maxwell's laws indicate clearly that an electric charge creates an electric potential, not a magnetic one, and a magnetic charge creates a magnetic potential. There could not be any interaction between electric and magnetic charge if an electric charge was not able to create a magnetic potential or if a magnetic charge was not able to create an A potential. Moreover only the potentials are present in the wave equations, fields are not. A magnetic monopole acts only by the potential B issued of its kind of gauge invariance. And since it does not have an electric charge, he cannot act via the potential A created by electric charges. The problem is yet solved through the rotation that realizes the Weinberg–Salam angle which rotates in the complex plane (A, Z^0) of (2.216) : $\mathrm{A} + iZ^0 = e^{-i\theta_W}(B + iW)$. It results that a potential $A^0 = e/r$ created by an electric charge e is accompanied by a potential:

$$B^0 = \cos(\theta_W)A^0 = \frac{\sqrt{3}e}{2r} = \frac{e'}{r};\ e' = \frac{\sqrt{3}}{2}e. \tag{4.133}$$

Then we obtain instead the Dirac relation the following formula:

$$\frac{1}{2} = \frac{e'g}{\hbar c};\ \frac{1}{\sqrt{3}} = \frac{eg}{\hbar c};\ \frac{e}{\sqrt{3}} = \frac{e^2 g}{\hbar c} = \alpha g, \tag{4.134}$$

where α is the fine structure constant. By squaring we get:

$$\frac{e^2}{3} = \alpha^2 g^2 = \frac{e^2}{3\hbar c}\hbar c = \frac{\alpha}{3}\hbar c;\ |g| = \sqrt{\frac{\hbar c}{3\alpha}} \approx 79.117771e;\ |e| = \sqrt{\alpha\hbar c}. \tag{4.135}$$

That value of the magnetic monopole charge is notably greater than the value resulting from Dirac's calculation.

4.3 Double link with the Lagrangian density

To be able to obtain the same properties as in flat space-time, it is necessary to replace everywhere the partial derivatives used in the first chapters by new derivations accounting for the covariance, or contravariance, or invariance of the objects on which the partial derivatives act. Similarly the form invariance of the wave equations needs the replacement of the ∇ operator by the invariant $\mathbf{D}$ in (4.120), with:

$$\begin{aligned}
X_0 &= a_0^0 + a_1^1 + a_2^2 + a_3^3;\ Y_0 = a_1^4 + a_2^5 + a_3^6,\\
X_1 &= a_0^1 + a_1^0 + a_3^5 - a_2^6;\ Y_1 = a_0^4 + a_2^3 - a_3^2,\\
X_2 &= a_0^2 + a_2^0 + a_1^6 - a_3^4;\ Y_2 = a_0^5 + a_3^1 - a_1^3,\\
X_3 &= a_0^3 + a_3^0 + a_2^4 - a_1^5;\ Y_3 = a_0^6 + a_1^2 - a_2^1.
\end{aligned} \tag{4.136}$$

For the left wave $L^1 = \phi(1-\sigma_3)/2$ and the right wave $R^1 = \phi(1+\sigma_3)/2$ of the electron, the form-invariant equation (1.149) becomes:

$$0 = \overline{L}^1(\mathbf{D}\widehat{L}^1)\sigma_{21} + \overline{L}^1(\frac{3\mathrm{b}}{2} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\widehat{L}^1, \tag{4.137}$$

$$0 = \overline{R}^1(\mathbf{D}\widehat{R}^1)\sigma_{21} + \overline{R}^1(\frac{5\mathrm{b}}{2} + \mathbf{r}\mathrm{v})\widehat{R}^1, \tag{4.138}$$

These equations read:

$$0 = [-i\eta^{1\dagger}(\nabla + X + iY)\eta^1 + \eta^{1\dagger}(\mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1](\sigma_1 - i\sigma_2), \tag{4.139}$$

$$0 = [i\widehat{\xi}^{1\dagger}(\nabla + X + iY)\widehat{\xi}^1 + \widehat{\xi}^{1\dagger}(2\mathrm{b} + \mathbf{r}\mathrm{v})\widehat{\xi}^1](\sigma_1 + i\sigma_2). \tag{4.140}$$

They are equivalent if $X = 0$ to:

$$0 = -i\eta^{1\dagger}(\nabla\eta^1) + \eta^{1\dagger}(Y + \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1, \tag{4.141}$$

$$0 = i\widehat{\xi}^{1\dagger}(\nabla\widehat{\xi}^1) + \widehat{\xi}^{1\dagger}(-Y + 2\mathrm{b} + \mathbf{r}\mathrm{v})\widehat{\xi}^1. \tag{4.142}$$

These equations obviously come from the wave equations with the form:

$$0 = -i\nabla\eta^1 + (Y + \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1, \tag{4.143}$$

$$0 = i\nabla\widehat{\xi}^1 + (-Y + 2\mathrm{b} + \mathbf{r}\mathrm{v})\widehat{\xi}^1. \tag{4.144}$$

On the contrary, if $X \neq 0$, these equations with the usual form do not come from equations with the invariant form (4.141) and (4.142), via the Lagrange equations: if we indeed consider second sides of the invariant equations as Lagrangian densities, these densities are no longer with real value, but with complex value; this comes from $\mathbf{D}^\dagger \neq \mathbf{D}$. Henceforth with $X = 0$ and with:

$$\mathbf{l}^1 := \sigma^\mu(Y_\mu + \mathrm{b}_\mu + 3\mathrm{w}^3_\mu + \mathbf{l}\mathrm{v}_\mu), \tag{4.145}$$

the left wave equation of the electron reads:

$$0 = (-i\nabla + \mathbf{l}^1)\eta^1. \tag{4.146}$$

Using the adjoint, we obtain:

$$0 = i(\nabla\eta^1)^\dagger + \eta^{1\dagger}\mathbf{l}^1. \tag{4.147}$$

If X is zero we are back in the case of the Lagrangian formalism studied in Chapter 2. Conserving the definition of the Lagrangian density, we have:

$$0 = \mathcal{L}^1_L = -i\eta^{1\dagger}(\nabla\eta^1) + \mathbf{l}^1_\mu \mathrm{D}^{1\mu}_L, \tag{4.148}$$

with a Lagrangian density which is complex, not only real. The real part, which is the Lagrangian density as before, satisfies: with a Lagrangian density which is complex, not only real. The real part, which is the Lagrangian density as before, satisfies:

$$2\Re(\mathcal{L}^1) = -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + 2\mathbf{l}^1_\mu \mathrm{D}^{1\mu}_L. \tag{4.149}$$

The double logical link between wave equations and the real Lagrangian density remains, because the X_μ terms are missing in the real Lagrangian density and the wave equation that is obtained by the use of the Lagrange equations, from the real Lagrangian density, is a complete invariant wave equation. The change from the flat space-time only comes from the imaginary part iX, the left current D^1_L is conserved.

15 similar equations exist, for the 15 other chiral spinors of the Standard Model. From one equation to the other the $X_\mu + iY_\mu$ terms are constant (universality of gravitation), and the η^n, ∇, σ^μ and $\mathrm{D}^{n\mu}_L$ must be replaced by some ξ^n, $\widehat{\nabla}$, $\widehat{\sigma}^\mu$, $\widehat{\mathrm{D}}^{n\mu}_R$ when left waves are replaced by right waves. After that change the double logical link between wave equation and Lagrangian density is conserved, as soon as X_μ are zero. Lagrange's equations allow us, as previously, to go from Lagrangian density to wave equation in ordinary form, while the multiplication on the left side by $\overline{\eta}^n$ or $\overline{\xi}^n$ allows us to obtain the wave equations in the completely invariant form. The Lagrangian mechanism thus remains a purely algebraic process and acts with any gravitational field such that $X_\mu = 0$.

With the seven other left waves it is enough to replace the $\mathbf{l}^1$ vector by the appropriate $\mathbf{l}^n$ vector:

$$\mathbf{l}^8 = Y + \mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v}, \tag{4.150}$$

$$\mathbf{l}^n = Y - \frac{\mathrm{b}}{3} + 3\mathrm{w}^3_n - 3\mathrm{h}^{d3}_{L\underline{n+1}} + 3\mathrm{h}^{d3}_{L\,n-1} + m_1\mathrm{v}_q, \tag{4.151}$$

$$\mathbf{l}^{3+n} = Y - \frac{\mathrm{b}}{3} - 3\mathrm{w}^3_n - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1} + m_3\mathrm{v}_q, \tag{4.152}$$

for $n = 2, 3, 4$. We thus obtain:

$$0 = -i\eta^{n\dagger}(\nabla\eta^n) + i(\nabla\eta^n)^\dagger\eta^n + 2\mathbf{l}^n_\mu \mathrm{D}^{n\mu}_L, \tag{4.153}$$

$$0 = -i\partial_\mu \mathrm{D}^{n\mu}_L, \tag{4.154}$$

$$0 = \nabla\eta^n + i\mathbf{l}^n\eta^n, \; n = 1, 2, \ldots, 8. \tag{4.155}$$

Next, for the right waves we simply replace $\widehat{L}^n$ with R^n, $\overline{L}^{4+n}$ with $\widetilde{R}^{4+n}$ for $n = 1, 2, 3, 4$ and η^n with ξ^n for $n = 1, \ldots, 8$, and more we have a sign change of Y. And we use the parity conjugation $P : M \mapsto \widehat{M}$, which is the main automorphism in Cl_3. We now let, for $n = 2, 3, 4$:

$$\mathbf{r}^1 = -Y + 2\mathrm{b} + \mathbf{r}\mathrm{v}; \; \mathbf{r}^8 = -Y - 4\mathrm{b} + m_r\mathrm{v}, \tag{4.156}$$

$$\mathbf{r}^n = -Y + \frac{2}{3}\mathrm{b} + 3\mathrm{h}^{d3}_{R\underline{n+1}} - 3\mathrm{h}^{d3}_{R\,n-1} + m_2\mathrm{v}_q, \tag{4.157}$$

$$\mathbf{r}^{3+n} = -Y - \frac{4}{3}\mathrm{b} + 3\mathrm{h}^{u3}_{R\underline{n+1}} - 3\mathrm{h}^{u3}_{R\,n-1} + m_4\mathrm{v}_q. \tag{4.158}$$

And we obtain:

$$0 = -i\xi^{n\dagger}(\widehat{\nabla}\xi^n) + i(\widehat{\nabla}\xi^n)^\dagger\xi^n + 2\mathbf{r}^n_\mu \mathrm{D}^{n\mu}_R, \tag{4.159}$$

$$0 = -i\partial_\mu \mathrm{D}^{n\mu}_R, \tag{4.160}$$

$$0 = \widehat{\nabla}\xi^n + i\widehat{\mathbf{r}}^n\xi^n, \; n = 1, \ldots, 8. \tag{4.161}$$

We see a total likeness between the left wave equations and those of right waves. They differ only by gauge terms, and the replacement of σ^μ with $\widehat{\sigma}^\mu$, and by an unexpected sign change for Y. The Lagrangian density relative to L^1 conserves exactly the form used in (2.172). Lagrange's equations thus show how the (2.176) and (2.177) equations are equivalent to the equation (2.178) of L^1. To the very old question: why does a Lagrangian mechanism exist? Does such an "extremal principle" exist, above physical laws? The answer is no, because what happens is very simple: since $\phi = R^1 + L^1$ is invertible,[4] and if the X_μ are zero, the wave equation of L^1 in the usual

4. The existence of the inverse is not general since the wave has value in a ring, which has zero divisors, not in a field. But the invertibility property is satisfied in any point for all calculated solutions of the improved wave equation. This property is strong, because the determinant of $\phi(\mathrm{x})$ is a modulus of complex number; its square is the sum of two squares, which is zero only if each of the two terms (the invariant Ω_1 and the invariant Ω_2) is zero. Moreover, the determinant being a continuous function, if it is nonzero at one point, it is necessarily nonzero in the neighborhood of this point-event.

form (2.180) is equivalent to the invariant form of the wave equation of L^1, where the real Cliffordian part satisfies the equation $0 = \mathcal{L}^1$. The usual form (2.180) of the wave equation is equivalent to the four real numerical equations (2.175) and following, which are exactly the Lagrange equations relative to the four real variables in L^1. This is possible for each spinor wave L^n and R^n. Consequently the Lagrangian mechanism is the automatic way to go from the Lagrangian equation to each wave equation in the usual form. This works without any supplementary justification coming from an integration by parts and the cancellation of terms at the boundary of the domain of integration. It is the simple consequence of the Clifford algebra structure, a purely algebraic property which depends only on the dimension and the signature of the space-time metric, and thus on the space-time geometry. And thus all that is consistent with the starting point of General Relativity, which is the link between gravitation and geometry of space-time. The space-time manifold inherits the Lagrangian mechanism for the electron (but only in the particular case $X = 0$) from the special relativity framework, because each tangent space-time to the space-time manifold, at any point-event, has same dimension and same signature as the space-time of special relativity.

4.4 Energy–momentum and kinetic momentum

Conservation properties of energy–momentum and of kinetic momentum come from the invariance of the Lagrangian density under the additive translations group and under the multiplicative Cl_3^* group generalizing $SU(2)$. Wave equations being form-invariant under both kinds of transformations, we have only a few things to change in comparison with previous chapters. In the case of the leptonic wave the T and V tensors remain defined by (2.229) and (2.230). We thus have:

$$T^\mu_\lambda = \frac{m}{k\mathbf{l}}T^{1\mu}_{L\lambda} + \frac{m}{k\mathbf{r}}T^{1\mu}_{R\lambda} + \frac{m}{km_l}T^{8\mu}_{L\lambda} + \frac{pm}{km_r}T^{8\mu}_{R\lambda}, \tag{4.162}$$

$$V^\mu_\lambda = \frac{m}{k\mathbf{l}}T^{1\mu}_{L\lambda} - \frac{m}{k\mathbf{r}}T^{1\mu}_{R\lambda} + \frac{m}{km_l}T^{8\mu}_{L\lambda} - \frac{pm}{km_r}T^{8\mu}_{R\lambda}. \tag{4.163}$$

The energy–momentum T is hence always the sum of four terms, one for each spinor of the leptonic wave (and the sum of twelve terms in the case of the quarks). It is enough to calculate one term and to transpose the procedure to the others. We calculate the left term of the electron, and the invariance of the Lagrangian density under translations implies:

$$\mathcal{L}^1_L = \Re[-i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1 + i\mathbf{l}^1_\mu\eta^1)], \tag{4.164}$$

$$\begin{aligned} T^{1\mu}_{L\lambda} &= \Re[-i\eta^{1\dagger}\sigma^\mu(\partial_\lambda\eta^1 + i\mathbf{l}^1_\lambda\eta^1)] + \delta^\mu_\lambda\mathcal{L}^1_L \\ &= \Re[-i\eta^{1\dagger}\sigma^\mu(\partial_\lambda\eta^1 + i\mathbf{l}^1_\lambda\eta^1)]. \end{aligned} \tag{4.165}$$

We hence have:

$$-i\eta^{1\dagger}(\nabla\eta^1) = -i\eta^{1\dagger}(-i\mathbf{l}^1\eta^1) = -\eta^{1\dagger}\mathbf{l}^1_\mu\sigma^\mu\eta^1 = -\mathbf{l}^1_\mu \mathrm{D}^{1\mu}_L, \tag{4.166}$$
$$2T^{1\mu}_{L\lambda} = -i\eta^{1\dagger}\sigma^\mu\partial_\lambda\eta^1 + i(\partial_\lambda\eta^{1\dagger})\sigma^\mu\eta^1 + 2\mathbf{l}^1_\lambda \mathrm{D}^{1\mu}_L. \tag{4.167}$$

Next we use the wave equation of η^1, which gives:

$$\nabla\eta^1 = -i\mathbf{l}^1\eta^1;\ \partial_\mu \mathrm{D}^{1\mu}_L = -X_\mu \mathrm{D}^{1\mu}_L, \tag{4.168}$$
$$\begin{aligned}2\partial_\mu T^{1\mu}_{L\lambda} &= -i(\nabla\eta^1)^\dagger\partial_\lambda\eta^1 - i\eta^{1\dagger}\partial_\lambda(\nabla\eta^1)\\ &\quad + i\partial_\lambda(\nabla\eta^1)^\dagger\eta^1 + i(\partial_\lambda\eta^{1\dagger})\nabla\eta^1 + (\partial_\mu\mathbf{l}^1_\lambda)\mathrm{D}^{1\mu}_L\end{aligned} \tag{4.169}$$

We thus get:

$$\partial_\mu T^{1\mu}_{L\lambda} = (\partial_\mu\mathbf{l}^1_\lambda - \partial_\lambda\mathbf{l}^1_\mu)\mathrm{D}^{1\mu}_L, \tag{4.170}$$
$$\partial_\mu T^{1\mu}_{L} = [(\partial_\mu\mathbf{l}^1_\lambda - \partial_\lambda\mathbf{l}^1_\mu)\mathrm{D}^{1\mu}_L]\sigma^\lambda. \tag{4.171}$$

Similarly, we obtain for the other parts of the lepton wave:

$$\partial_\mu T^{1\mu}_{R} = [(\partial_\mu\mathbf{r}^1_\lambda - \partial_\lambda\mathbf{r}^1_\mu)\mathrm{D}^{1\mu}_R]\sigma^\lambda, \tag{4.172}$$
$$\partial_\mu T^{8\mu}_{L} = [(\partial_\mu\mathbf{l}^8_\lambda - \partial_\lambda\mathbf{l}^8_\mu)\mathrm{D}^{8\mu}_L]\sigma^\lambda, \tag{4.173}$$
$$\partial_\mu T^{8\mu}_{R} = [(\partial_\mu\mathbf{r}^8_\lambda - \partial_\lambda\mathbf{r}^8_\mu)\mathrm{D}^{8\mu}_R]\sigma^\lambda. \tag{4.174}$$

Adding the four parts of the lepton wave, we obtain:

$$\begin{aligned}\partial_\mu T^\mu &= \frac{m}{k\mathbf{l}}[(\partial_\mu\mathbf{l}^1_\lambda - \partial_\lambda\mathbf{l}^1_\mu)\mathrm{D}^{1\mu}_L]\sigma^\lambda + \frac{m}{k\mathbf{r}}[(\partial_\mu\mathbf{r}^1_\lambda - \partial_\lambda\mathbf{r}^1_\mu)\mathrm{D}^{1\mu}_R]\sigma^\lambda \\ &\quad + \frac{m}{km_l}[(\partial_\mu\mathbf{l}^8_\lambda - \partial_\lambda\mathbf{l}^8_\mu)\mathrm{D}^{8\mu}_L]\sigma^\lambda + \frac{pm}{km_r}[(\partial_\mu\mathbf{r}^8_\lambda - \partial_\lambda\mathbf{r}^8_\mu)\mathrm{D}^{8\mu}_R]\sigma^\lambda\end{aligned} \tag{4.175}$$

$$\begin{aligned}&= \big[(\partial_\mu Y_\lambda - \partial_\lambda Y_\mu)\big(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L - \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R + \frac{m}{km_l}\mathrm{D}^{8\mu}_L - \frac{pm}{km_r}\mathrm{D}^{8\mu}_R\big)\\ &+ [\partial_\mu(\mathrm{b}_\lambda + 3\mathrm{w}^3_\lambda) - \partial_\lambda(\mathrm{b}_\mu + 3\mathrm{w}^3_\mu)]\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L\\ &+ [\partial_\mu(2\mathrm{b}_\lambda) - \partial_\lambda(2\mathrm{b}_\mu)]\frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R\\ &+ [\partial_\mu(\mathrm{b}_\lambda - 3\mathrm{w}^3_\lambda) - \partial_\lambda(\mathrm{b}_\mu - 3\mathrm{w}^3_\mu)]\frac{m}{km_l}\mathrm{D}^{8\mu}_L\\ &+ [\partial_\mu(-4\mathrm{b}_\lambda) - \partial_\lambda(-4\mathrm{b}_\mu)]\frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R\\ &+ \frac{m}{k}(\partial_\mu\mathrm{v}_\lambda - \partial_\lambda\mathrm{v}_\mu)(\mathrm{D}^{1\mu}_L + \mathrm{D}^{1\mu}_R + \mathrm{D}^{8\mu}_L + \mathrm{D}^{8\mu}_R)\big]\sigma^\lambda.\end{aligned} \tag{4.176}$$

Only one term more appears, compared with what we have obtained in Chapter 2: the first term, with a curvature field:

$$C_{\mu\nu} = \partial_\mu Y_\lambda - \partial_\lambda Y_\mu. \tag{4.177}$$

This field is not linked to the probability current, but to a similar current, distinguishing the role of right and left waves:

$$\mathrm{K}_l := \frac{m}{k\mathbf{l}}\mathrm{D}^1_L - \frac{m}{k\mathbf{r}}\mathrm{D}^1_R + \frac{m}{km_l}\mathrm{D}^8_L - \frac{m}{km_r}\mathrm{D}^8_R. \tag{4.178}$$

We thus get, on the place of (2.257):

$$\partial_\mu T^\mu = \Big[qF^e_{\mu\lambda}(\underline{\mathrm{J}}^\mu + \frac{mp}{km_r}\mathrm{D}^{8\mu}_R) + C_{\mu\lambda}\mathrm{K}^\mu_l \tag{4.179}$$
$$+ iqF^m_{\mu\lambda}(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L - \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R - 2\frac{m}{km_l}\mathrm{D}^{8\mu}_L - p\frac{m}{km_r}\mathrm{D}^{8\mu}_R) + \frac{m}{k}G_{\mu\lambda}\mathrm{J}^\mu_l)\Big]\sigma^\lambda.$$

If the electron is lone, if weak interactions are not at play, and neither C nor G fields, it remains:

$$\partial_\mu T^\mu = qF^e_{\mu\lambda}(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L + \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R)\sigma^\lambda, \tag{4.180}$$

This gives the Lorentz force (1.328) acting on the electric current $\mathrm{j}_e = e(\frac{m}{k\mathbf{l}}\mathrm{D}^1_R + \frac{m}{k\mathbf{r}}\mathrm{D}^1_L)$ of the electron. We truly obtain classical electromagnetism at the limit of low gravitational field.

4.4.1 Probability density

The T^0_0 component of the energy–momentum tensor satisfies:

$$kT^0_0 = \Re\Big[-i\Big(\frac{m}{\mathbf{l}}\eta^{1\dagger}D_0\eta^1 + \frac{m}{\mathbf{r}}\xi^{1\dagger}D_0\xi^1 + \frac{m}{m_l}\eta^{8\dagger}D_0\eta^8 + \frac{m}{m_r}\xi^{8\dagger}D_0\xi^8\Big)\Big]. \tag{4.181}$$

For a solution of the wave equation with energy E of the whole wave, such as:

$$-iD_0\xi^1 = \frac{E}{\hbar c}\xi^1(\mathrm{x});\ -iD_0\xi^8 = \frac{E}{\hbar c}\xi^8(\mathrm{x}), \tag{4.182}$$
$$-iD_0\eta^1 = \frac{E}{\hbar c}\eta^1(\mathrm{x});\ -iD_0\eta^8 = \frac{E}{\hbar c}\eta^8(\mathrm{x}). \tag{4.183}$$

We then have, like in Chapter 2:

$$T^0_0 = \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\xi^1 + \frac{m}{km_l}\eta^{8\dagger}\eta^8 + \frac{m}{km_r}\xi^{8\dagger}\xi^8\Big)$$
$$= \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}D^1_L + \frac{m}{k\mathbf{r}}\mathrm{D}^1_R + \frac{m}{km_l}\mathrm{D}^8_L + \frac{m}{km_r}\mathrm{D}^8_R\Big)^0 = \frac{E}{\hbar c}\underline{\mathrm{J}}^0, \tag{4.184}$$

noting always the weighted currents $\underline{\mathbf{J}}$ with relative weights $\frac{m}{k\mathbf{l}}$, $\frac{m}{k\mathbf{r}}$, $\frac{m}{km_l}$ and $\frac{m}{km_r}$. The reason of the existence of a probability in quantum mechanics remains thus the equivalence between inertial and gravitationnal mass, which implies:

$$E = \iiint dvT^0_0\ ;\quad \iiint \frac{\underline{\mathbf{J}}^0}{\hbar c}dv = 1. \tag{4.185}$$

We will note that, if the X_μ terms are no longer negligible, which make the wave equations go out of the Lagrangian case, neither energy–momentum nor currents remain conservative. These X_μ terms may be at play with the strong gravitational field around black holes. Even in the case of a weak field, the fact that the C field acts on a difference between left and right currents may have been important for the preference of weak interactions for left waves.

4.4.2 Quantization of the kinetic momentum

The approach is exactly that of section 2.5. The invariance under the Cl_3^* group of the energy–momentum tensor V, together with the normalization of the probability current, leads to the quantization of the kinetic momentum with the value $\hbar/2$, in conformity with what we know since 1926.

We saw in Chapter 3 how we may extend this quantization to the quarks. To account for gravitation it is enough to replace d_μ with D_μ in (4.120), in the case where the X_μ are negligible. The energy–momentum tensors and the quantization of the kinetic momentum do not change with respect to Chapter 3.

4.4.3 Equivalence principle

About the extremal principle which until now guided the set of mechanics and optics laws we explained in 2.3.4 how that principle is not above physical laws, how its comes out of the Clifford structure of the Cl_3 algebra. We see now how the equivalence principle also, is a consequence of the properties of the wave with spin 1/2.

In the previous discussion of the Pauli exclusion principle (1.12) we saw that the mass term of the wave equation is variable, depending on the number of particles at play. And the energy of emitted or absorbed photons is exactly the difference between the energy levels of the considered system of particles (atoms, molecule, ...) before or after the emission or the absorption. Denoting by m_b the mass of the system considered before and m_a the mass of the transformed system, we necessarily consider the Lagrangian density as a difference. We have two other reasons for this difference: the potential b is $-a^7$ and the differential term is also easily expressed as a difference. We recall that the Lagrangian density, in the lepton case, is the sum of four terms, and in the quark case the sum of twelve terms. For the left wave of the electron, with (4.120) and (4.126) and supposing the cancellation of the X_μ, we have:

$$\begin{aligned} 0 = \mathcal{L}_L^1 &= -i\eta^{1\dagger}\sigma^\mu\Big[\partial_\mu\eta^1 + i\Big(\frac{3}{2}\mathrm{b}_\mu + 3\mathrm{w}_\mu^3 + \frac{a_\mu^7}{2} + \frac{1}{2}\mathrm{Y}_\mu + \mathbf{l}\mathrm{v}_\mu\Big)\Big]\eta^1 \\ &= -\mathcal{L}_i^1 + \mathcal{L}_g^1 \end{aligned} \tag{4.186}$$

$$\mathcal{L}_i^1 = \frac{i}{2}\eta^{1\dagger}\nabla\eta^1 - \eta^{1\dagger}\Big(\frac{3}{2}\mathrm{b} + 3\mathrm{w}^3 - m_a\mathrm{v}\Big)\eta^1, \tag{4.187}$$

$$\mathcal{L}_g^1 = \frac{i}{2}(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}\Big(\frac{1}{2}a_\mu^7 + \frac{1}{2}Y_\mu\sigma^\mu + m_b\mathrm{v}\Big)\eta_1, \tag{4.188}$$

$$\mathbf{l} = m_b - m_a;\ \mathcal{L}_i^1 = \mathcal{L}_g^1. \tag{4.189}$$

We may notice that we did not group together the two terms containing $\mathrm{b} = -a^7$ on only one side. The fact that this potential belongs to each one of the two parts comes from the following property: the multiplication by i works in the same manner whether from the right side or from the left. Thus the potential is naturally present both in the $\mathcal{L}_i^1$ part that allows us to get the forces acting on the electron, and in the $\mathcal{L}_g^1$ part that holds the a^n giving the Christoffel symbols. To these two parts of the Lagrangian density are attached two tensors of energy–momentum, equal from their definition:

$$T_{Li\lambda}^{1\mu} = \frac{i}{2}\eta^{1\dagger}\sigma^\mu(\partial_\lambda\eta^1) - \eta^{1\dagger}\sigma^\mu\Big(\frac{3}{2}\mathrm{b}_\lambda + 3\mathrm{w}_\lambda^3 - m_a\mathrm{v}_\lambda\Big)\eta^1, \tag{4.190}$$

$$T_{Lg\lambda}^{1\mu} = \frac{i}{2}(\partial_\lambda\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}\sigma^\mu\Big(\frac{1}{2}a_\lambda^7 + \frac{1}{2}Y_\lambda + m_b\mathrm{v}_\lambda\Big)\eta^1, \tag{4.191}$$

$$0 = -T_{Li\lambda}^{1\mu} + T_{Lg\lambda}^{1\mu};\ T_{Li\lambda}^{1\mu} = T_{Lg\lambda}^{1\mu} \tag{4.192}$$

Since this may be generalized for all parts of the Lagrangian density we obtain in a very general manner an equality between the inertial tensor T_i and the gravitational tensor T_g: this is the equivalence principle.

4.4.4 Mössbauer effect

A photon may be absorbed or emitted without any recoil of the nucleus, exactly as if it was emitted or absorbed by the whole crystal containing the atom and its nucleus, in spite of the fact that the frequency of the photon corresponds to a difference between energy levels of one nucleus. The understanding of this effect induces us to admit that not only is the energy–momentum tensor a difference, but also the proper mass at play in the definition of the quantum wave:

$$m_a = m_T;\ m_b = m_{ST}, \tag{4.193}$$

where m_T is the total mass (eventually that of the whole laboratory if it is necessary) and m_{ST} the mass of the subtotal, which is the previous sum minus that of the system that is emitting or receiving. It is well known that a frame of reference usable in quantum mechanics must be neither too massive, if we want to sidestep gravitation, nor too light because it is then impossible to neglect phenomena of recoil due to the momentum of emitted or absorbed photons[7]. It is always possible to take as the total mass that of the reference frame in which the measurements are made. Since only a difference is useful there is no problem with the immensity of masses, even if we must include stars and galaxies.

If we study a particular electron, the mass at play is the proper mass of this single electron. If the electron belongs to a system of two electrons, the mass used in the double equality $E = mc^2 = h\nu$ is the mass of the system. It is the same for the protons or the neutrons in a nucleus, or even for a crystal. This explains why the properties of a nucleus are different according to whether the nucleus is surrounded by an electron cloud or not, particularly for the probability of radioactive decay [74].

We then see that it is possible to analyze all particles and systems of relativistic quantum mechanics with physical waves propagating in a space-time whose properties are determined by these physical waves. It remains for physics indeed to go from theory to practice.

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We thus see that it is possible to analyze all particles and systems of relativistic quantum mechanics with physical waves propagating in a space-time whose properties are determined by these physical waves. It remains for physics indeed to go from theory to practice.

4.5 The whole space-time manifold

4.5.1 Local and global structure of space-time

By writing $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu$, quantum mechanics actually includes the set of x, which is the space-time manifold, in Cl_3. Moreover, the space-time length is given by $\det(\mathrm{x}) = \mathrm{x}\overline{\mathrm{x}} = \mathrm{x} \cdot \mathrm{x}$. This equality implies that multiplication is the single operation to be considered. Thus we must also consider the physics point-of-view about length and use [42] a unit length l_u:

$$\mathbf{x} := \frac{\mathrm{x}}{l_u};\ \mathbf{x} \in Cl_3^*. \tag{4.194}$$

The first difference with classical geometry is that the origin of the measure of time and space is at $\mathbf{x} = 1$, not 0. Second difference, Cl_3 is the Lie algebra of the Cl_3^* multiplicative group. This means that the neighborhood of any point O is isomorphic to Cl_3. This set is a linear space which contains two subsets: Cl_3^*, which is the set of $\mathbf{x}$ satisfying $\det(\mathbf{x}) \neq 0$, and the light cone, which is the set of $\mathbf{x}$ satisfying $\det(\mathbf{x}) = 0$. Third, these conditions $\neq 0$ and $= 0$ exclude themselves, therefore the light cone is included in each (local) Lie algebra, not in the (global) Lie group Cl_3^*. Consequently the space-time manifold is a manifold with boundary. Fourth, the only link between each

Lie algebra and the whole Lie group is the exponential function, which we calculate as follows:

$$\mathbf{x} = a + b\mathbf{u};\ \mathbf{u} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3;\ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1,$$

$$\mathbf{x}^n = \frac{1}{2}\Big[(a+b)^n(1+\mathbf{u}) + (a-b)^n(1-\mathbf{u})\Big] \tag{4.195}$$

$$\exp(\mathbf{x}) = \sum_{n=0}^{\infty}\frac{\mathbf{x}^n}{n!} = \frac{1}{2}\Big[e^{a+b}(1+\mathbf{u}) + e^{a-b}(1-\mathbf{u})\Big]$$

$$= e^a[\cosh(b) + \sinh(b)\mathbf{u}]. \tag{4.196}$$

Moreover we have:

$$\det[\exp(\mathbf{x})] = \exp[\mathrm{tr}(\mathbf{x})] = e^{2a}. \tag{4.197}$$

Hence with $\exp(\mathbf{x}) = A + B\mathbf{u} = A + B(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)$ we get:

$$e^{2a} = \det[\exp(\mathbf{x})] = (A + B\mathbf{u})(A - B\mathbf{u}) = A^2 - B^2, \tag{4.198}$$

which implies that the light cone ($A^2 = B^2$) is the boundary of the space-time manifold and that nothing exists outside this boundary, since $e^{2a} > 0$.[5] From this sign we may see the purely theoretical and local character of the Schwarzschild solution in general relativity. Moreover we obtain:

$$e^a = \sqrt{A^2 - B^2};\ \cosh(b) + \sinh(b)\mathbf{u} = \frac{A + B\mathbf{u}}{\sqrt{A^2 - B^2}}.$$

$$a = \ln(\sqrt{A^2 - B^2}) = \frac{1}{2}[\ln(A+B) + \ln(A-B)], \tag{4.199}$$

$$b = \sinh^{-1}\Big[\frac{B}{\sqrt{A^2 - B^2}}\Big] = \frac{1}{2}[\ln(A+B) - \ln(A-B)],$$

$$a + b = \ln(A+B);\ A + B = e^{a+b}. \tag{4.200}$$

But the confusion between global space-time and local space-time is easily done, since the difference between the neutral respective elements 1 and 0 of the Lie group and the Lie algebra was extraordinarily small in comparison with the immensity of the whole space-time. Nevertheless this distinction is necessary, as we see now.

4.5.2 The EPR paradox

Two photons are emitted at the point-event O. We suppose (only simplifying the calculation) that they are emitted in two orthogonal directions, σ_1 and σ_2, of the tangent space-time at O. They are absorbed at the same time $y > 0$, also to simplify the calculation. The photon emitted in the direction σ_1 is absorbed at the point-event:

$$\mathbf{x}_1 = a_1 + b_1\mathbf{u}_1 = (a+y) + (bx^1 + y)\sigma_1 + b(x^2\sigma_2 + x^3\sigma_3), \tag{4.201}$$

5. That implies that the idea of an interior place into black holes is nonsense.

and we have:

$$\begin{aligned} &a_1 = a + y;\ \mathbf{u}_1 = x_1^1\sigma_1 + x_1^2\sigma_2 + x_1^3\sigma_3;\ (x_1^1)^2 + (x_1^2)^2 + (x_1^3)^2 = 1, \\ &(x^1 + y/b)^2 + (x^2)^2 + (x^3)^2 = 1 + 2x^1 y/b + (y/b)^2, \\ &b_1 = b\sqrt{1 + 2x^1 y/b + (y/b)^2};\ \mathbf{u}_1 = \frac{(x^1 + y/b)\sigma_1 + x^2\sigma_2 + x^3\sigma_3}{\sqrt{1 + 2x^1 y/b + (y/b)^2}}. \end{aligned} \tag{4.202}$$

The photon emitted in the direction σ_2 is absorbed at the point-event:

$$\mathbf{x}_2 = a_2 + b_2\mathbf{u}_2 = (a + y) + bx^1\sigma_1 + (bx^2 + y)\sigma_2 + bx^3\sigma_3. \tag{4.203}$$

And we also have:

$$\begin{aligned} &a_2 = a + y;\ \mathbf{u}_2 = x_2^1\sigma_1 + x_2^2\sigma_2 + x_2^3\sigma_3;\ (x_2^1)^2 + (x_2^2)^2 + (x_2^3)^2 = 1, \\ &(x^1)^2 + (x^2 + y/b)^2 + (x^3)^2 = 1 + 2x^2 y/b + (y/b)^2, \\ &b_2 = b\sqrt{1 + 2x^2 y/b + (y/b)^2};\ \mathbf{u}_2 = \frac{x^1\sigma_1 + (x^2 + y/b)\sigma_2 + x^3\sigma_3}{\sqrt{1 + 2x^2 y/b + (y/b)^2}}. \end{aligned} \tag{4.204}$$

On the space-time manifold, the point event O is at $\mathbf{X} = O/l_u = A + B\mathbf{u} = \exp(\mathrm{x})$ while the photon emitted in the direction σ_1 is absorbed at the point-event $\mathbf{X}_1 = M/l_u = \exp(\mathrm{x}_1)$. The photon emitted in the direction σ_2 is absorbed at the point-event $\mathbf{X}_2 = P/l_u = \exp(\mathrm{x}_2)$. The position of the point event P, seen from 1, is:

$$\mathbf{x}_2^0 = [\exp(\mathbf{x})]^{-1/2}\exp(\mathbf{x}_2)[\exp(\mathbf{x})]^{-1/2}. \tag{4.205}$$

The position of the point event P, now seen from M, is:

$$\mathbf{x}_2^1 = [\exp(\mathbf{x}_1)]^{1/2}[\exp(\mathbf{x})]^{-1/2}\exp(\mathbf{x}_2)[\exp(\mathbf{x})]^{-1/2}[\exp(\mathbf{x}_1)]^{1/2}. \tag{4.206}$$

The position of the point event M, seen from 1, is:

$$\mathbf{x}_1^0 = [\exp(\mathbf{x})]^{-1/2}\exp(\mathbf{x}_1)[\exp(\mathbf{x})]^{-1/2}. \tag{4.207}$$

The position of the point event M, seen from P, is:

$$\mathbf{x}_1^2 = [\exp(\mathbf{x}_2)]^{1/2}[\exp(\mathbf{x})]^{-1/2}\exp(\mathbf{x}_1)[\exp(\mathbf{x})]^{-1/2}\exp[(\mathbf{x}_2)]^{1/2}. \tag{4.208}$$

And we have, since the determinant of a product is the product of the determinants:

$$\begin{aligned} \det(\mathbf{x}_2^1) &= e^{a+y}e^{-a}e^{2(a+y)}e^{-a}e^{a+y} = e^{2(a+y+y)}, \\ \det(\mathbf{x}_1^2) &= e^{a+y}e^{-a}e^{2(a+y)}e^{-a}e^{a+y} = e^{2(a+y+y)}. \end{aligned} \tag{4.209}$$

Therefore at each point-event, when a photon is absorbed at the local time $a + y$, **each observer** sees the absorption of his photon as preceding, with the same length of time y, the arrival of the photon for the other observer: the absorption of the other photon is in the future of each observer, not just at the moment of the photon arrival. This strange result seems very similar to the fact that each observer sees any length shorter for a moving object: an observer in the moving object also sees the other observer as moving, thus with shorter length. The paradox is that a measurement made on either of the particles apparently collapses the state of the entire entangled system and does so *instantaneously*, before any information about the measurement result could have been communicated to the other particle [4]. Our previous calculation shows a key to this strange and very effective paradox: the instantaneous character of the measurement is simply false, an illusion. The "collapse" only results from the supposition that this situation may be described in the space-time of restricted relativity, not really in the space-time manifold. The wave train of the other photon reaches the other observer in the future of each observer of his own photon.

Look out! We don't deny quantum entanglement, a well establish experimental phenomenon. It says to us that the quantum wave is not only a local phenomenon: the normalization of the wave, always possible from the moment the Lagrangian formalism acts, is fully non-local. We may say that the paradox is only in the interpretation of this situation by a non fully relativistic theory, whereas physics must account for the fact that each "fixed" observer is journeying in time on the space-time manifold, even if he does not travel in space. The understanding of the true geometry of space-time simply requires the use of the space-time manifold itself, not merely the use of the flat tangent space-time at any particular point-event.

Einstein, Podolsky and Rosen said [69]: "From this follows that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. For if both of them had simultaneous reality — and thus definite values — these values would enter into the complete description, according to the condition of completeness."

Experiments with the polarization of two photons simultaneously emitted (very fine indeed and meriting the Nobel prize for Aspect) can neither prove (1) nor (2) because the absorption of these photons cannot be simultaneous at the points where each absorption is effective. The quantum wave used here, with value in $\text{End}(Cl_3)$, and not just in $\mathbb{C}$, is enough to prove that (1) was true in 1935 (the description of the quantum wave is not complete), independently of what we now think about (2) (the limitation coming from Heisenberg inequalities is absolute). More generally no contradiction can exist between General Relativity and quantum mechanics. **Any apparent contradiction results from bad approximations of relativistic laws**.

4.5.3 The arrow of time, the expansion of the universe

Any point of the space-time manifold is at a position:

$$X = l_u \exp(a + b\mathbf{u}) = l_u(A + B\mathbf{u});\ A = e^a \cosh(b);\ B = e^a \sinh(b). \quad (4.210)$$

Thus the time position $l_u e^a \cosh(b)$ is the product of positive real numbers: time is an oriented quantity, **the arrow of time has a geometric root**. The time variable goes from 0 to $+\infty$.

Now we consider a photon received at this position X, coming from a distant galaxy, for instance along the σ_1 direction. It was emitted at the position:

$$l_u \exp[a - y + (bx^1 - y)\sigma_1 + b(x^2\sigma_2 + x^3\sigma_3)] = l_u \exp(a_1 + b_1\mathbf{u}_1), \quad (4.211)$$

with [6]

$$\begin{aligned} a_1 &= a - y;\ \mathbf{u}_1 = x_1^1\sigma_1 + x_1^2\sigma_2 + x_1^3\sigma_3;\ (x_1^1)^2 + (x_1^2)^2 + (x_1^3)^2 = 1, \\ (x^1 - y/b)^2 &+ (x^2)^2 + (x^3)^2 = 1 - 2x^1 y/b + (y/b)^2, \\ b_1 &= b\sqrt{1 - 2x^1 y/b + (y/b)^2};\ \mathbf{u}_1 = \frac{(x^1 - y/b)\sigma_1 + x^2\sigma_2 + x^3\sigma_3}{\sqrt{1 - 2x^1 y/b + (y/b)^2}}. \end{aligned} \quad (4.212)$$

The photon was emitted at:

$$\mathrm{x}_e = l_u\, e^{a_1}[\cosh(b_1) + \sinh(b_1)\mathbf{u}_1]. \quad (4.213)$$

At this point-event the local time was $t_e = l_u e^{a_1} \cosh(b_1) \approx l_u e^{a_1 + b_1}/2$. The same photon is absorbed at the point-event X, then at the local time $t_a = l_u e^a \cosh(b) \approx l_u e^{a+b}/2$. The only constant object of this geometry is the Lie algebra: each local tangent space, in each point of a manifold that is also a Lie group, is isomorphic to the Lie algebra of the group. We will then suppose that:

$$d(a_1 + b_1) = d(a + b);\ \frac{dt_e}{t_e} = \frac{dt_a}{t_a} \quad (4.214)$$

And we have:

$$\frac{\nu_a}{\nu_e} = \frac{dt_e}{dt_a}. \quad (4.215)$$

In the first approximation, $b_1 \approx b$, we obtain:

$$\begin{aligned} \frac{1}{1+z} = \frac{\nu_a}{\nu_e} = \frac{dt_e}{dt_a} &= \frac{d[l_u e^{a_1} \cosh(b_1)]}{d[l_u e^a \cosh(b)]} \\ &\approx \frac{l_u\, da\, e^{a-y} \cosh(b)}{l_u\, da\, e^a \cosh(b)} = \frac{1}{e^y} \approx \frac{1}{1+y}. \end{aligned} \quad (4.216)$$

6. Since we now look at past, $a_1 < a$.

This means that the redshift due to the expansion of the universe, previously interpreted as a Doppler effect, is a direct effect of the geometry of space-time, and the z parameter, defined as $(\nu_e - \nu_a)/\nu_a$, is almost equal to y. But this is true only as a crude approximation, or as a false velocity. When y is small this redshift seems proportional to y. The Hubble parameter (73.3 ± 1.4km/s/Mpc) gives for the distance 1 Mpc the value $z = 0.0002443$, thus giving $R = l_u\, e^{a+b}/2 \approx 6.3 \times 10^{25}$m.

Using the purely geometric condition (4.214), independent on the material density of space-time, which results from the Lie algebra as the only fixed object, independent from the space-time position on the manifold, we may calculate more precisely the ratio dt_e/dt_a in the case where y is small. We have:

$$\frac{d[l_u e^{a_1}\cosh(b_1)]}{d[l_u e^{a}\cosh(b)]} = \frac{d[e^{a-y}\cosh(b_1)]}{d[e^{a}\cosh(b)]} = \frac{e^{-y}\cosh(b_1)}{\cosh(b)} = \frac{1}{f(y)} \quad (4.217)$$

$$f(y) := e^{y}\frac{\cosh(b)}{\cosh(b_1)} \approx f(0) + yf'(0) + y^2\frac{f''(0)}{2} + ... \quad (4.218)$$

We use:

$$b_1 := bg(y) = \sqrt{b^2 - 2x^1 by + y^2};\ g(y) = \sqrt{1 - 2\frac{x^1}{b}y + (\frac{y}{b})^2},$$
$$g(y) \approx 1 - \frac{x^1}{b}y + \frac{1-(x^1)^2}{2b^2}y^2 + \frac{x^1[1-(x^1)^2]}{2b^3}y^3 + \dots . \quad (4.219)$$

And we get:

$$f(y) \approx e^{y}\frac{e^{b}}{e^{b_1}} = e^{a(y)} \quad (4.220)$$
$$a(y) = y + b - b_1 \approx (1+x^1)y - \frac{1-(x^1)^2}{2b}y^2 - \frac{x^1[1-(x^1)^2]}{2b^2}y^3 + \dots,$$
$$f'(y) \approx a'(y)e^{a(y)} = (1+x^1)[1 - \frac{1-x^1}{b}y - \frac{3x^1(1-x^1)}{2b^2}y^2 + \dots]e^{a(y)}. \quad (4.221)$$

From the values of the Hubble parameter we obtain $a + b \approx 142$. We only know that $a > b > 0$. The ratio a/b is unknown. If our position in the manifold is anywhere, for instance is $(a+b)/a \approx a/b$, we could have $a \approx 88$ and $b \approx 54$. This should give a ratio B/A very close to 1. We now look at the acceleration or deceleration of the expansion.

4.5.4 Beginning of the acceleration

Defining h such that $h(y) := f(y)/y$ the redshift seems accelerated if and only if h is increasing (because y grows in reversed time), hence if $h'(y) > 0$.

We obtain:

$$\begin{aligned} y^2h'(y) &= yf'(y) - f(y) \approx [ya'(y) - 1]e^{a(y)} \\ &= [-1 + (1 + x^1)y - \frac{1-(x^1)^2}{b}y^2 - \frac{3x^1[1-(x^1)^2]}{2b^2}y^3 + \dots]e^{a(y)}. \end{aligned} \tag{4.222}$$

For instance if $b = 40$ and $x^1 = 0.6$ we have:

$$y^2h'(y) \approx [-1 + 1.6y - 0.016y^2 - 0.00036y^3 + \dots]e^{a(y)} \tag{4.223}$$

Hence in that case, $h'(y) < 0$ if and only if:

$$y < y_0, \; y_0 \approx 0.63. \tag{4.224}$$

Moreover, the sign of the coefficient of y^3 indicates a sign change for a large y, but the method of calculation used here cannot give a precise calculation of this new change of sign. Hence the acceleration of the expansion seems to begin near y_0, with possible differences depending on the direction of observation with regard to the whole space-time manifold. And the expansion seems to decelerate for very large redshifts.

What we obtain here is completely different from the cosmology developed since Einstein's works on relativistic gravitation. We think that it is much more satisfactory than the ΛCDM model, previously promoted by most cosmologists. First we do not need to suppose a homogeneous distribution of matter, never observed at small (stars and galaxies) nor very large scale (billions of light years). Second, there is no need for a gigantic amount of unknown black matter to obtain the uniformity of each tangent space-time.[7] Third we do not need a cosmological constant to obtain the acceleration of the expansion. The smallness of that cosmological constant was always a big problem. Introduced by Einstein as the only constant that may be added to the conservative Ricci tensor, this constant Λ is used to justify the huge amount of unknown black energy necessary to explain the recent acceleration of the expansion. That Λ is nowadays in contradiction with the new experimental data on the expansion rate [71]. Furthermore we have obtained two results that Einstein should have very much liked: first, a space-time which, as a whole manifold, is invariant (for a given cosmological time t, space is not an hypersphere S^3 with growing radius, it is a $\mathbb{R}^3$ unlimited and globally invariant). Only the elements in the whole manifold are variable. Second, the geometry of the cosmos, globally, is independent of matter, integrating to the geometry both inertia and gravitation. Last but not least advantage: whatever the smallness of the probability of existence of an intelligent species on a planet able to protect life during billions of years, in an unlimited space-time including an infinite number of stars, the existence of such a planet somewhere becomes possible.

7. The movement of stars in galaxies and the movement of galaxies in galaxy clusters is another question. Indeed the absence of necessity for dark matter and dark energy does not prove their non-existence. The simple name "black hole" is enough to prove that some objects may exist and be unable to directly send light.

Chapter 5

Why?

Thousands of years ago, physics began with the questions of our ancestors: why the regular return of the Sun, why the phases of the Moon? Why the wind, the rain, why the rainbow after a thunderstorm? When physics started to progress more and more quickly, understanding the motion of the planets, linking together all these “whys” into a theory of gravitation, of electricity and of light, gave rise to many other whys.

Take the instance of light. Physicists began by understanding some of its **properties**, like the fact that it originated in the Sun before coming to the retina of our eyes, and not the reverse, as was long believed. Going a little further they understood some **laws** governing these properties, for instance the law of refraction when light passes from one medium to another with a different refractive index. These laws are described with mathematical tools, such as sines of angles of refraction. Next these laws depend on **principles** which are, in a sense, laws governing laws. Concerning light, Pierre Fermat understood that the law of refraction comes from the following simple physical principle: light automatically chooses the path of shortest duration. In the previous chapters we did not only study **properties** of quantum waves (for the electron, they are functions of space-time with values in Cl_3^*), we have studied **laws**: partial differential equations for the wave, also the orthonormalization of the electron wave and the existence of a probability density. We also obtained the laws of motion for a charged fluid. We have even explained how these laws come from **principles**: the wave equations arise through the Lagrangian mechanism from an extremal principle resulting from a Lagrangian density. The orthonormalization of the wave comes from the principle of equivalence between inertial mass and gravitational mass. What is newest here: to end up with a causal loop, by the deduction of these principles from the properties of matter waves themselves. We completely dissected how the extremal principle was, for quantum waves, a consequence of properties of the quantum wave as a function with value in a particular Lie group included into its Lie algebra.

These properties are linked to the structure of space-time, the fact that time is 1-dimensional and space is 3-dimensional. We have also explained how the equivalence principle comes from the properties of all densities of energy–momentum.

What we have continued here is proper to the building up of science: to search for laws from properties of physical objects, and not beyond these objects. The causal loop that we just described is hence a successful realization of the scientific process for this field of science that studies matter, called physics. And as a loop closes on itself, that rounds out the process, even if a loop may indeed be extended, doubled or integrated into other similar loops.

The double equality $E = mc^2 = h\nu$ is an essential component to these parts of our causal loop which were progressively improved over time. The equality $E = mc^2$ comes from the electrodynamics of matter in motion, obtained by Albert Einstein in 1905. Straight after sending his article for publication he accounted for this: if all matter has an electromagnetic origin, then $E = mc^2$. This equality is extremely well established by experiment, but as a consequence the if-then nature of the statement is somewhat forgotten. Physicists no longer asked: why does all matter must have an electromagnetic origin? Here we have carried our knowledge on this point a little further: all fundamental objects of physics are fermions obeying the same laws; thus saying that all matter has the same origin is equivalent to this: any mass-energy in the universe comes from fermions. Hence if any boson seems to have a proper mass, it is composed of fermions that possess that mass.

After his discovery of the electrodynamic laws of matter in motion, Einstein reconsidered gravitation, aiming to transform the gravitation law into a field theory. He started from the identity between inertial mass and gravitational mass. This identity implies that the gravitational field is an acceleration field, not a force field, unlike electromagnetic field, which acts through the Lorentz force. He thus understood that gravitation was a completely geometric phenomenon, linked to the structure itself of space-time, its curvature. But then why was gravitation this way? Why this identity between inertial mass and gravitational mass? We also advanced a little further here by showing that Lagrangian densities of fermions may naturally be interpreted as a null difference between gravitational terms and inertial terms. We have even further advanced the next question, "Why is this the case?" The different terms of the wave equations are the only possible ones, able to exist in a manner compatible with the form invariance of those wave equations. Furthermore these wave equations are form-invariant due to the properties of the structure of waves themselves. This causal loop goes through the Lagrangian mechanism that we dissected and which contains no metaphysical principle. Everything arises from the algebraic structure automatically associated to the geometric structure of space-time. This structure is itself linked to the quantum wave, which has value on the Cl_3^*

manifold that includes both $SU(2)$ and space-time.

The second equality of the pair, $E = h\nu$, was first obtained by Max Planck in his studies of the laws of the radiation issuing from a material heated at a high temperature. The equality contains a constant which is rightfully named after its discoverer. This was extended twice, first by Albert Einstein who introduced the wave-particle dualism for light as early as 1905, then by Louis de Broglie who a century ago extended this dualism to all matter[55]. Between then and now, multiple discoveries of this quantum world took place. They are nowadays described by the Standard Model of quantum physics, basis of the present text.

5.1 Einstein was right

Despite his discoveries, both of wave-particle dualism and of gravitation as geometry of space-time, Einstein ended up being isolated from the scientific community: A quantum physics was developed in a very different way from the physics of gravitation. Einstein continued to search for a unifying synthesis aiming to encompass electromagnetism and quantum physics along with the physics of gravitation. He was chasing after what was so characteristic of his beautiful theory of gravitation: a completely relativistic physics, with a field following a partial differential equation, deterministic, and able to yield the laws of motion of field sources.

That is exactly what comprises the set of partial differential equations that we obtained for the fermionic waves: they are completely deterministic and they allow us to derive the laws of motion of these sources of gauge fields, which fermions actually are. Einstein was thus right to attempt such a synthesis, as our previous chapters show it to be workable.

Why was Einstein misunderstood? The first reason was the novelty of his understanding of the space and time nature, particularly his rejection of an absolute time. Schrödinger, who himself perfectly understood the relative time of Einstein's gravitation, first found a nonrelativistic wave equation for de Broglie's wave. This wave equation, plus Pauli's exclusion principle, resulted in a wave which does not have direct physical reality. This wave does not propagate in space-time but in an absolute time and in configuration space. Einstein himself was the first to understand and use the geometric properties of configuration spaces.

From 1917 until his death Einstein made many attempts to reconcile gravitation, electromagnetism and quantum physics. He tried for instance a manifold with torsion, in a manner very close to our calculations. But his starting point could not be the chiral right and left waves issuing from the discovery, just after his death, of maximal parity violation in weak interactions. Moreover non local properties of quantum waves (he first conceived of their existence) were not yet understood. Einstein was indeed not truly happy with his first theory of gravitation. He had serious doubts about the longevity of that theory, notably because the left side of his equation, which is purely geometric, is much stronger than the right side, which was not well defined. And the geometric side may or may not include a cosmological constant.

Einstein was not fortunate with the cosmos. He thought the whole space-time as necessarily invariant and steady-state, while astronomers discovered the immensity of space that appeared to be expanding. There also Einstein's view was right, since the space-time manifold, invariant in its totality, is perfectly compatible with the redshift of distant galaxies, and moreover with a recent acceleration of that redshift, an acceleration measured today by astronomers, and which can be explained as a purely global geometric effect (see 4.7.3).

5.1.1 "There is no alternative"

Certainly there is no way to escape the double equality $E = h\nu = mc^2$, and it should be stupid to claim the opposite. Certainly, it is impossible to avoid Heisenberg's inequalities since the kinetic momentum of all fermions is quantized. We agree all the more since we know where it comes from. But for many other things an alternative exists, and the proof is precisely our work: we have worked out laws of nonrelativistic quantum mechanics and obtained better results. An opinion very common in physics believes that the Dirac equation is "a kind of Schrödinger equation." This is false! The first who explained that was Louis de Broglie in his second book about the Dirac equation [60], when he began to question the quantum theory issued from his discovery of wave mechanics. The electromagnetic interaction is part of a gauge interaction described by a noncommutative $U(1) \times SU(2)$ gauge group: it is therefore impossible to dissociate this interaction from other electroweak interactions.

Quantum field theory was developed from a wave with only one phase. But the electron always has two phases. Certainly the second phase (the Yvon-Takabayasi angle), which appears in magnetic phenomena and in electroweak interactions, is very difficult to see in many situations: Only then does quantum electrodynamics work perfectly, even with its most surprising predictions.

In physics the universe is what it is. We have changed the title of this work from "Developing a Theory of Everything" in the first edition to "Developing *the* Theory of Everything"; it is another way of saying that there is no alternative. Time must be ordered, thus time is necessarily unidimensional. Space is 3-dimensional, thus the algebra of space is Cl_3. The rotational invariance of the laws of mechanics (there is no privileged direction in space) was replaced in quantum mechanics by the invariance under the $SU(2)$ Lie group. That leads physicists to consider (since nearly a century ago!) space (and later space-time) as included in the part of Cl_3 containing $SU(2)$, which can only be the whole multiplicative group Cl_3^*: the space-time of general relativity is a four-dimensional manifold, and Cl_3^* is just big enough (Whitney's theorem), with its eight dimensions, to include any four-dimensional manifold. The pseudo-metric of space-time comes from the determinant, and thus the signature of space-time is $+, -, -, -$. Cl_3^* is a Lie group, each Lie group is associated to a unique Lie algebra, and the Lie algebra of Cl_3^* is Cl_3: there is no alternative. A Lie group is a manifold and the tangent space at any point of the Lie group is isomorphic to the tangent space at the neutral element of the group, which is the Lie algebra: no alternative!

5.1.2 After this work

When we state why the Weinberg–Salam angle θ_W exactly satisfies the equality $\sin(\theta_W) = 1/2$, or why the charge of the d quark is exactly a third of the charge of the electron, the precision is certainly above eleven significant digits since it is exact. The advantages of a correct understanding concern not only the precision of predictions: Understanding why two colored quarks exist in each generation, why leptons are insensitive to strong interactions, why a Lagrangian mechanism exists, how the electromagnetic field is directly linked to the energy–momentum of the quantum wave – these are definitive advancements. The same predictive power must be expected for any proposed alternative. If such an attempt obtains, for instance, better predictive power from a Lagrangian with both an independent fermion part and a boson part, the existence of the Lagrangian itself would need to be accounted for. Indeed we gave an explanation for the existence of Lagrange equations in the fermion wave case: Thus any attempt to build a ToE will be asked to do the same. We arrived at a simple origin of light polarization: Any further ToE attempt will be asked for its ability to derive the link between the electromagnetic field and energy–momentum of the fermion wave. We also understood the geometric reason for the time arrow, and the redshift of light coming from very distant stars, including the recent acceleration of this redshift, all from the very structure of the space-time manifold: any further ToE attempt will be asked for such a simple explanation of this "expansion".

The most important and novel understanding brought by the present work is the quantization of the kinetic momenta of the electron, neutrino, proton and neutron with the same $\hbar/2$ value. From this quantization of the kinetic momentum come both Heisenberg's inequalities and the quantization of the electric charge. Any attempt to build a ToE will also be expected to obtain this quantization and with the true value, fully established experimentally.

5.2 de Broglie was right

Einstein and de Broglie were right, because the quantum wave exists and it is fundamentally relativistic. With the electron wave, whether in the case of low or of high velocity, the D_R and D_L currents, respectively formed by the right and left parts of the quantum wave, are on the light cone. These currents indeed have a sum which is the probability current, linked to the invariance of the electric gauge. That J current is the only one visible in the version of quantum mechanics at the basis of QFT. But the D_R and D_L currents also have a difference which is the second K current. This current is as important as the first current.

It is also linked to the chiral gauge, and thus linked to magnetism and to weak interactions. The dependence of tensor densities on the wave chirality concerns not only currents; it is also extended to densities of energy–momentum and of kinetic momentum, and thus to the electromagnetic field. By accounting for this dependence, we obtained in 2.5 and 3.7 the quantization of the kinetic momentum. Since the quantum wave is fundamentally relativistic, the replacement of the Dirac equation by the Pauli equation is untenable. And the integration of the electron into Hamiltonian physics does not work out well. This is why difficulties arise in all QFT calculations, such as the infinite quantities that must be gotten around, renormalization and anomalies that must be tamed. And all this turns out to be impossible for gravitation, and justly, because gravitation is completely relativistic.

In his time Einstein could not elaborate a better theory, as he could not guess what was discovered after his death. And that discoveries later allowed the building of the Standard Model. The most important discovery of the second half of the previous century (according to Lochak) was the violation of parity in weak interactions. The different roles of right and left waves is important as well for the Standard Model which carefully accounts for this difference, as for general relativity. This is due to the orientation of space, placed again at the center of physics. This orientation of "space" is a convenient shorthand; it is actually the orientation of space-time and the arrow of time which are conserved, and therefore the orientation of space.

De Broglie was very much aware of the defects of quantum theory that stems from nonrelativistic wave equations. He thoroughly studied relativistic Dirac theory twice, and published two books [56, 60]. In the meantime he also used the Dirac wave as the starting point for his theory of light [57, 58]. These works were not understood; they were too far ahead of his time. And de Broglie could know neither the existence of quarks nor their chromodynamics. The present work was mainly developed by two persons who met in the seminar organized by de Broglie himself in the "Fondation Louis de Broglie" created in the "Académie des Sciences" in Paris, to continue his scientific work. The director of this private foundation was G. Lochak who discovered the leptonic magnetic monopole [90, 91]. His monopole wave equation was the starting point of our work.

De Broglie bequeathed us his very deep knowledge of the various domains of classical and quantum physics. He also advised us to exercise our freedom in criticizing the fashions of a system where everyone takes too many unsound habits for granted, also in physics.

5.3 Bohr was also (partly) right

During the early development of quantum mechanics, the universality of Heisenberg's inequalities and the implied limits of our knowledge were not at all evident. Einstein, who had encountered nothing similar to his theory of gravitation, entered into a great debate with Bohr, but Bohr's arguments prevailed, and justly because it was he who made best use of the universality of Einstein's relativistic physics. The generalization of relativistic invariance to the Cl_3^* group again takes up this universality and allows us to obtain, as a consequence of fermion wave properties, the quantization of the kinetic momentum with a value of $\hbar/2$ (see 2.5 and 3.7).

In his second book on the Dirac theory (see [60] 2.6), from the quantization of the kinetic momentum, de Broglie deduced the precise form of the uncertainty relations for two quantities A and B canonically conjugated (like x and p_x): $\sigma_A \cdot \sigma_B \geqslant \hbar/2$. Before this book, de Broglie had studied Heisenberg's inequalities in an earlier work written in 1950-1951 but edited only thirty years later [61] thanks to Lochak. The derivation of the quantization of kinetic momentum allows us to obtain the fourth uncertainty relation in the form proved by de Broglie: $\sigma_t \cdot \sigma_E \geqslant \hbar/2$, where σ_t is the uncertainty in the temporal coordinate of an event and σ_E is the uncertainty of the energy at work in this event.

5.4 Intrinsic or statistically random?

Einstein was the first to understand Brownian motion as the random movement of a particle constantly colliding with molecules, and obviously had nothing against probabilities. What he questioned was the intrinsic randomness attributed to the quantum wave.

When physicists can afford to suppress the "small components" of the Dirac wave, just because electron velocity is low and because two of the complex components then have a small modulus in comparison with the other two, they not only completely destroy the relativistic invariance of the wave equation, but also return to the Hamiltonian pattern inhabited by the Schrödinger and Pauli equations. In that case time plays a different role in comparison with space, and the equation takes the Hamiltonian form of the Schrödinger equation $i\hbar\partial_t\psi = H(\psi)$. The wave equation obtained by this suppression is seldom presented as merely a Pauli equation. In the end, everyone believes that the Dirac equation is "a kind of Schrödinger equation." That shrinks the Dirac wave to the general probabilistic schema of nonrelativistic quantum theory: the only things we may calculate are probabilities. And since there is no physical reality beyond what is measurable, the research of other ideas – the understanding of what actually happens – are considered useless, and even harmful.

Have we in the present work moved outside this probabilistic schema? At first it may seem not, since the quantum wave with spin $1/2$ is always associated with a probability: by dividing the local energy density by the total energy we naturally arrive at a density whose summation over the whole space takes the value one. It is thus a probability density. In the case of several indistinguishable fermions we also obtain a measure which gives the number of these fermions. But certainly the answer is yes, we have gone outside the purely probabilistic schema because the wave does not give *only* probabilities. In the wave we can find the origin of all the so-called "quantum numbers" such as the baryon number, lepton number, weak hypercharge and so on. We are now able to understand the value of each elementary charge. We are also able to derive the Lorentz law of motion for the density of electric charge and for other currents. We are able to understand how electrons essentially differ from neutrinos and quarks. Thus in the wave with spin $1/2$ there are some elements of physical reality rather than mere probabilities. The quantum wave is not reduced to mere amplitude and phase.

Certainly a large part of quantum mechanics is reduced to the study of the couple amplitude and phase, that part in which **the** phase (thought to be single), which always means the angle of the electric gauge associated by Noether's theorem to the probability current, is dominant and overrides all other currents.

Even in this case – meaning in the realm of QFT, which is indeed vast because most fermions have an electric charge, but which does not encompass all physics – the quantum wave follows an equation with partial derivatives just as deterministic and relativistic as Einstein's gravitational equations. It is the extended relativistic invariance which gives the quantization of kinetic momentum and this explains Heisenberg's inequalities, which means the limitation of our knowledge about position–energy–momentum (in space-time). Moreover, among principles that may be consequently derived from properties of quantum waves is the exclusion principle expounded by Pauli. This principle states that the occupation number of an electron wave can only be either 0 or 1. The "probability of presence" concept is thus non-verifiable: the experimental validation of any probabilistic law is necessarily made via the convergence of statistical frequencies onto the probability law. And statistics is impossible with only one object. Statistics based on n electrons also includes n electron waves. The probability that in a domain of space D the "electron–particle" is present in D and nowhere else is always calculable but not statistically verifiable from the wave of this lone electron. The situation is absolutely different for a photon because an electromagnetic wave may accommodate myriads of photons. The spatial density of these photons on the wave is proportional to the magnitude of the electric field; this is statistically verifiable.

The concept of probability has two kinds of justifications, *a priori* or *a posteriori*. The concept of probability *a priori*, theorized by Kolmogorov's axioms, defines probability as an additive measure on a family of sets such that the probability of the whole is 1. It is this kind of probability that we encounter in the present book. Very different, the concept of *a posteriori* probability is based on randomness, which means an intervention of causes of which we know nothing: For instance a uranium-238 atom exists since its creation, billions of years ago, but suddenly a nucleus of helium is ejected and the nucleus transforms into a thorium-234 nucleus. We do not know what happens, how this process begins, or how it evolves. We only know the end of the process, when the two nuclei separate. Statistics that physicists carry out on an enormous number of uranium-238 atoms allows them to establish probabilistic laws. The probability of decay is constant in time. Its half-life, the duration such that only half of the uranium remains, is 4.4688 billion years. Can this probability be linked to the wave of the protons, neutrons and electrons of this kind of atom? We do not know. And QFT does not know either, despite attempts to causally relate the temporal probability of decay to the nonzero spatial probability of presence beyond the potential barrier. QFT must obviously justify how a spatial probability can yield a temporal probability. This has been discussed by many physicists [108].

Astonishing implications of these probabilities, such as entanglement, Bell's inequalities and Aspect's experiment, are always interpreted with the idea of a quantum wave following Hamiltonian relativistic dynamics. That is in the framework of a theory which replaces the necessary definition of mathematical objects by a set of postulates supposedly universal. But these postulates are not universal, because waves of different fermions of the Standard Model have left and right waves. And never was it proved that these left and right waves would obey the postulates of quantum theory. Certainly what we introduce here, using a wave function of space-time with value in $\text{End}(Cl_3)$, can be used to support the mathematical foundations of second quantization. But problems of Hamiltonian dynamics would remain entirely.

Why are we entitled to doubt the possibility of a Hamiltonian relativistic dynamics for the spinor wave of the electron? The problem comes from time, which is revertible in Hamiltonian dynamics and which is not revertible for the invariance under Cl_3^*. Since QFT admits the universal validity of CPT symmetry along with the violation of P-symmetry and of CP-symmetry, this is equivalent to the violation of T-symmetry. It is thus logical to think that the dynamics of fermions in the Standard Model cannot be Hamiltonian. Moreover, we have explained in Chapter 1 how the first Hamiltonian form of the Dirac equation is both nonrelativistic and nonequivalent to the second form of the Dirac equation, which is relativistic. It is thus false to consider two nonequivalent wave equations as describing the same particle!

We now have a much stronger reason, knowing that ordinary time is expressed through the exponential function which applies the Lie algebra Cl_3 on the Cl_3^* group, particularly $\mathbb{R}$ onto $\mathbb{R}^{+*}$: time is oriented by the structure of the whole space-time. The measurements of space and time used in the interpretation of Aspect's experiment and of Bell's inequalities are interpreted in the frame of the space-time of special relativity. All these measurements should account for the inclusion of space and time in the true space-time manifold (would say Bohr). We explain in 4.5.2 how events which seem to coincide, as observed in a particular frame, in fact may be in the future of each final observer.

In the previous discussion of the decay of a uranium atom, and with the emission of a photon as well, it is essential to understand the all-or-nothing character of quantum phenomena that is the main feature of quantization. Surely, for any quantum phenomenon the kinetic momentum comes in *integer* multiples of $\hbar/2$. Yet making use of ergodic properties, it is perfectly possible to link temporal probabilities of seemingly random events, to a continuous distribution of spatial probabilities. This is of course something to elaborate on, to account for probabilities used by Einstein for his physics of the black body.

5.5 The nearly forgotten Dirac equation

That is from two parts of Einstein's scientific work that our work is built. A first part of his work was interpreted as the replacement of the invariance group of Newtonian physics laws by another invariance group called the Poincaré group, which is 10-dimensional. That group contains space-time translations and Lorentz transformations. Relativistic quantum mechanics replaced the restricted Lorentz group by the $SL(2,\mathbb{C})$ group. And this group, 6-dimensional, was extended by us to the $GL(2,\mathbb{C}) = Cl_3^*$ group. This extension is justified by the spin 1/2 of all the fundamental objects of quantum physics: fermions. These are named after Fermi who worked out the statistics indicated by Pauli's exclusion principle. Regarding the exclusion principle, we also went further since it is now linked to the additivity of the fermion mass-energy, through the orthonormalization of the fermion waves. This additivity is not an exact law and is only due to the extremely tiny masses of particles, which makes the nonlinearity of gravitation negligible.

Why was quantum mechanics essentially built from the Schrödinger equation, when only a few months after this first discovery the Dirac equation was also available? De Broglie explained how after the 1927 Solvay Conference, having been appointed professor at the Sorbonne, and aware of the obstacles to his idea of the wave guiding the particle, he began teaching the works of the other quantum physicists, not his own theory. He returned to the ideas of his youth only many years later. Yet a long time prior to this change, he was already interested in the Dirac equation because the equation was relativistic, like his initial concept of a wave associated to the movement of any material particle [55]. But by the time he changed his mind about the explanatory power of quantum mechanics, the Dirac equation was already considered outdated, seldom taught. This area of quantum physics was slowly disappearing from the physics curricula of universities.

Among the reasons for this decline is the great difference introduced by the spin $1/2$, between what is called a physical quantity in classical physics and what is called a physical quantity in quantum mechanics. In classical mechanics the quantities are numbers, for instance a temperature of 302 Kelvin. Other quantities are components of vectors like velocity or force, where the components are real numbers. Others, slightly more difficult to understand, are tensors such as the inertia tensor or the electromagnetic tensor. Still, all these quantities are real numbers. The wave equation found by Schrödinger, that of Pauli and more so the Dirac equation, all introduced a deep change: quantum states have no direct link with the quantities of classical physics. To each classical quantity is associated, everywhere in quantum mechanics, an operator acting on the linear space of states; and it is the proper value of this operator which gives the real number of classical physics. According to this line of reasoning, the ultimate physical reality of the electromagnetic field comes from creation and annihilation operators which add or subtract a unity to the number of photons present in the electromagnetic wave.

On the contrary, we explain in Appendix C how all quantum numbers of solutions in the hydrogen case are obtained with only the condition of the normalization of the wave. This does not contradict quantum mechanics, because we may show adequate operators such that each solution is automatically a proper vector of these operators. Nevertheless, the general theory of Hermitian operators is simply useless.

De Broglie remarked early on [56] that with the Dirac wave equation it was still different: certainly the idea of classical numbers as proper values of operators is conserved, but it is not these quantities that have true relativistic variance; it is the tensor densities which transform following the law established for relativistic physics. Several arguments were brought up against the Dirac wave, one of which is that the matrices used in the wave equation are only defined up to an arbitrary matrix factor. It is thus difficult to consider the wave as having any element of physical reality, and the wave appears to be merely a tool for calculations, nonphysical. We resolved this difficulty by defining the Dirac matrices from the Pauli matrices in a unique manner, and by defining the Pauli matrices from the canonical basis of $GL(2, \mathbb{C})$. They are the same for two observers in relative motion, and thus the wave with spin $1/2$ may have the status of physical reality, in the same way as for instance an electric field.

Some other difficulties are only historical; they were resolved when the study of the tensors in the theory was improved: Hestenes introduced new methods of calculation, much more efficient. They allowed him to prove that the densities of electric charge and of electric current nearly follow the Lorentz force law [1]. Only one other theory derived the laws of motion from field equations: General Relativity. This strongly impressed de Broglie when Einstein managed to prove the derivation (de Broglie needed this nonlinearity to link his particle to its wave). And we may say that the improved wave equation is even stronger than Einstein's gravitation equations, which gives the law of the movement only for a singularity of the field, while **the wave equation of the electron gives the Lorentz force for any solution of the wave equation**.

Another reason for theoretical physics to discountenance the spin 1/2 wave is that the Dirac equation is only a linear equation. Thus its worth is much less than that of general relativity, which is nonlinear. It is also only a theory for a single electron, and in an exterior potential which is nonsense in a field theory. Yet this criticism applies to the Dirac equation as formulated in 1928, not to our work: the improved equation obtained in Chapter 1 and its subsequent generalizations in the next chapters are nonlinear, both in mass terms and in gauge terms where potentials are dependent on the wave. Algebraic identities suppress the effect of each chiral current on itself. This eliminates the self-effect, without destroying the effect. It is seen only if we consider the entire wave altogether and not merely the different pieces. Furthermore the more useful form of the fermion wave equation is its invariant form, which is not at all linear. The wave is a well-defined function of space-time (not configuration space) with value onto a set of operators acting on themselves. This is the only possible justification for second quantization.

With Lorentz' electron-particle model, the mass-energy is the sum over all space of the energy density of the electromagnetic field. If the electron is exactly a point, this energy is infinite. It the electron is extended, the repulsive force due to the electric field of the charge must necessarily be compensated by other unknown forces. This led us to separately consider the exterior field created by the other charges. In the previous chapters the energy density of the electron is no longer the energy density of the electric field; it is the temporal component of the energy–momentum density linked to the Lagrangian density of the electron. It was previously known that the energy density linked to the electromagnetic field $W = \frac{1}{2}(E^2+H^2)$ was problematic: the mass of this energy depends on how energy is defined from the mechanical point of view [8]. We see in Chapter 1 that it is the electromagnetic field itself which is the energy–momentum tensor. The mass-energy of the electron is exactly the sum of the energy density of the electron wave. That tensor density of energy–momentum in quantum physics is linked by Noether's theorem to the invariance of the Lagrangian density under space-

1. Our improved equation gives the Lorentz force exactly, see 1.10.

time translations. Since we only needed the fermion part of the Lagrangian density of the Standard Model, and since wave equations of bosons were derived from those of fermions by the recursion on wave equations, we conclude that we need only the fermion part of the Lagrangian density.

This part of the Lagrangian density is derived from the wave equations, and the wave equations are derived from the Lagrangian density. This suggests that gauge fields have no proper energy. Phenomena where gauge fields seem to own a proper energy are phenomena where it is always possible to reallocate this energy to the fermions that give or receive this energy.

This leads to a first prediction: **As strong as a magnetic field may be around a star (including neutron stars and black holes) or a galaxy, this field, despite its bipolar and multi-polar structure, has absolutely zero effect on the geometry of the gravitational field which can remain perfectly spherical**.

5.6 Why those wave equations and not others?

The global wave equation for all fermions of the first generation separates into 16 equations corresponding to 16 spinors, eight left and eight right, making up the wave. This splitting is what allows us to distinguish each of these objects from others. But the separation is only partial: wave equations are all constructed in the same manner, with a differential part (the only part of the equation that totally distinguishes parts of the wave), a mass term and a gauge term. The mass term and the gauge term contain space-time vectors that are themselves functions of left and right spinors. This dependence of the gauge and mass terms on spinors reveals that the wave equations are highly nonlinear. We again look at the three parts of our wave equation: The whole equation is constrained by the invariance under the Cl_3^* group that governs the whole of the Standard Model and gravitation. We consider the homothety ratio in terms of the invdim (see 1.7) that we use to distinguish contravariance from covariance.

1. Spinors have a invdim 1. Partial derivatives acting on them give terms with invdim -1.

2. Thus the other terms must have the same -1 invdim. And they contain a multiplication by the spinor wave function, with $+1$ invdim. Thus the other factors must together bring a -2 invdim. Therefore a single spinor factor is inconvenient, and it is impossible to have quadratic terms with regard to spinors; only cubic terms are possible. These cubic terms bring a supplementary invdim of $+2$, not -2, and thus we have a difference of $+4$ to compensate.

3. This may be done in only two ways, either bringing a -4 invdim or bringing two -2 invdims. The first possibility is what the gauge term brings, where the lone charge (actually g_1, g_2 and g_3 constants) brings a -4 invdim.

4. The second possibility, $-4 = -2 - 2$ is actually what m/ρ brings to the mass term because m brings a -2 invdim and $1/\rho$ also brings a -2 invdim. All in all, there are two, and only two, possible terms in addition to the differential term because there are exactly two possibilities to express 2 as an ordered sum of integers: $0 + 2$ and $1 + 1$. Moreover this justifies the difference between mass and charge, which certainly give both potentials in $1/r$. They are different only from the point of view of the extended invariance.

Why do we not obtain derivatives of higher order? First the wave equation Dirac envisioned must have similar partial derivatives for time and space coordinates: that is required by special relativity. And it is necessary to only have first-order derivatives so as to obtain a conservative probability density. First-order derivatives are also the terms of the first approximation. In the study of manifolds, by distinguishing the variation of points and the variation of a mobile basis, it is possible to avoid the writing of differential terms with higher degrees. It is similar to the systems of first-order equations that are obtained in mechanics (where second-order derivatives are natural) when velocities are used as auxiliary variables. Hence the use of only first-order derivatives does not restrict the generality of our wave equations. Furthermore the recursion takes place in the wave equations. Second-order derivatives allow us the definition of the gauge bosons. And similarly, terms of higher order are included in the relations linking gauge fields to potentials, and currents to gauge fields. Lastly the system is closed for another reason: the null invdim of all gauge fields. Consequently, by multiplying operators acting on these gauge fields we still obtain such an operator. The quantum wave gives two connections on the space-time manifold: a connection linked to the currents of the quantum wave (inertia) and another linked to its invariance group (geometry)(see chapter 4). The identity of these connections is exactly the equivalence principle between inertia and gravitation. And the reason for this identity is: the space-time manifold is a hypersurface of dimension four itself included in the 8-dimensional Cl_3^* Lie group, which is also a manifold. Since the connection acts by the wave equation, it concerns directly the Lagrangian density, it also concerns the energy–momentum. The proper mass of quantum wave equations is thus a difference between inertial and gravitational mass, though not noticeable because Avogadro's number is too high. This mass is not defined by the particle alone; it is proper to the particle interacting with a material system great enough to allow measurements.

5.7 Treasure hunt

In the vast "treasure hunt" that is scientific research, it is very easy to let oneself be rerouted by coincidences, the main reason for believing we are following the right track when actually the track is already lost. And there have been several coincidences throughout the history of physics. For instance the wave equation of the electron was discovered at just the same time as the spin $1/2$, and at the time, there was yet no direct relation. Another coincidence concerns mathematical tools: the Clifford algebra of 3-dimensional space is also the algebra of complex 2×2 matrices (but only as algebras on the real field!). This contributes to justify the habit of quantum mechanics to only use functions with value in the complex field. A third coincidence: the Lie group of rotations in 3-dimensional space has the same Lie algebra as the group of the 2×2 unitary complex matrices, denoted as $SU(2)$ (but the groups themselves are different!). That gave an additional justification to the sole use of functions with complex values, and the primacy of unitary transformations that conserve the probability. These coincidences, which are accidental from the mathematical point of view, are reasons that led physicists to consider the theory of operators on quantum states as a tool that is all at once necessary, sufficient, and impregnable – but all the while: that was a false track!

The human spirit always tries to reduce novelties down to what is already known: it's our nature. There still are some people today who persist in restraining the study of electromagnetism to an absolute time, which is only time as perceived by their internal biological clock. In the same line the concept of spinors was systematically reinterpreted, distorted, in order to reduce the new concept to something previously known: tensor physics. Therefore the novelty of the situation was not received, like the infinite kinds of tensor densities that may be constructed from spinor waves. Similarly, only tensor densities which are invariant under the electric gauge were considered, as if the electron could not also be affected by weak interactions.

5.8 Physics and mathematics

Mathematics and physics are closely related sciences, both work on geometric and numeric data and integrate those into an orderly body of knowledge. But these two sciences, both extensively developed, are nowadays so vast that it is impossible for a young scientist to master the whole of physics or the whole of mathematics and even more to master both domains fully.

Galileo pointed out that the language of physics was mathematics, and since then, the connection between physics and mathematics grew ever closer. But misunderstandings got significantly worse since the beginning of quantum physics.

Those misunderstandings, as is often the case, can in part be ascribed to both parties. The evolution of mathematics towards greater abstraction and generality is natural but ill-adapted to physicists' needs: the theory of linear spaces is naturally made with a general n-dimensional space, but what is interesting for physics is simply the 3-dimensional space and the $3+1$-dimensional space-time. Only with three dimensions does a cross-product exist, which is so useful in physics. Properties specific to a 3-dimensional space (scalar and cross products, curl, and also the electromagnetic field as a field of energy–momentum densities) do not interest most mathematicians. The particular properties of the algebra of the 2×2 Pauli matrices, like the fact that the co-matrices are complex numbers, act only if $n = 2$. Hence the use of general n-dimensional linear spaces, so natural in mathematics, is in practice detrimental for physics.

Physicists are also partly responsible for these misunderstandings. It is impossible to take advantage of the strength of mathematical results when their constraints are disregarded: for instance, the necessary definition of mathematical objects for which the reasoning can be applied, and the importance of theorems of existence and of impossibility. [2]So it is the Pauli algebra, with its modest 8 dimensions on $\mathbb{R}$, which is important, while anything else seems to lead the Dirac theory to resort to the use of $M_4(\mathbb{C})$ or at most the subalgebras $Cl_{1,3}$ and $Cl_{3,1}$. It is also necessary to distinguish similar yet nevertheless different concepts such as a Lie group and a Lie algebra.

Quantum theory was built on the basis of mathematics of the century that preceded its beginnings. For instance the concept of function was purely computational, and the questions of limit, of topology and even mere concern for the set of departure and the set of values, or the usefulness of Clifford algebras: all this was misunderstood by physicists who were old enough, despite their remarkable youth, to know only the mathematics of the nineteenth century. Afterward physicists were even misled by the power of the distribution theory, which made more effective the use of Fourier and Laplace transformations. Thus when QFT was introduced, most people were quite sure that the needed theorems would be necessarily soon demonstrated. But the expected proofs never came: physicists had too much confidence in the power of mathematics.

5.9 Understanding and predicting

Among the ideas that we now understand better, several were known for a long time. The existence of Planck's constant in physics is more than a century old, and the quantization of kinetic momentum arises from there. Here we also explain how this constant ratio between energy and

2. In his second book on the Dirac theory ([60] Chapter II, section 2) de Broglie clearly explained the following impossibility: if we consider three operators m_x, m_y and m_z satisfying anti-commutation relations of 3-dimensional rotations, all possible proper values of m_z are $-j, -j+1, \dots, j-1, j$ where $j(j+1)$ is proper value of $m_x^2 + m_y^2 + m_z^2$, and all possible values of j are $0, 1/2, 1, 3/2, 2, 5/2 \dots$. But if m_x, m_y and m_z are angular momentum operators ($m_x = i(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$ and so on), the only possible values of j are $0, 1, 2, 3, \dots$. Consequently the operators of the Dirac theory, with values $j = 1/2, 3/2, 5/2, \dots$ are not angular momentum operators! (Thus we named these operators as "kinetic momentum" operators).

frequency comes from the invariance of quantum laws under Cl_3^*. This invariance emerges from the mathematical structure of the fermionic waves. And the mathematical structure of the fermion waves in turn comes from this invariance. [3]This is manifested in the equivalence between two forms of the wave equations, due to the invertible character of each value of the wave function. Furthermore this equivalence gives rise to the Lagrangian mechanism. It is an extremal principle: Noether's theorem associates the translational invariance of wave equations to the existence of conservative densities of energy–momentum. This theorem also associates a conservative kinetic momentum to the invariance under Cl_3^*. We consequently obtain the quantization of kinetic momentum with the expected value $\hbar/2$. And since the kinetic momentum is quantized, the Planck constant appears fixed. The orthonormalization of the wave and the resulting quantization of the kinetic momentum justify the use of kinetic momentum operators that give the different states of electrons in atoms.

Pauli's exclusion principle has also been known for nearly a century. We link this principle to the necessary orthonormalization of states in the case of the electron in an atom. This orthonormalization is itself linked to the additivity of the mass-energy, and hence to the properties of gravitational sources, since masses of microphysical objects are very small in comparison with the masses necessary to reveal the nonlinear character of gravitation. So the energies of the various electrons in an atom are additive. This additivity of the energy is itself related to the additivity of the gauge potentials. It is enough to justify that the gravity around a star is proportional to the total mass of the star, the sum of the masses of all its components. This is only a linear approximation, legitimate in the case of a low gravitational field.

We also linked the equivalence principle to this weak-field approximation, through Lagrangian densities that may be written as the difference of an inertial part and a gravitational part. Noether's theorem hence gives two equal energy–momentum tensors, which thus have the same temporal component. By integrating over space we thus obtain the equality between inertial mass and gravitational mass. This was the starting point of Einstein's general relativity.

The inclusion of the space-time manifold into the Lie group Cl_3^* is enough to explain homogeneity and isotropy of our space-time: in a Lie group, the vicinity of any element of the group is similar to the vicinity of the neutral element. This is largely established from the experimental point of view, where the cosmic background radiation is still today very close to homogeneity (one part in 10^5). Moreover the geometry of the physical space naturally tends to infinity (like that of $\mathbb{R}^3$).

3. This looping causality is the only reason explaining why metaphysics is unnecessary.

5.10 Falsifiability

Any scientific theory must be falsifiable: it should be possible to prove that the theory is false. Conversely it is impossible to prove *definitively* that the theory is true. Hence this can only humble the authors of this work. Will the best theory someday completely do without the Dirac equation in understanding the properties of electrons, neutrinos, quarks and other "particles"? Even if the Dirac equation gives all known results for electrons in atoms, we proved that it is possible to get the same results from another point of departure [34].

From Fermat's principle through Lagrangian mechanics and up to the Standard Model, the whole of physical theory has been developed from an extremal principle. Is this principle fundamental in physics? The answer that we gave in 2.3.4 is clearly: No! We detailed how the algebraic structure of Cl_3^* gives the double logical link between the wave equation and Lagrangian density. Thus the extremal principle is not fundamental, though it is very efficient because the invariance of a Lagrangian density gives, through Noether's theorem, conservative quantities. And that which conserves, which is stable, is much easier to understand than that which is furtive, unstable, changing, unpredictable. Furthermore the extremal principle is the reason for the unity of all matter-energy, because each fermion contributes to this energy–momentum tensor, whose temporal component gives the energy of matter. In addition, the electromagnetic field itself is this energy–momentum. Moreover only fermions contribute; photons only transport the energy–momentum between two fermions. Nevertheless in a regime dominated by gravitation there is no longer a Lagrangian formalism (see 4.3) and thus no laws of conservation.

The great debate of quantum physics was around the question: what *is* the quantum object? A particle? (A very small object, even an infinitely small point?) A wave? A wave *and* a particle, as de Broglie thought? Any phenomenon in quantum physics that is adequately described with particles can also be adequately described with waves, and conversely. And it is also possible to describe the same phenomenon with objects that are both waves and particles [100]. Here we began from the Dirac wave. And we even claimed: an electron is an orthonormalized quantum wave. Is the electron also a point object? Nothing forbids this! It is possible that our orthonormalized quantum waves may include singularities, or even must include singularities. To ascertain this it will be necessary to solve the wave equations, study the solutions carefully and understand in particular the emission and absorption of photons and of the other bosons. Note also that the solutions calculated, among which are our solutions for an electron in a hydrogen atom, may be qualified as "solitons": the appearance of the radial functions means that these solutions are similar to solitons. These solutions are completely stable, definitely. Their "loneliness" is simply less visible because these waves are not separated in ordinary space, but orthogonal for a scalar product concerning the Cl_3^* manifold where our space-time is only a 4-dimensional sub-manifold.

The great debate began at a time when only one elementary particle was really known: the electron. Moreover the wave of the electron is not elementary but double, made of a left and a right wave: it is a complicated object. Other particles known in the early years of nuclear physics – protons and neutrons – are no longer considered elementary since they are made of three quarks, hence of six waves, three left and three right. They are still much more complicated objects.

We are very far from a complete exploration of all the consequences of the extension of the invariance group. Consider once more a common phenomenon like the transition of an electron of the solar photosphere from one energy state to another, followed by the travel of a photon to our eye and its absorption by an electron in our retina. We interpret this chain of events by attributing an electromagnetic wave to the photon. This allows us to neglect both the emitting electron and the receiving electron during the transport. But the duration of this transport, from the point of view of the photon, is exactly null. We might as well describe the event as follows: the electron wave in the photosphere produces a energy–momentum tensor which is also the electromagnetic field that propagates towards an electron in our retina, giving a direct interaction of two fermion waves, that of an electron in the Sun and that of an electron in the retina.

A great number of things remain to be understood. To give the wave equations of the quarks is only a first step. It will also be necessary to know how to calculate magnetic moments of the proton and the neutron, and to study what new equations would allow nuclear physicists to understand. And it may happen that many other consequences exist which we did not even think of.

What we already obtained fully justifies the extension of the invariance group from $SL(2,\mathbb{C})$ into Cl_3^*. **Without this extension there is neither quantization of the kinetic momentum, nor the double presence of the chiral invariance both in the gauge symmetries and in the geometry of gravitation. Without this extension physics can understand neither the reason for the existence of the neutrino–monopole, nor the values for the electric charge of leptons and quarks of each generation of the Standard Model, nor the specificity of gauge fields**: their invdim is null, they are sensible only to the part containing Lorentz transformations of the similitude group. The products of such fields also have a null invdim, and themselves behave as gauge fields: this makes possible the construction of creation and annihilation operators.

In metrology, physicists are nowadays working to replace the old Standard Kilogram by this standard of action which is the Planck constant. This is perfectly compatible with the extended invariance: when a similitude multiplies all lengths by r, the length of all standard meters are also multiplied by r. Thus the physicist who always locally measures only the ratio between the length of the measured object and the length of the standard meter cannot see the homothety. We may say the same thing for proper masses or for actions, replacing only r with r^3 and r^4 respectively.

Chapter 6

Epilogue

Thank you for your attention on this work which is the result of thirty years of research. We present the following as a recap of the most salient features. We introduced five novelties:

1. The natural mathematical framework of the quantum wave is the Cl_3 algebra (instead of space-time algebra); it is enough to describe both the quantum wave and the space-time manifold.

2. The invariance group (form invariance which in quantum physics replaces the Lorentz group of special relativity), is extended from $SL(2,\mathbb{C})$ to the $GL(2,\mathbb{C}) = Cl_3^*$ Lie group.

3. The linear Dirac equation is replaced by our improved (and nonlinear) wave equation, obtained by simplifying the mass term of the Lagrangian density, with a possible double proper mass.

4. Space-time is not the starting point, but it is a consequence of the field of values of the fermion waves.

5. The space-time manifold is included in Cl_3^* (invariance group of all physical laws) as its auto-adjoint part.

The calculations with Cl_3 are much simpler than those with 4×4 complex matrices. The first yield of these simplifications is a better understanding:

1. Why there is an extremal principle in quantum mechanics.

2. Why there is an equivalence principle in General Relativity.

3. Why there is the double equality $E = mc^2 = h\nu$ (Einstein's relation and existence of the quantum of action).

4. Why action and electric charge are both quantized.

5. Why the maximal violation of parity in weak interactions.

6. Why the spin $1/2$ and not integer values for kinetic momentum operators.

7. Where the exclusion principle comes from.

8. Why this kind of wave equations and how these equations can be linked to the geometry of space-time.

9. What charge conjugation is and how the puzzle of the negative energies is solved.

10. How the fermion part and the boson part of the standard model are connected.

11. Why the baryonic number is conserved.

12. Why all leptons are insensitive to strong interactions.

All these results are obtained without any metaphysical principle.

By enlarging the linear space of values of the quantum wave to $Cl_{3,3}$, the fermion wave integrates most of the novelties introduced by the Standard Model:

1. The existence of exactly two quarks with three color states, for each generation.

2. The linked existence of three generations and of charge conjugation.

3. The gauge invariance under the $U(1) \times SU(2) \times SU(3)$ group of the Standard Model, and the impossibility of a different or greater gauge group.

4. The distinction between leptons insensitive to strong interactions and quarks linked by strong interactions with color. For the lepton sector the existence of one particle with an electric charge and of one with a magnetic charge, the magnetic-monopole–neutrino with a total of four waves linked to the four kinds of representations of the Cl_3^* group. The neutrino with a right and a left wave, and Lochak's magnetic monopole are the same object. For each of three generation, a charged lepton and a neutrino exist, plus a fourth neutrino which is its own anti-particle.

5. For quarks the existence of twelve elementary waves, three for each of the four kinds of representations of Cl_3^*, with six of these waves, three left and three right, forming the three quarks of a proton or of a neutron.

6. The quantization of kinetic momentum with the value $\hbar/2$ for each elementary particle (it is precisely for this reason that they may be considered as particles), namely: the proton, neutron, electron and neutrino-monopole

7. The magnitude of the electric charges of all particles (electrons, neutrinos, quarks...) and of their antiparticles.

8. The origin of the preference for left waves (see 3.8). The inclusion of Cl_3 into End(Cl_3) also explains why the electron wave in second quantization can account for all results given by its wave of first quantization.

9. About the geometry of space-time, we also resolve the ambiguity of the signature of space-time in special relativity. Since the quadratic form giving the space-time metric comes from the determinant in (1.33), the signature is necessarily $+ \; - \; - \; -$.

10. The equivalence between wave equations in the usual form and wave equations in the completely invariant form requires the cancellation of the X term in 4.3.

11. The existence in the electromagnetic field of quanta of energy–momentum (photons), because the electromagnetic field is made of components of the energy–momentum densities of the fermion field.

12. The existence in quantum physics of a probability density and the necessity to normalize the quantum wave. That results from the equivalence principle between gravitation and inertia.

The inclusion of the space-time manifold into the Cl_3^* Lie group brings:

1. The geometric origin of the arrow of time.

2. A better understanding of non-simultaneity in optics.

3. A mainly geometric origin for the expansion of the universe, and its recent acceleration.

All this work could not have even begun without the creation, by Louis de Broglie himself, of a free foundation with the aim of continuing his physics research. The head of this “Fondation Louis de Broglie” was Georges Lochak (1931-2021), who discovered the leptonic magnetic monopole, starting point of our work.

If you have questions or comments you may use our email addresses.

The path is arduous, but Louis de Broglie declared the necessity of both freedom and imagination. He passed on his device "**Pour l’avenir**" to his Foundation.

Appendix A

Clifford algebras

We present what a Clifford algebra is. We study the algebra of the Euclidean plane and in great detail the algebra of 3-dimensional space. This algebra is also generated by Pauli matrices. We include here space-time and relativistic invariance. We study different tensor densities of the electron wave. We prove identities necessary to obtain the form invariance. We study left and right currents, potential vectors and the electromagnetic field.

Clifford algebra is a useful tool: the physics of light, of gravitation, and quantum physics need waves; thus they need trigonometric functions. Trigonometry is highly simplified with the use of the exponential function. This exponential function needs addition and multiplication: it is necessary to may add and multiply vectors. Mathematics then provides the structure of algebra. Here we present this algebra at a level of minimal difficulty. As this is a presentation for physics needs, we expect our pedagogical decision to be met with some criticism from mathematicians. For instance we choose to speak only about real Clifford algebras, though algebras on the complex field also exist. We might think that they ought to be essential in quantum physics since the most frequently used Clifford algebra is also a complex algebra. But it is actually its structure as a real algebra which is useful in quantum physics.[1] We may also consider that it is not the algebra which is important but only the ring structure, and even only the multiplicative Lie group structure. The presentation here is intentionally made for beginners, not for theorists of Lie groups.

1. A real Clifford algebra has vectors whose components are real numbers and which are never multiplied by i. A complex Clifford algebra has vectors whose components are complex numbers which may be multiplied by i. You can also refer to Doran and Lasenby [66], which is more oriented towards space-time algebra. The disadvantage of that approach, yet natural with relativistic physics, comes from the fact that 2×2 matrices have very particular properties, specific compared to greater dimensions. Those properties play an important role in quantum physics.

A.1 What is a Clifford algebra?

1. It is an algebra [9][22], and there are two operations: an addition denoted by $A+B$ and a multiplication denoted by AB, such that for any A, B, C:

$$\begin{aligned} &A+(B+C)=(A+B)+C \quad ; \quad A+B=B+A, \\ &A+0=A \quad ; \quad A+(-A)=0, \\ &A(B+C)=AB+AC \quad ; \quad (A+B)C=AC+BC, \\ &A(BC)=(AB)C. \end{aligned} \tag{A.1}$$

This last equality (associativity of multiplication) allows us to suppress parentheses. The product is thus simply denoted as ABC.

2. The algebra contains a set of vectors, denoted with arrows, in which a scalar product exists and the Clifford multiplication $\vec{u}\vec{v}$ is supposed to satisfy for any vector $\vec{u}$ the identity:

$$\vec{u}\vec{u}=\vec{u}\cdot\vec{u}. \tag{A.2}$$

where $\vec{u}\cdot\vec{v}$ points out the scalar product [2] of these vectors. This implies, since $\vec{u}\cdot\vec{u}$ is a real number, that the algebra which contains the vectors also contains the real numbers.

3. Real numbers commute with any member of the algebra: if a is a real number and if A is any element in the algebra:

$$aA=Aa; \quad 1A=A. \tag{A.6}$$

These algebras exist for any finite-dimensional linear space. The smallest algebra is unique, up to an isomorphism.

Relations (A.1) and (A.6) imply that the algebra is also a linear space which must be distinguished from the initial linear space. If we start from an n-dimensional linear space the dimension of the algebra is 2^n. We will see for instance in A.3 that the algebra of the usual space, 3-dimensional, is an 8-dimensional linear space.

If $\vec{u}$ et $\vec{v}$ are two orthogonal vectors (that means if $\vec{u}\cdot\vec{v}=0$) the equation $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})=(\vec{u}+\vec{v})(\vec{u}+\vec{v})$ implies

$$\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}=\vec{u}\vec{u}+\vec{u}\vec{v}+\vec{v}\vec{u}+\vec{v}\vec{v},$$

2. The scalar product satisfies, for any vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and any real number a:

$$\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}, \tag{A.3}$$
$$(a\vec{u})\cdot\vec{v}=a(\vec{u}\cdot\vec{v}), \tag{A.4}$$
$$(\vec{u}+\vec{v})\cdot\vec{w}=(\vec{u}\cdot\vec{w})+(\vec{v}\cdot\vec{w}). \tag{A.5}$$

We recall also that the scalar product of two vectors is the product of the lengths of these vectors by the cosine of the angle that they form.

hence we have:

$$0 = \vec{u}\vec{v} + \vec{v}\vec{u} \quad ; \quad \vec{v}\vec{u} = -\vec{u}\vec{v}. \tag{A.7}$$

This is the major difference compared to usual rules of calculation with numbers: the multiplication is not commutative and anyone must be as cautious as for matrix calculus. It is besides always possible to perform all calculations using an algebra of squared matrices. The addition is defined in the whole algebra, which contains both numbers and vectors. Then we will get sums of numbers and vectors: $3 + 5\vec{i}$ is **authorized**. This may perhaps seem strange or disturbing, but it is not any different from $3 + 5i$ and everyone using complex numbers finally gets used to it. The two following definitions are important and general:

Even subalgebra: The even subalgebra is the subalgebra generated by all products of an even number of vectors: : $\vec{u}\vec{v}$, $\vec{e}_1\vec{e}_2\vec{e}_3\vec{e}_4$, and so on.

Reversion: The reversion $A \mapsto \widetilde{A}$ changes the order of all products. Reversion does not change numbers a nor vectors: $\tilde{a} = a$, $\widetilde{\vec{u}} = \vec{u}$, and we get, for any $\vec{u}$ and $\vec{v}$, or A and B:

$$\widetilde{\vec{u}\vec{v}} = \vec{v}\vec{u} \ ; \quad \widetilde{AB} = \tilde{B}\tilde{A} \ ; \quad \widetilde{A+B} = \tilde{A} + \tilde{B}. \tag{A.8}$$

A.2 Clifford algebra of a Euclidean plane

The algebra of the Euclidean plane Cl_2 contains all real numbers and all vectors of an Euclidean plane, $\vec{u} = x\vec{e}_1 + y\vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ form a direct orthonormal basis of the plane: this means that they are two vectors with length 1, orthogonal to each other; they satisfy: $\vec{e}_1{}^2 = \vec{e}_2{}^2 = 1$, $\vec{e}_1 \cdot \vec{e}_2 = 0$. Usually we let: $i := \vec{e}_1\vec{e}_2$. The general element of the algebra of the plane is expressed as:

$$A = a + x\vec{e}_1 + y\vec{e}_2 + b\vec{e}_1\vec{e}_2 = a + x\vec{e}_1 + y\vec{e}_2 + ib, \tag{A.9}$$

where a, x, y and b are real numbers. Those 4 real numbers are enough because:

$$\begin{aligned} &\vec{e}_1 i = \vec{e}_1(\vec{e}_1\vec{e}_2) = (\vec{e}_1\vec{e}_1)\vec{e}_2 = 1\vec{e}_2 = \vec{e}_2; \quad \vec{e}_2 i = -\vec{e}_1; \quad i\vec{e}_2 = \vec{e}_1 \\ &i\vec{e}_1 = -\vec{e}_2; \quad i^2 = ii = i(\vec{e}_1\vec{e}_2) = (i\vec{e}_1)\vec{e}_2 = -\vec{e}_2\vec{e}_2 = -1. \end{aligned} \tag{A.10}$$

We have two remarks:
1. The even subalgebra Cl_2^+ is the set formed by all $a + ib$; thus it is the complex field. This subalgebra is commutative. We can say that complex numbers are underlying as soon as the dimension of the linear space is greater or equal to 2.
2. Here the reverse is the complex conjugate $\tilde{i} = \widetilde{\vec{e}_1\vec{e}_2} = \vec{e}_2\vec{e}_1 = -i$.

We hence obtain for any $\vec{u} = x\vec{e}_1 + y\vec{e}_2$ and any $\vec{v} = x'\vec{e}_1 + y'\vec{e}_2$ in the plane: $\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + i \det(\vec{u}, \vec{v})$ where $\vec{u} \cdot \vec{v} = xx' + yy'$ and where $\det(\vec{u}, \vec{v})$ is the determinant $xy' - yx'$ of both vectors in the basis $(\vec{e}_1,\ \vec{e}_2)$.

A.3 Clifford algebra of 3-dimensional space

The dimension 3 of our space is the main reason for the significance of that algebra. We explain in Chapter 1 why other reasons exist for preferring this framework for quantum physics.

The algebra, denoted as Cl_3, contains [3] all real numbers and all vectors of the geometry of space which read:

$$\vec{u} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 =: x^j\vec{e}_j, \tag{A.11}$$

where x^1, x^2, x^3 are 3 real numbers. $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ form an orthonormal basis. The second equality is the usual Einstein summation convention, with Latin indices from 1 to 3. The scalar product satisfies:

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_1 = 0 \;\; ; \;\; \vec{e}_1{}^2 = \vec{e}_2{}^2 = \vec{e}_3{}^2 = 1. \tag{A.12}$$

We let:

$$i_1 := \vec{e}_2\vec{e}_3 \;\; ; \;\; i_2 := \vec{e}_3\vec{e}_1 \;\; ; \;\; i_3 := \vec{e}_1\vec{e}_2 \;\; ; \;\; i := \vec{e}_1\vec{e}_2\vec{e}_3. \tag{A.13}$$

That gives:

$$i_1^2 = i_2^2 = i_3^2 = i^2 = -1, \tag{A.14}$$

$$i\vec{u} = \vec{u}i \;\; ; \;\; i\vec{e}_j = i_j \;,\; j = 1,\ 2,\ 3. \tag{A.15}$$

In the calculation of squares we may use (A.10). To obtain the commutation of i with all vectors we may begin to prove that i commutes with each $\vec{e}_j$. General element in Cl_3 is: $A = a + \vec{u} + i\vec{v} + ib$. For Cl_3 that gives $1 + 3 + 3 + 1 = 8 = 2^3$ dimensions. (third line of the arithmetical triangle). We have five remarks:

1. The center of Cl_3, set of all elements commuting with all other, is the set of the $a + ib$ terms. They are the only elements commuting with all the others in the algebra. That center is hence the complex field: $\mathbb{C}$. This is the main reason for the important role of complex numbers in quantum physics. In a Cl_n with n even, the center of the real Clifford algebra Cl_n is only the real field. The larger center, in Cl_3, has many consequences that we encounter later.

2. The even subalgebra Cl_3^+ is the set of $a + i\vec{v}$ which is isomorphic to the quaternion field $\mathbb{H}$. Using the quaternion field we automatically use the Cl_3 algebra which is sometimes called algebra of biquaternions.

3. $\widetilde{A} = a + \vec{u} - i\vec{v} - ib$: The reversion is for the center $\mathbb{C}$, the complex conjugation and also for quaternions in the even sub-algebra $Cl_3^+ = \mathbb{H}$.

4. The $i\vec{v}$ term is usually called an axial vector or pseudo-vector, while $\vec{u}$ is the true vector or (in short) vector. It is well known that this situation is proper to the dimension 3, only dimension for which there is equality

between the second and the third number on a line in the arithmetical triangle.

5. 4 different and independent terms with square -1 exist: 4 different ways to get complex numbers. Nonrelativistic quantum theory always uses a unique term with square -1, which is the case with Cl_2, and hence works like a 2D software. What physics actually needs since we live in a 3-dimensional space is thus the 3D software of the Cl_3 algebra.

A.3.1 Vector product, orientation

Given vectors $\vec{u}$ and $\vec{v}$, the vector product $\vec{u}\times\vec{v}$ is the vector orthogonal to $\vec{u}$ and to $\vec{v}$, with a length equal to the product of the lengths of $\vec{u}$ and $\vec{v}$ by the sine of their angle, and such that the basis $(\vec{u},\,\vec{v},\,\vec{u}\times\vec{v})$ is direct. Using coordinates in the basis $(\vec{e}_1,\,\vec{e}_2,\,\vec{e}_3)$, the following properties are proved, for any $\vec{u}$ and $\vec{v}$:

$$\vec{u}\vec{v} = \vec{u}\cdot\vec{v} + i\ \vec{u}\times\vec{v}, \tag{A.16}$$

$$(\vec{u}\cdot\vec{v})^2 + (\vec{u}\times\vec{v})^2 = \vec{u}^{\,2}\vec{v}^{\,2}. \tag{A.17}$$

From (A.16) we deduce:

$$\vec{u}\cdot\vec{v} = \frac{1}{2}(\vec{u}\vec{v}+\vec{v}\vec{u});\ \ \vec{u}\times\vec{v} = \frac{1}{2i}(\vec{u}\vec{v}-\vec{v}\vec{u}). \tag{A.18}$$

By dividing (A.17) by the right term, and taking θ to be a measure of the angle $(\vec{u},\vec{v})$, we get:

$$1 = \frac{(\vec{u}\cdot\vec{v})^2}{(\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel)^2} + \frac{(\vec{u}\times\vec{v})^2}{(\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel)^2} = \cos^2(\theta) + \Big(\frac{\vec{u}\times\vec{v}}{\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel}\Big)^2,$$

$$\frac{\vec{u}\times\vec{v}}{\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel} = \sqrt{1-\cos^2(\theta)} = |\sin(\theta)|, \tag{A.19}$$

$$\parallel\vec{u}\times\vec{v}\parallel = \parallel\vec{u}\parallel\,\parallel\vec{v}\parallel\ \ |\sin(\theta)|. \tag{A.20}$$

Next $\det(\vec{u},\vec{v},\vec{w})$ refers to the determinant whose columns contain the coordinates of the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, in the basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$. Using these coordinates again it is possible to prove, for any $\vec{u}$, $\vec{v}$, $\vec{w}$:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \det(\vec{u},\vec{v},\vec{w}), \tag{A.21}$$

$$\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{w}\cdot\vec{u})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}, \tag{A.22}$$

$$\vec{u}\vec{v}\vec{w} = i\det(\vec{u},\vec{v},\vec{w}) + (\vec{v}\cdot\vec{w})\vec{u} - (\vec{w}\cdot\vec{u})\vec{v} + (\vec{u}\cdot\vec{v})\vec{w}. \tag{A.23}$$

From the mixed product (A.21) we deduce that $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and to $\vec{v}$. The determinant (A.21) gives the orientation. We recall that a basis $(\vec{u},\vec{v},\vec{w})$ is said to be direct (which means to have the same orientation as $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$) if $\det(\vec{u},\vec{v},\vec{w}) > 0$, and is said to be inverse (which means to have the contrary orientation) if $\det(\vec{u}, \vec{v}, \vec{w}) < 0$. The rule (A.23) allows us to establish that, $B = (\vec{u}, \vec{v}, \vec{w})$ being any orthonormal basis, then $\vec{u}\vec{v}\vec{w} = i$ if and only if B is direct, and $\vec{u}\vec{v}\vec{w} = -i$ if and only if B is inverse. In the case where $\vec{u}\vec{v}\vec{w} = i$ we also have:

$$\vec{w} = \vec{u} \times \vec{v};\ \vec{u} = \vec{v} \times \vec{w};\ \vec{v} = \vec{w} \times \vec{u}. \tag{A.24}$$

On the contrary, with the other orientation where $\vec{u}\vec{v}\vec{w} = -i$, we have:

$$\vec{w} = \vec{v} \times \vec{u};\ \vec{u} = \vec{w} \times \vec{v};\ \vec{v} = \vec{u} \times \vec{w}. \tag{A.25}$$

Therefore i is fully linked to the orientation of space. Changing i into $-i$ is equivalent to changing the space orientation. The fact that i determines the space orientation plays an essential role in physics of magnetism and of weak interactions.

All calculations in Cl_3 result from the sum (where we add numbers to numbers, vectors to vectors and so on) and the product (product of two numbers, product of a number and a vector, product of two or three vectors), through the scalar product, the vector product and the mixed product, all well known to physicists and engineers. In Cl_3 algebra there are no mysteries nor undue complications. This should be taught in any scientific and technical university, to spare time for all students.

A.3.2 Pauli algebra

This algebra, introduced in physics as early as 1926 to account for the spin 1/2 of the electron, is the algebra $M_2(\mathbb{C})$ formed by 2×2 complex matrices. It is equal – isomorphic, to be precise – to Cl_3, but only as an algebra on the real field [3]. Identifying the complex numbers with the scalar matrices, and the basis vectors e_j with the Pauli matrices σ_j is enough to determine this identification [4]. And it is fully compatible with our previous calculations because:

$$\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = i, \tag{A.26}$$

$$\sigma_1\sigma_2 = i\sigma_3 \quad ; \quad \sigma_2\sigma_3 = i\sigma_1 \quad ; \quad \sigma_3\sigma_1 = i\sigma_2. \tag{A.27}$$

Consequently the reverse is identical to the adjoint (transposed conjugate matrix):

$$\tilde{A} = A^\dagger = (A^*)^t. \tag{A.28}$$

3. Pauli algebra has a dimensionality of 8 on the real field, and only 4 on the complex field.

4. The identifying process could be considered a lack of rigor, but in fact it is a frequent and well-known process in mathematics. The same process allows us to include integer numbers into relative numbers, or real numbers into complex numbers. To do without this process results in very complicated notations. This process considers $(\sigma_1,\ \sigma_2,\ \sigma_3)$ as a direct basis in the usual space.

We will thus refer to this algebra as either Cl_3 or Pauli algebra. Some Cliffordians who did not seem understand the concept of isomorphism refuse to use matrix calculus. As for physicists, who always used Pauli algebra with complicated notations, the full use of Cl_3 brings only great simplifications to their calculations, without changing their results.

A.3.3 Three conjugations are useful

$A = a+\vec{u}+i\vec{v}+ib$ is the sum of the even element $A_1 = a+i\vec{v}$ (quaternion) and of the odd part $A_2 = \vec{u}+ib$. We define the P conjugation (called "parity" in quantum physics) such that:

$$P : A \mapsto \widehat{A};\ \widehat{A} = A_1 - A_2 = a - \vec{u} + i\vec{v} - ib. \tag{A.29}$$

For any elements A and B in Cl_3 this parity conjugation satisfies:

$$\widehat{A+B} = \widehat{A} + \widehat{B}\ \ ;\ \ \widehat{AB} = \widehat{A}\widehat{B}. \tag{A.30}$$

P is the main automorphism of the algebra. Any Clifford algebra possesses a similar involutive (meaning: PP is the identity) automorphism. From this conjugation and from the reversion, we can define a third conjugation:

$$\overline{A} := \widehat{A}^\dagger = a - \vec{u} - i\vec{v} + ib\ \ :\ \ \overline{A+B} = \overline{A} + \overline{B}\ \ ;\ \ \overline{AB} = \overline{B}\,\overline{A}. \tag{A.31}$$

The composition, in any order, of two among these three conjugations gives the third one. Only P conserves the order of the products, while $A \mapsto \overline{A}$ and $A \mapsto A^\dagger$ inverse the order of the factors. Now a, b, c, d being any complex numbers and $\overline{a} = a^*$ the conjugate complex [5] of a, we can prove that for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the Pauli algebra [6] we have:

$$\widetilde{A} = A^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\ \ ;\ \ \widehat{A} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}\ \ ;\ \ \overline{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{A.32}$$

$$A\overline{A} = \overline{A}A = \det(A) = ad - bc\ ;\ \ \widehat{A}A^\dagger = A^\dagger\widehat{A} = [\det(A)]^*;\ A + \overline{A} = \mathrm{tr}(A).$$

If $\det(A) \neq 0$ we then get:

$$[\det(A)]^{-1}\overline{A}A = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ A^{-1} = [\det(A)]^{-1}\overline{A}. \tag{A.33}$$

5. The notation $\bar{a}$ for the conjugate is today the only notation used in mathematics, though the notation a^* was commonly used in course books for quantum physics. Thus we allow ourselves the use of either notation when it is clear that there can be no confusion due to $\overline{A} = \widehat{A}^\dagger$.

6. The equality $A\overline{A} = \overline{A}A$ is general in Cl_3. The equality $A\overline{A} = \det(A)$ uses the identification between real numbers and scalar matrices, which means the inclusion of the real numbers in the Clifford algebra.

A.3.4 Gradient, divergence and curl

In Cl_3 we use the differential operator:

$$\vec{\partial} = \vec{e}_1\partial_1 + \vec{e}_2\partial_2 + \vec{e}_3\partial_3 = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}, \tag{A.34}$$

with [7]:

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \ ; \ \partial_j = \frac{\partial}{\partial x^j}. \tag{A.35}$$

The Laplacian is simply the square of $\vec{\partial}$:

$$\Delta = (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 = \vec{\partial}\vec{\partial}. \tag{A.36}$$

When applied to a scalar, $\vec{\partial}a$ is the gradient of a, and when applied to a vector $\vec{u}$ we get both the divergence and the curl:

$$\vec{\partial}a = \vec{\text{grad}}\ a = (\partial_1 a)\sigma_1 + (\partial_2 a)\sigma_2 + (\partial_3 a)\sigma_3, \tag{A.37}$$

$$\vec{\partial}\vec{u} = \vec{\partial}\cdot\vec{u} + i\,\vec{\partial}\times\vec{u};\ \vec{\partial}\cdot\vec{u} = \text{div}\vec{u} = \partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3, \tag{A.38}$$

$$\vec{\partial}\times\vec{u} = \text{curl}(\vec{u}) = (\partial_2 u^3 - \partial_3 u^2)\sigma_1 + (\partial_3 u^1 - \partial_1 u^3)\sigma_2 + (\partial_1 u^2 - \partial_2 u^1)\sigma_3.$$

Thus, for any function with a scalar value $a = a(\vec{x})$ and for any function with a vector value $\vec{v} = \vec{v}(\vec{x})$ we have:

$$\vec{\partial}(\vec{\partial}a) = (\vec{\partial}\vec{\partial})a = \Delta a;\ \vec{\partial}(\vec{\partial}\vec{v}) = (\vec{\partial}\vec{\partial})\vec{v} = \Delta\vec{v}, \tag{A.39}$$

$$\vec{\partial}\cdot(\vec{\partial}\times\vec{v}) = 0;\ \vec{\partial}\times(\vec{\partial}a) = 0, \tag{A.40}$$

$$\vec{\partial}\times(\vec{\partial}\times\vec{v}) = \vec{\partial}(\vec{\partial}\cdot\vec{v}) - \Delta\vec{v}. \tag{A.41}$$

A.3.5 Center of Cl_3^*, centraliser of σ_3

The center of a group is the set of all elements commuting with each element of this group. In Cl_3^* the center is the set of scalar matrices $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ where z is non zero. That group is isomorphic to $\mathbb{C}^*$, that means to the set of complex non zero numbers. To simplify the notations it is convenient to identify complex numbers and scalar matrices ($z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$).

The centraliser $\mathbb{A}^*$ of σ_3 is the set of all elements commuting with σ_3. That is the set of invertible diagonal matrices:

$$a \in \mathbb{A}^* \Leftrightarrow a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},\quad a_1 a_2 \in \mathbb{C}^*. \tag{A.42}$$

7. This operator $\vec{\partial}$ is usually denoted in quantum mechanics as a scalar product, for instance $\vec{\sigma}\cdot\vec{\nabla}$. From there results much confusion. Simple notations are very useful for optimizing calculations.

That commutative group, is 4-dimensional on $\mathbb{R}$. The Lie algebra of the group is the sub-algebra $\mathbb{A}$ of diagonal matrices. That set is all at once a 2-dimensional linear space on $\mathbb{C}$ and a 4-dimensional linear space on $\mathbb{R}$, and a commutative ring with unity. $\mathbb{A}$ is a ring: the sum and the product of two elements in $\mathbb{A}$ is in $\mathbb{A}$. That ring is commutative: le product AB of two diagonal matrices is equal to BA. That ring has a unit which is the unit matrix I_2. That ring is not a field because $\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The elements in $\mathbb{A}$ which are not zero divisor are invertible:

$$a^{-1} = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{pmatrix}. \tag{A.43}$$

Moreover we have:

$$\phi_d := x - i_2 y := \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & -y_2 \\ y_1 & x_2 \end{pmatrix}. \tag{A.44}$$

And similarly we have:

$$\phi_g := x + \overline{y}(-i_2) := \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} + \begin{pmatrix} y_2 & 0 \\ 0 & y_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_1 & -y_2 \\ y_1 & x_2 \end{pmatrix} = \phi_d. \tag{A.45}$$

ϕ_d allows us to consider Cl_3 as a right modulus on $\mathbb{A}$, (the x and y scalar being to the right of $-i_2$), while ϕ_g allows us to consider Cl_3 as a left modulus on $\mathbb{A}$, (the x and $\overline{y}$ being to the left of $-i_2$). We use as well the structure of right modulus as the structure of left modulus, it is enough to use the conjugation $y \mapsto \overline{y}$ to go from a structure to the other. For:

$$\begin{aligned} v = (v_1,\ v_2) := v_1 + (-i_2)v_2 = \begin{pmatrix} x_1 & -\bar{y}_2 \\ x_2 & \bar{y}_1 \end{pmatrix}, \\ v_1 := \begin{pmatrix} x_1 & 0 \\ 0 & \bar{y}_1 \end{pmatrix};\ \ v_2 := \begin{pmatrix} x_2 & 0 \\ 0 & \bar{y}_2 \end{pmatrix}, \end{aligned} \tag{A.46}$$

we have:

$$v^\dagger = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ -y_2 & y_1 \end{pmatrix} = v_1^\dagger + v_2^\dagger i_2 = v_1^\dagger + i_2\widehat{v}_2, \tag{A.47}$$

$$\overline{v} = \begin{pmatrix} \bar{y}_1 & \bar{y}_2 \\ -x_2 & x_1 \end{pmatrix} = \overline{v}_1 + \overline{v}_2 i_2 = \overline{v}_1 + i_2 v_2, \tag{A.48}$$

$$\widehat{v} = \widehat{v}_1 - i_2\widehat{v}_2 = \begin{pmatrix} y_1 & -\bar{x}_2 \\ y_2 & \bar{x}_1 \end{pmatrix}. \tag{A.49}$$

That gives, for the addition and the multiplication in Cl_3 :

$$\begin{aligned}(v_1,\ v_2)+(w_1,\ w_2) &= v_1 - i_2 v_2 + w_1 - i_2 w_2 = (v_1+w_1) - i_2(v_2+w_2)\\ &= (v_1+w_1,\ v_2+w_2), \qquad\qquad (A.50)\\ (v_1,\ v_2)(w_1,\ w_2) &= (v_1 - i_2 v_2)(w_1 - i_2 w_2)\\ &= v_1 w_1 - v_1 i_2 w_2 - i_2 v_2 w_1 + i_2 v_2 i_2 w_2\\ &= v_1 w_1 - i_2 \overline{v}_1 w_2 - i_2 v_2 w_1 + i_2^2 \overline{v}_2 w_2\\ &= v_1 w_1 - \overline{v}_2 w_2 - i_2(\overline{v}_1 w_2 + v_2 w_1)\\ &= (v_1 w_1 - \overline{v}_2 w_2,\ \overline{v}_1 w_2 + v_2 w_1). \qquad\qquad (A.51)\end{aligned}$$

A.3.6 Space-time in the Pauli algebra

That inclusion was implicitly made with the Dirac equation (1928):

$$\mathrm{x} = \mathrm{x}^\mu \sigma_\mu = \begin{pmatrix} \mathrm{x}^0+\mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2\\ \mathrm{x}^1+i\mathrm{x}^2 & \mathrm{x}^0-\mathrm{x}^3\end{pmatrix};\ \sigma_0=\sigma^0=I;\ \mathrm{x}^0 = ct, \qquad (A.52)$$

where c is the speed of light and t is time. Using the scalar in $\mathbb{A}$, we have:

$$\begin{aligned}\mathrm{x} = (x,\ y) = x - i_2 y;\ x &= \mathrm{x}^0 + \mathrm{x}^3\sigma_3 = \begin{pmatrix}\mathrm{x}^0+\mathrm{x}^3 & 0\\ 0 & \mathrm{x}^0-\mathrm{x}^3\end{pmatrix}\\ -i_2 y &= \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 = \sigma_1\sigma_3(\mathrm{x}^1\sigma_3 + i\mathrm{x}^2) = -i_2(\mathrm{x}^1\sigma_3 + i\mathrm{x}^2),\\ y &= i\mathrm{x}^2 + \mathrm{x}^1\sigma_3 = \begin{pmatrix} i\mathrm{x}^2+\mathrm{x}^1 & 0\\ 0 & i\mathrm{x}^2-\mathrm{x}^1\end{pmatrix}. \qquad (A.53)\end{aligned}$$

Any element M of the Pauli algebra is sum of a vector v and of the product of a second vector w by i:

$$M = v + iw;\ v = \frac{1}{2}(M + M^\dagger)\ ;\quad v^\dagger = v \qquad (A.54)$$

$$iw = \frac{1}{2}(M - M^\dagger)\ ;\quad w^\dagger = w. \qquad (A.55)$$

These two space-time vectors v and w are single. Since $\mathrm{x} = \mathrm{x}^\dagger$, space-time is the set of $M = v + iw$ such that $w = 0$. We call this set the self-adjoint part of Cl_3. In this framework we need two differential operators:

$$\nabla = \partial_0 - \vec{\partial}\ \ ;\ \ \widehat{\nabla} = \partial_0 + \vec{\partial}. \qquad (A.56)$$

They allow us to calculate the D'Alembertian:

$$\nabla\widehat{\nabla} = \widehat{\nabla}\nabla = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 =: \Box. \qquad (A.57)$$

The main reason for the use of Cl_3 is in these equations. They means that the D'Alembertian includes the parity transformation $P:\ M \mapsto \widehat{M}$. (We see its implications in Chapter 1). Now we use the structure of modulus on the ring $\mathbb{A}$:

$$\nabla = (\nabla_1,\ -\nabla_2) = \nabla_1 + i_2\nabla_2;\ \ \widehat{\nabla} = (\widehat{\nabla}_1, -\widehat{\nabla}_2) = \widehat{\nabla}_1 + i_2\widehat{\nabla}_2, \quad \text{(A.58)}$$

$$\nabla_1 = \partial_0 - \sigma_3\partial_3 = \begin{pmatrix} \partial_0 - \partial_3 & 0 \\ 0 & \partial_0 + \partial_3 \end{pmatrix}, \quad \text{(A.59)}$$

$$\widehat{\nabla}_1 = \partial_0 + \sigma_3\partial_3 = \begin{pmatrix} \partial_0 + \partial_3 & 0 \\ 0 & \partial_0 - \partial_3 \end{pmatrix} = \overline{\nabla}_1, \quad \text{(A.60)}$$

$$\nabla_2 = \sigma_3\partial_1 + i\partial_2 = \begin{pmatrix} \partial_1 + i\partial_2 & 0 \\ 0 & -\partial_1 + i\partial_2 \end{pmatrix}, \quad \text{(A.61)}$$

$$\widehat{\nabla}_2 = \sigma_3\partial_1 + i\partial_2 = \nabla_2, \quad \text{(A.62)}$$

$$\widehat{\nabla} = (\widehat{\nabla}_1, -\nabla_2) = \widehat{\nabla}_1 + i_2\nabla_2. \quad \text{(A.63)}$$

Operators ∇_1 and ∇_2 are derivations, in the sense that they are linear and satisfy the Leibniz condition: let $a \in \mathbb{A}$, v and w two scalar fields, functions of space-time with value in $\mathbb{A}$. a_1 and a_2 are complex numbers, v_1 and v_2, w_1 and w_2 are functions of space-tide with complex value. We may read:

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix};\ \ v = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix};\ \ w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix},$$

$$\begin{aligned} \nabla_1[v+w] &= \begin{pmatrix} \partial_0 - \partial_3 & 0 \\ 0 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} v_1 + w_1 & 0 \\ 0 & v_2 + w_2 \end{pmatrix} \\ &= \begin{pmatrix} \partial_0 v_1 + \partial_0 w_1 - \partial_3 v_1 - \partial_3 w_1 & 0 \\ 0 & \partial_0 v_2 + \partial_0 w_2 + \partial_3 v_2 + \partial_3 w_2 \end{pmatrix} \\ &= \nabla_1[v] + \nabla_1[w]. \end{aligned} \quad \text{(A.64)}$$

Similarly we have:

$$\begin{aligned} \nabla_2[v+w] = \nabla_2[v] + \nabla_2[w];\ \ \nabla_1[av] = \nabla_1[va] = a\nabla_1[v] = \nabla_1[v]a \\ \nabla_2[av] = \nabla_2[va] = a\nabla_2[v] = \nabla_2[v]a, \end{aligned} \quad \text{(A.65)}$$

That proves the linearity. For the multiplication we get the Leibniz rule:

$$\begin{aligned} \nabla_1[vw] &= \begin{pmatrix} \partial_0 - \partial_3 & 0 \\ 0 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} v_1 w_1 & 0 \\ 0 & v_2 w_2 \end{pmatrix} \\ &= \begin{pmatrix} (\partial_0 v_1)w_1 + v_1(\partial_0 w_1) & 0 \\ 0 & (\partial_0 v_2)w_2 + v_2(\partial_0 w_2) \end{pmatrix} \\ &+ \begin{pmatrix} -[(\partial_3 v_1)w_1 + v_1(\partial_3 w_1)] & 0 \\ 0 & (\partial_3 v_2)w_2 + v_2(\partial_3 w_2) \end{pmatrix} \\ &= \nabla_1[v]w + v\nabla_1[w]. \end{aligned} \quad \text{(A.66)}$$

And also:

$$\nabla_2[vw] = \nabla_2[v]w + v\nabla_2[w]. \quad \text{(A.67)}$$

Besides we may prove the linearity and the Leibniz rule for:

$$\overline{\nabla}_1 = \begin{pmatrix} \partial_0 + \partial_3 & 0 \\ 0 & \partial_0 - \partial_3 \end{pmatrix}; \quad \overline{\nabla}_2 = -\begin{pmatrix} -\partial_1 + i\partial_2 & 0 \\ 0 & \partial_1 + i\partial_2 \end{pmatrix}, \tag{A.68}$$

and also for $\widehat{\nabla}_1$, $\widehat{\nabla}_2$, $\nabla_1^\dagger$ and $\nabla_2^\dagger$.

Let A and B be two space-time vectors: $A = A^0 + \vec{A}$, $B = B^0 + \vec{B}$. The scalar product in space-time $A \cdot B$ is:

$$A \cdot B = \frac{1}{2}(A\widehat{B} + B\widehat{A}) = \frac{1}{2}(\widehat{A}B + \widehat{B}A). \tag{A.69}$$

We indeed have:

$$\begin{aligned} A\widehat{B} + B\widehat{A} &= (A^0 + \vec{A})(B^0 - \vec{B}) + (B^0 + \vec{B})(A^0 - \vec{A}) \\ &= A^0B^0 - A^0\vec{B} + B^0\vec{A} - \vec{A}\vec{B} + A^0B^0 + A^0\vec{B} - B^0\vec{A} - \vec{B}\vec{A} \\ &= 2(A^0B^0 - \vec{A}\cdot\vec{B}) = 2A\cdot B, \end{aligned} \tag{A.70}$$

$$\widehat{A}B + \widehat{B}A = \widehat{A\widehat{B} + B\widehat{A}} = \widehat{2A\cdot B} = 2A\cdot B. \tag{A.71}$$

A.3.7 The space algebra with 4×4 real matrices

When using a matrix definition for complex numbers, the complex field $\mathbb{C}$ is associated to the algebra $M_2(\mathbb{R})$ as follows:

$$\begin{aligned} z = x + iy &= \begin{pmatrix} x & -y \\ y & x \end{pmatrix}; \quad \bar{z} = x - iy = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = z^t, \\ \det(z) &= x^2 + y^2 = |z|^2; \quad x = \Re(z) = \frac{1}{2}\mathrm{tr}(z). \end{aligned} \tag{A.72}$$

That association acts both on the algebraic structure and the topological structure of $\mathbb{C}$: addition and multiplication are associated. The norm defining the distance is the same. The unity 1 of $\mathbf{C}$ is associated to the unity matrix I_2 in $M_2(\mathbb{R})$. That returns to identify real numbers and scalar matrices. That association allows us to build Cl_3 as the algebra of the M and of the $\mathbf{M}$ such as:

$$\begin{aligned} M &= s + \vec{u} + i\vec{v} + ip \\ &= s + u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 + iv_1\sigma_1 + iv_2\sigma_2 + iv_3\sigma_3 + ip \\ &= \begin{pmatrix} s + u_3 + i(v_3 + p) & u_1 + v_2 + i(v_1 - u_2) \\ u_1 - v_2 + i(u_2 + v_1) & s - u_3 + i(p - v_3) \end{pmatrix} =: \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, \end{aligned} \tag{A.73}$$

$$\widehat{M} = s - \vec{u} + i\vec{v} - ip; \; M^\dagger = s + \vec{u} - i\vec{v} - ip; \; \overline{M} = s - \vec{u} - i\vec{v} + ip, \tag{A.74}$$

$$\mathbf{M} = \begin{pmatrix} s + u_3 & -v_3 - p & u_1 + v_2 & u_2 - v_1 \\ v_3 + p & s + u_3 & v_1 - u_2 & u_1 + v_2 \\ u_1 - v_2 & -u_2 - v_1 & s - u_3 & v_3 - p \\ u_2 + v_1 & u_1 - v_2 & p - v_3 & s - u_3 \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1 & \mathbf{z}_2 \\ \mathbf{z}_3 & \mathbf{z}_4 \end{pmatrix}, \tag{A.75}$$

with :

$$\mathbf{z}_1 = \begin{pmatrix} a & -c \\ c & a \end{pmatrix};\ \mathbf{z}_2 = \begin{pmatrix} f & -h \\ h & f \end{pmatrix};\ \mathbf{z}_3 = \begin{pmatrix} e & -g \\ g & e \end{pmatrix};\ \mathbf{z}_4 = \begin{pmatrix} b & -d \\ d & b \end{pmatrix}$$

$$a = s + u_3;\ b = s - u_3;\ c = p + v_3;\ d = p - v_3,$$
$$e = u_1 - v_2;\ f = u_1 + v_2;\ g = u_2 + v_1;\ h = v_1 - u_2. \tag{A.76}$$

We have:

$$\overline{\mathbf{M}} = \begin{pmatrix} \mathbf{z}_4 & -\mathbf{z}_2 \\ -\mathbf{z}_3 & \mathbf{z}_1 \end{pmatrix};\ \mathbf{M}^t = \begin{pmatrix} \mathbf{z}_1^t & \mathbf{z}_3^t \\ \mathbf{z}_2^t & \mathbf{z}_4^t \end{pmatrix}, \tag{A.77}$$

$$\widehat{\mathbf{M}} = \overline{\mathbf{M}}^t = \begin{pmatrix} \mathbf{z}_4^t & -\mathbf{z}_3^t \\ -\mathbf{z}_2^t & \mathbf{z}_1^t \end{pmatrix}. \tag{A.78}$$

The identification $z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ between $\mathbb{C}$ and the sub-algebra of $M_2(\mathbb{R})$ satisfying (A.72) may be extended to an identification between M and $\mathbf{M}$, that means between Cl_3 and the sub-space $\mathcal{A}$ of $M_4(\mathbb{R})$ formed by the matrices such as (A.75). Moreover since the matrix product may be calculated by blocks, we may identify Cl_3 to $\mathcal{A}$ as unitary rings. They thus are unitary algebras that we may identify. That identification is extended to the calculation, for any invertible element of the inverse, since the associativity of the multiplication implies the unicity of the inverse. We have:

$$M\overline{M} = \overline{M}M = \det(M) = \Omega_1 + i\Omega_2 = z_1 z_4 - z_2 z_3 = \rho e^{i\beta}, \tag{A.79}$$

$$\mathbf{M}\overline{\mathbf{M}} = \begin{pmatrix} \Omega_1 + i\Omega_2 & 0 \\ 0 & \Omega_1 + i\Omega_2 \end{pmatrix} = \begin{pmatrix} \Omega_1 & -\Omega_2 & 0 & 0 \\ \Omega_2 & \Omega_1 & 0 & 0 \\ 0 & 0 & \Omega_1 & -\Omega_2 \\ 0 & 0 & \Omega_2 & \Omega_1 \end{pmatrix}. \tag{A.80}$$

We obtain:

$$\widehat{M}M^\dagger = \Omega_1 - i\Omega_2 = \rho e^{-i\beta};\ \mathbf{M}\overline{\mathbf{M}}\widehat{\mathbf{M}}\mathbf{M}^t = \rho^2 I_4, \tag{A.81}$$

$$\mathbf{M}^{-1} = \rho^{-2}\overline{\mathbf{M}}\widehat{\mathbf{M}}\mathbf{M}^t \tag{A.82}$$

That calculation of the inverse is linked to the calculation of determinants thus to the characteristic polynomial. There are two ways of calculation. With Cl_3 as $M_2(\mathbb{C})$ we have:

$$P_M(X) = \det(XI_2 - M) = \begin{vmatrix} X - z_1 & -z_2 \\ -z_3 & X - z_4 \end{vmatrix}$$
$$= X^2 - (z_1 + z_4)X + z_1 z_4 - z_2 z_3, \tag{A.83}$$
$$0 = M^2 - (z_1 + z_4)M + (z_1 z_4 - z_2 z_3)I_2;\quad z_1 + z_4 = 2(s + ip).$$

And we have:

$$\det(M) = z_1z_4 - z_2z_3 = [s + u_3 + i(p + v_3)][s - u_3 + i(p - v_3)] \\ - [u_1 + v_2 + i(v_1 - u_2)][u_1 - v_2 + i(u_2 - v_1)] \quad \text{(A.84)}$$

$$\rho e^{i\beta} = \Omega_1 + i\Omega_2 := \det(M) = s^2 - \vec{u^2} + \vec{v}^2 - p^2 + 2i(sp - \vec{u}\cdot\vec{v}) \\ = (s + ip)^2 + (\vec{u} + i\vec{v})^2 \quad \text{(A.85)}$$

That gives:

$$M^{-1} = \frac{1}{\Omega_1 + i\Omega_2}[-M + 2(s + ip)I_2] = \frac{e^{-i\beta}\overline{M}}{\rho^2} \quad \text{(A.86)}$$

$$\overline{M} = \begin{pmatrix} s - u_3 + i(p - v_3 & -(u_1 + v_2) + i(u_2 - v_1) \\ v_2 - u_1 - i(v_1 + u_2) & s + u_3 + i(p + v_3) \end{pmatrix}. \quad \text{(A.87)}$$

We have also:

$$\Omega_1 = ab - cd - ef + gh; \quad \Omega_2 = ad + bc - eh - fg, \quad \text{(A.88)}$$

$$M\overline{M} = \overline{M}M = \det(M); \ \overline{\mathbf{M}} = \begin{pmatrix} z_4 & -z_2 \\ -z_1 & z_1 \end{pmatrix} \quad \text{(A.89)}$$

With Cl_3 as sub-algebra of $M_4(\mathbb{R})$ the characteristic polynomial of $\mathbf{M}$ reads:

$$\det(XI_4 - \mathbf{M}) = \begin{vmatrix} a - X & -c & f & -h \\ c & a - X & h & f \\ e & -g & b - X & -d \\ g & e & d & b - X \end{vmatrix} \quad \text{(A.90)}$$

$$= X^4 - k_1X^3 + k_2X^2 - k_3X^3 + k_4; \ k_4 = \det(\mathbf{M})$$

$$k_1 = 2(a + b) = 4s, \quad \text{(A.91)}$$

$$k_2 = (a + b)^2 + (c + d)^2 + 2\Omega_1 = 4s^2 + 4p^2 + 2\Omega_1, \quad \text{(A.92)}$$

$$k_3 = 2[(a + b)\Omega_1 + (c + d)\Omega_2] = 4(s\Omega_1 + p\Omega_2), \quad \text{(A.93)}$$

$$\det(\mathbf{M}) = \rho^2 = \Omega_1^2 + \Omega_2^2 = |\det(M)|^2. \quad \text{(A.94)}$$

Therefore it is better to avoid the equality $M = \mathbf{M}$, we will prefer:

$$M \equiv \mathbf{M}; \ M^\dagger \equiv \mathbf{M}^t. \quad \text{(A.95)}$$

For any $\mathbf{M}$ the characteristic equation reads:

$$\mathbf{M}^4 - 4s\mathbf{M}^3 + (4s^2 + 4p^2 + 2\Omega_1)\mathbf{M}^2 - 4(s\Omega_1 + p\Omega_2)\mathbf{M} + \rho^2 = 0 \quad \text{(A.96)}$$

Multiplying by the inverse we get the identity:

$$-M^3 + 4sM^2 - (4s^2 + 4p^2 + 2\Omega_1)M + 4(s\Omega_1 + p\Omega_2) = \overline{M}(\Omega_1 - i\Omega_2). \quad \text{(A.97)}$$

A.3.8 The exponential function in Cl_3

The Lie algebra of the Lie group Cl_3^*, group formed by the invertible elements of Cl_3, is the Cl_3 algebra itself. The exponential function applies the general element M of the algebra on the Lie group. Consider:

$$M = z_0 + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3;\ z_j = x_j + iy_j, j = 0,1,2,3, \tag{A.98}$$

where x_j and y_j are any real numbers. We let:

$$N = z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3 \tag{A.99}$$

Since complex numbers commute, we have:

$$\begin{aligned} e^M &= e^{z_0+N} = e^{z_0}e^N;\ e^{z_0} = e^{x_0+iy_0} = e^{x_0}(\cos y_0 + i\sin y_0), \\ e^N &= \sum_{j=0}^{\infty}\frac{N^j}{j!}. \end{aligned} \tag{A.100}$$

And we have:

$$\begin{aligned} N^2 &= z_1^2 + z_2^2 + z_3^2 = n = n_1 + in_2, \\ n_1 &= x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 - y_3^2;\ n_2 = 2(x_1y_1 + x_2y_2 + x_3y_3). \end{aligned} \tag{A.101}$$

Now we use a square root r of n (defined up to a sign):

$$\begin{aligned} n = r^2;\ r = r_1 + ir_2;\ r_1 &= \sqrt{\frac{n_1 + \sqrt{n_1^2 + n_2^2}}{2}};\ r_2 = \sqrt{\frac{-n_1 + \sqrt{n_1^2 + n_2^2}}{2}}, \\ N^2 &= r^2 = n;\ N^3 = N^2N = nN;\ N^4 = n^2; N^5 = n^2N;\ N^6 = n^3, ... \\ e^N &= \mathbf{C}(n) + \mathbf{S}(n)N, \end{aligned} \tag{A.102}$$

With:

$$\begin{aligned} \mathbf{C}(n) &:= \sum_{j=0}^{\infty}\frac{n^j}{(2j)!} = \sum_{j=0}^{\infty}\frac{r^{2j}}{(2j)!} = \cosh(r) = \frac{e^r + e^{-r}}{2} \\ &= \cosh r_1 \cos r_2 + i \sinh r_1 \sin r_2 \end{aligned} \tag{A.103}$$

$$\begin{aligned} \mathbf{S}(n) &:= \sum_{j=0}^{\infty}\frac{n^j}{(2j+1)!} = \frac{1}{r}\sum_{j=0}^{\infty}\frac{r^{2j+1}}{(2j+1)!} = \frac{1}{r}\sinh(r) \\ &= \frac{e^r - e^{-r}}{2r} = \frac{1}{r}(\sinh r_1 \cos r_2 + i\cosh r_1 \sin r_2). \end{aligned} \tag{A.104}$$

And we have:

$$\begin{aligned} e^M &= e^{z_0}[\mathbf{C}(n) + \mathbf{S}(n)(z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3)] \\ &= \begin{pmatrix} e^{z_0}[\mathbf{C}(n) + \mathbf{S}(n)z_3] & e^{z_0}\mathbf{S}(n)(z_1 - iz_2) \\ e^{z_0}\mathbf{S}(n)(z_1 + iz_2) & e^{z_0}[\mathbf{C}(n) - \mathbf{S}(n)z_3] \end{pmatrix}. \end{aligned} \tag{A.105}$$

And we have:

$$1 = \cosh^2(r) - \sinh^2(r) = \mathbf{C}^2(n) - n\mathbf{S}^2(n), \tag{A.106}$$

$$\begin{aligned} \det(e^M) &= e^{z_0}[\mathbf{C}(n) + \mathbf{S}(n)z_3]e^{z_0}[\mathbf{C}(n) - \mathbf{S}(n)z_3] \\ &\quad - e^{z_0}\mathbf{S}(n)(z_1 + iz_2)e^{z_0}\mathbf{S}(n)(z_1 - iz_2) \\ &= e^{2z_0}[\mathbf{C}^2(n) - \mathbf{S}^2(n)z_3^2 - \mathbf{S}^2(n)(z_1^2 + z_2^2)] \\ &= e^{2z_0}[\mathbf{C}^2(n) - n\mathbf{S}^2(n)] = e^{2z_0}, \end{aligned} \tag{A.107}$$

$$\det(e^N) = \mathbf{C}^2(n) - n\mathbf{S}^2(n) = 1 \tag{A.108}$$

Hence e^N is an element of $SL(2,\mathbb{C})$ and $2y_0$ is the Yvon-Takabayasi angle of e^M. The conjugations give:

$$\widehat{M} = \bar{z}_0 - \bar{z}_1\sigma_1 - \bar{z}_2\sigma_2 - \bar{z}_3\sigma_3;\ \bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2 = \bar{n} = \bar{r}^2,$$

$$\widehat{e^M} = e^{\widehat{M}} = e^{\bar{z}_0}[\mathbf{C}(\bar{n}) - \mathbf{S}(\bar{n})(\bar{z}_1\sigma_1 + \bar{z}_2\sigma_2 + \bar{z}_3\sigma_3)] \tag{A.109}$$

$$\overline{M} = z_0 - z_1\sigma_1 - z_2\sigma_2 - z_3\sigma_3;\ M + \overline{M} = 2z_0 = \mathrm{tr}(M),$$

$$\overline{e^M} = e^{\overline{M}} = e^{z_0}[\mathbf{C}(n) - \mathbf{S}(n)(z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3)], \tag{A.110}$$

$$\widetilde{M} = M^\dagger = \widehat{\overline{M}} = \bar{z}_0 + \bar{z}_1\sigma_1 + \bar{z}_2\sigma_2 + \bar{z}_3\sigma_3;\ \bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2 = \bar{n} = \bar{r}^2,$$

$$\widetilde{e^M} = e^{M^\dagger} = e^{\bar{z}_0}[\mathbf{C}(\bar{n}) + \mathbf{S}(\bar{n})(\bar{z}_1\sigma_1 + \bar{z}_2\sigma_2 + \bar{z}_3\sigma_3)]. \tag{A.111}$$

For $\phi = e^\chi$ and $\chi = \sqrt{2}\begin{pmatrix} \xi_1 & -\bar{\eta}_2 \\ \xi_2 & \bar{\eta}_1 \end{pmatrix}$ we have:

$$z_0 + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3 = \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \xi_1 & -\bar{\eta}_2 \\ \xi_2 & \bar{\eta}_1 \end{pmatrix}$$

$$\begin{aligned} z_0 &= \frac{1}{\sqrt{2}}(\xi_1 + \bar{\eta}_1);\quad z_3 = \frac{1}{\sqrt{2}}(\xi_1 - \bar{\eta}_1), \\ z_1 &= \frac{1}{\sqrt{2}}(\xi_2 - \bar{\eta}_2);\quad z_2 = \frac{1}{i\sqrt{2}}(\xi_2 + \bar{\eta}_2). \end{aligned} \tag{A.112}$$

Since ρ is the modulus and β is an argument of $\det(\phi)$, we have:

$$\begin{aligned} \rho &= e^{2x_0};\quad 2x_0 = z_0 + \bar{z}_0 = \frac{1}{\sqrt{2}}(\xi_1 + \bar{\xi}_1 + \eta_1 + \bar{\eta}_1) \\ 2y_0 &= \frac{i}{\sqrt{2}}(-\xi_1 + \bar{\xi}_1 + \eta_1 - \bar{\eta}_1), \end{aligned} \tag{A.113}$$

$$n = z_1^2 + z_2^2 + z_3^2 = -2\xi_2\bar{\eta}_2 + \frac{1}{2}(\xi_1^2 - 2\xi_1\bar{\eta}_1 + \bar{\eta}_1^2). \tag{A.114}$$

A.3.9 Projectors

The center of Cl_3, that means the set of elements which commute which all others, is $\mathbb{C}$. The result is that Cl_3 is also a complex linear space noted

$M_2(\mathbb{C})$ when we look at the matrices, and noted $GL(2,\mathbb{C})$ when we look at the linear applications from $\mathbb{C}^2$ into $\mathbb{C}^2$. A basis of that linear space is made of four elements:

$$p_r := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1+\sigma_3}{2}; \quad p_l := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1-\sigma_3}{2}, \tag{A.115}$$

$$p_b := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{\sigma_1 - i\sigma_2}{2}; \quad p_h := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{\sigma_1 + i\sigma_2}{2}. \tag{A.116}$$

The two first terms of that basis (p_r, p_l, p_b, p_h) are projectors, since they satisfy:

$$p_r^2 = p_r; \quad p_l^2 = p_l; \quad p_r p_l = p_l p_r = 0; \quad p_r + p_l = 1; \quad p_r - p_l = \sigma_3. \tag{A.117}$$

Those projectors are parts of the centraliser $\mathbb{A}$ of σ_3, hence we have:

$$a \in \mathbb{A} \Leftrightarrow a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = a_1 p_r + a_2 p_l \tag{A.118}$$

$$e^a = e^{a_1 p_r + a_2 p_l} = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix} = e^{a_1} p_r + e^{a_2} p_l. \tag{A.119}$$

So those projectors linearize (partially) the exponential function. Two other projectors are important for calculations with spherical coordinates:

$$p^+ := \frac{1+\vec{u}}{2}; \quad p^- := \frac{1-\vec{u}}{2}, \tag{A.120}$$

where we use the following vectors:

$$\vec{\mathrm{x}} := \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3 = r\vec{u},$$
$$\vec{u} := \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}; \quad r := \sqrt{(\mathrm{x}^1)^2 + (\mathrm{x}^2)^2 + (\mathrm{x}^3)^2}. \tag{A.121}$$

We have:

$$\vec{u}^2 = 1; \quad \vec{u} = S\sigma_3 S^{-1}, \quad S := e^{-\frac{\varphi}{2} i_3} e^{-\frac{\theta}{2} i_2}; \quad p^+ - p^- = \vec{u}, \tag{A.122}$$

$$p^+p^+ = p^+; \quad p^-p^- = p^-; \quad p^+p^- = p^-p^+ = 0; \; p^+ + p^- = 1, \tag{A.123}$$

$$p^+p_l = p_r p^- - \frac{u_3+\sigma_3}{2}; \quad p^-p_l = p_r p^- + \frac{u_3 - \sigma_3}{2},$$
$$p^+p_r = p_l p^+ + \frac{u_3+\sigma_3}{2} : \quad p^-p_r = p_l p^- - \frac{u_3-\sigma_3}{2}. \tag{A.124}$$

We also have:

$$p^+p_lp^+ = \frac{1-u_3}{2}p^+; \quad p^+p_lp^- = \frac{1}{4}[-\sigma_3 + u_3\vec{u} + i(u_1\sigma_2 - u_2\sigma_1)],$$
$$p^-p_lp^+ = \frac{1}{4}[-\sigma_3 + u_3\vec{u} - i(u_1\sigma_2 - u_2\sigma_1)]; \quad p^-p_lp^- = \frac{1+u_3}{2}p^-,$$
$$p^+p_rp^+ = \frac{1+u_3}{2}p^+; \quad p^+p_rp^- = \frac{1}{4}[\sigma_3 - u_3\vec{u} - i(u_1\sigma_2 - u_2\sigma_1)], \tag{A.125}$$
$$p^-p_rp^+ = \frac{1}{4}[\sigma_3 - u_3\vec{u} + i(u_1\sigma_2 - u_2\sigma_1)]; \quad p^-p_rp^- = \frac{1-u_3}{2}p^-.$$

A.3.10 Krüger's identities

These identities are necessary to solve, in Chapter C, the improved wave equation in spherical coordinates. We let:

$$
\begin{aligned}
&x^1 =: r\sin\theta\cos\varphi;\quad x^2 =: r\sin\theta\sin\varphi;\quad x^3 =: r\cos\theta,\\
&i_1 = \sigma_{23} := i\sigma_1\ ;\quad i_2 = \sigma_{31} := i\sigma_2\ ;\quad i_3 = \sigma_{12} := i\sigma_3,\\
&u_1 : = \sin\theta\cos\varphi;\quad u_2 := \sin\theta\sin\varphi;\quad u_3 = \cos\theta;\quad \vec{u} := u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3,\\
&S := e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2}\ ;\quad \Omega = \widehat{\Omega} := r^{-1}(\sin\theta)^{-\frac{1}{2}}S, \qquad \text{(A.126)}\\
&\vec{\partial}' := \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi;\quad \vec{\partial} := \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3.
\end{aligned}
$$

With (A.59) and (A.61), remark that we have $\nabla = \partial_0 - \vec{\partial} = \nabla_1 + i_2\nabla_2$. The derivation of a composite function gives:

$$
\partial_j = \frac{\partial r}{\partial \mathrm{x}^j}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \mathrm{x}^j}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial \mathrm{x}^j}\frac{\partial}{\partial \varphi}. \qquad \text{(A.127)}
$$

We have:

$$
\begin{aligned}
&r = [(\mathrm{x}^1)^2 + (\mathrm{x}^2)^2 + (\mathrm{x}^3)^2]^{\frac{1}{2}},\\
&\theta = \mathrm{Arctan}\Big[\frac{[(\mathrm{x}^1)^2 + \mathrm{x}^2)^2]^{\frac{1}{2}}}{\mathrm{x}^3}\Big];\quad \varphi = \mathrm{Arctan}\Big[\frac{\mathrm{x}^2}{\mathrm{x}^1}\Big], \qquad \text{(A.128)}\\
&\frac{\partial r}{\partial \mathrm{x}^j} = \frac{\mathrm{x}^j}{r};\quad \frac{\partial\theta}{\partial \mathrm{x}^1} = \frac{\cos\theta\cos\varphi}{r};\quad \frac{\partial\theta}{\partial \mathrm{x}^2} = \frac{\cos\theta\sin\varphi}{r};\quad \frac{\partial\theta}{\partial \mathrm{x}^3} = -\frac{\sin\theta}{r},\\
&\frac{\partial\varphi}{\partial \mathrm{x}^1} = -\frac{\sin\varphi}{r\sin\theta};\quad \frac{\partial\varphi}{\partial \mathrm{x}^2} = \frac{\cos\varphi}{r\sin\theta};\quad \frac{\partial\varphi}{\partial \mathrm{x}^3} = 0. \qquad \text{(A.129)}
\end{aligned}
$$

We then get:

$$
\vec{\partial} = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix};\quad \partial_3 = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \text{(A.130)}
$$

$$
\begin{aligned}
&\partial_1 + i\partial_2 = e^{i\varphi}\Big(\partial_r + \frac{\cos\theta}{r}\partial_\theta + \frac{i}{r\sin\theta}\partial_\varphi\Big),\\
&\partial_1 - i\partial_2 = e^{-i\varphi}\Big(\partial_r + \frac{\cos\theta}{r}\partial_\theta - \frac{i}{r\sin\theta}\partial_\varphi\Big). \qquad \text{(A.131)}
\end{aligned}
$$

That gives:

$$
\vec{\partial} = S\sigma_3S^{-1}\partial_r + S\sigma_1S^{-1}\frac{1}{r}\partial_\theta + S\sigma_2S^{-1}\frac{1}{r\sin\theta}\partial_\varphi. \qquad \text{(A.132)}
$$

Next we get:

$$
\begin{aligned}
\vec{\partial}(\Omega\widehat{\phi}) = {} & S\sigma_3S^{-1}\partial_r\Big(\frac{1}{r\sqrt{\sin\theta}}S\widehat{\phi}\Big)\\
& + S\sigma_1S^{-1}\frac{1}{r}\partial_\theta\Big(\frac{1}{r\sqrt{\sin\theta}}S\widehat{\phi}\Big) \qquad \text{(A.133)}\\
& + S\sigma_2S^{-1}\frac{1}{r\sin\theta}\partial_\varphi\Big(\frac{1}{r\sqrt{\sin\theta}}S\widehat{\phi}\Big).
\end{aligned}
$$

That gives:

$$\begin{aligned}\vec{\partial}(\Omega\widehat{\phi}) &= \Omega\Big(\sigma_3\frac{-1}{r} - \sigma_1\frac{e^\theta i_2}{2r\sin\theta} + \sigma_2\frac{S^{-1}(-i_3)S}{2r\sin\theta}\Big)\widehat{\phi} \\ &+ \Omega\Big(\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big)\widehat{\phi} \\ &= 0 + \Omega\Big(\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big)\widehat{\phi} \\ &= \Omega\vec{\partial}'\widehat{\phi}. \end{aligned} \tag{A.134}$$

Next, since Ω does not depend on $\mathrm{x}^0 = ct$, and with:

$$\nabla = \partial_0 - \vec{\partial};\ \ \nabla' = \partial_0 - \vec{\partial}' = \partial_0 - (\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big)$$

$$\nabla' = \begin{pmatrix} \partial_0 - \partial_r & -\frac{1}{r}\big(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi\big) \\ -\frac{1}{r}\big(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\big) & \partial_0 + \partial_r \end{pmatrix} = \nabla'_1 + i_2\nabla'_2,$$

$$\nabla'_1 = \begin{pmatrix} \partial_0 - \partial_r & 0 \\ 0 & \partial_0 + \partial_r \end{pmatrix} = \partial_0 - \sigma_3\partial_r, \tag{A.135}$$

$$\nabla'_2 = \begin{pmatrix} \frac{i}{r\sin\theta}\partial_\varphi + \frac{1}{r}\partial_\theta & 0 \\ 0 & \frac{i}{r\sin\theta}\partial_\varphi - \frac{1}{r}\partial_\theta \end{pmatrix} = \frac{i}{r\sin\theta}\partial_\varphi + \frac{1}{r}\sigma_3\partial_\theta. \tag{A.136}$$

Thus we obtain the Krüger identities [86]:

$$\nabla(\Omega\widehat{\phi}) = \Omega\nabla'\widehat{\phi} = \Omega(\nabla'_1 + i_2\nabla'_2)\widehat{\phi}. \tag{A.137}$$

Laplacian with spherical coordinates

The Laplacian of a A term satisfies:

$$\begin{aligned}\Box A &= (\nabla\widehat{\nabla})A = \nabla(\widehat{\nabla}A) \\ &= (\partial_0 - \vec{\partial})(\partial_0 A + \vec{\partial}A) = \partial_0\partial_0 A - \vec{\partial}\partial_0 A + \partial_0\vec{\partial}A - \vec{\partial}\vec{\partial}A \\ &= \partial_0\partial_0 A - \vec{\partial}\vec{\partial}A, \end{aligned} \tag{A.138}$$

We also have:

$$\begin{aligned}&\vec{\partial}\vec{\partial}A = \vec{\partial}[\vec{\partial}(\Omega B)] = \vec{\partial}[\Omega(\vec{\partial}B)];\ A = \Omega B;\ B = \Omega^{-1}A, \\ &\vec{\partial}\vec{\partial}A = \Omega\vec{\partial}'(\vec{\partial}'B) = \Omega[(\vec{\partial}'\vec{\partial}')B] = \Omega\vec{\partial}'\vec{\partial}'(\Omega^{-1}A) \end{aligned} \tag{A.139}$$

$$\begin{aligned}\vec{\partial}'\vec{\partial}' &= \Big(\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big)\Big(\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big) \\ &= \sigma_3^2\partial_r\partial_r + \sigma_3\Big(\frac{1}{r}\sigma_1\partial_\theta\Big) + \sigma_3\partial_r\Big(\frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big)\Big] \\ &+ \frac{1}{r}\Big[\sigma_1\sigma_3\partial_\theta\partial_r + \frac{1}{r}\sigma_1^2\partial_\theta\partial_\theta + \sigma_1\partial_\theta\Big(\frac{\sigma_2}{r\sin\theta}\partial_\varphi\Big) \\ &+ \frac{1}{r\sin\theta}\Big[\sigma_2\sigma_3\partial_\varphi\partial_r + \frac{\sigma_1}{r}\partial_\varphi\partial_\theta + \frac{1}{r\sin\theta}\sigma_2^2\partial_\varphi\partial_\varphi\Big]. \end{aligned} \tag{A.140}$$

We then obtain:

$$\vec{\partial}'\vec{\partial}' = \partial_r\partial_r + \frac{1}{r^2}\partial_\theta\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\varphi\partial_\varphi - \frac{i_2}{r^2}\partial_\theta - \frac{i_3}{r^2\sin^2\theta}e^{\theta i_2}\partial_\varphi. \quad (A.141)$$

For a physical quantity A which depends neither on time nor on the φ angle, we have:

$$\begin{aligned}\Box A &= \Omega\Big[-\partial_r\partial_r(\Omega^{-1}A) - \frac{1}{r^2\sin^2\theta}\partial_\theta\partial_\theta(\Omega^{-1}A) + \frac{i_2}{r^2\sin^2\theta}\partial_\theta(\Omega^{-1}A)\Big]\\ &= \Omega\Big[-\partial_r\partial_r(\Omega^{-1}A) + \frac{i_2}{r^2\sin^2\theta}\partial_\theta[(\Omega^{-1}A) + i_2\partial_\theta(\Omega^{-1}A)]\Big]. \quad (A.142)\end{aligned}$$

A.3.11 Laws of electromagnetism with Cl_3

The simplest framework to express the complicated laws of electromagnetism is also Cl_3: We call $A = A^\dagger$ a space-time vector "potential" and we calculate, with the Dirac operator $\nabla = \sigma^\mu\partial_\mu = \nabla^\dagger$, as well as $A = A^0 + \vec{A}$, the electromagnetic field $F = \vec{E} + i\vec{H}$ associated to this potential. It is purely a 2-vector in space-time, the sum of the electric field $\vec{E}$ and the magnetic field $i\vec{H}$, an axial vector. The derivation of the potential A gives:

$$\begin{aligned}\vec{E} + i\vec{H} = F = \nabla\widehat{A} &= (\partial_0 - \vec{\partial})(A^0 - \vec{A}) = \partial_0 A^0 - \partial_0\vec{A} - \vec{\partial}A^0 + \vec{\partial}\vec{A}\\ &= (\partial_0 A^0 + \vec{\partial}\cdot\vec{A}) + (-\partial_0\vec{A} - \vec{\partial}A^0) + i\vec{\partial}\times\vec{A},\\ \vec{E} &= -\partial_0\vec{A} - \vec{\partial}A^0;\ \vec{H} = \vec{\partial}\times\vec{A}. \quad (A.143)\end{aligned}$$

Thus we obtain a bivector F, sum of only a vector $\vec{E}$ and of a pseudovector $i\vec{H}$, if and only if the Lorentz gauge condition $0 = \partial_0 A^0 + \vec{\partial}\cdot\vec{A}$ is satisfied. We also have:

$$\begin{aligned}\overline{F} &= (\partial_0 A^0 + \vec{\partial}\cdot\vec{A}) - (-\partial_0\vec{A} - \vec{\partial}A^0) - i\vec{\partial}\times\vec{A},\\ F - \overline{F} &= 2[(-\partial_0\vec{A} - \vec{\partial}A^0) + i\vec{\partial}\times\vec{A}] = 2(\vec{E} + i\vec{H}) = 2F. \quad (A.144)\end{aligned}$$

We hence let:

$$F := \vec{E} + i\vec{H} := \frac{1}{2}(\nabla\widehat{A} - \overline{\nabla\widehat{A}}) = \frac{1}{2}(\nabla\widehat{A} - A^\dagger\overline{\nabla}). \quad (A.145)$$

That implies:

$$\widehat{F} = -F^\dagger = -\vec{E} + i\vec{H} = \frac{1}{2}(\widehat{\nabla}A - \widehat{A}\nabla). \quad (A.146)$$

$$\nabla\widehat{F} = \frac{1}{2}(\nabla\widehat{\nabla}A - \nabla\widehat{A}\nabla) = \frac{1}{2}(\Box A - \nabla\widehat{A}\nabla), \quad (A.147)$$

$$\begin{aligned}\Box := \nabla\widehat{\nabla} &= (\partial_0 - \vec{\partial})(\partial_0 + \vec{\partial}) = \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3\\ &= (\partial_0 + \vec{\partial})(\partial_0 - \vec{\partial}) = \widehat{\nabla}\nabla. \quad (A.148)\end{aligned}$$

Moreover we get:

$$\nabla\widehat{F} = \nabla(\widehat{\nabla}A) = (\nabla\widehat{\nabla})A = \Box A = A\Box, \tag{A.149}$$

$$\begin{aligned}(\nabla\widehat{F})^\dagger &= \frac{1}{2}(\nabla\widehat{\nabla}A - \nabla\widehat{A}\nabla)^\dagger = \frac{1}{2}(\Box A - \nabla\widehat{A}\nabla)^\dagger \\ &= \frac{1}{2}(A\Box - \nabla\widehat{A}\nabla) = \frac{1}{2}(\Box A - \nabla\widehat{A}\nabla) = \nabla\widehat{F}. \end{aligned}\tag{A.150}$$

Therefore $\mathrm{j} = \nabla\widehat{F}$ is also a space-time covariant vector, called a "current". We also have:

$$\mathrm{j} = \mathrm{j}^0 - \vec{\mathrm{j}} = \nabla\widehat{F} = (\partial_0 - \vec{\partial})(-\vec{E} + i\vec{H}) \tag{A.151}$$

$$= \vec{\partial}\cdot\vec{E} + (-\partial_0\vec{E} + \vec{\partial}\times\vec{H}) + i(\partial_0\vec{H} + \vec{\partial}\times\vec{E}) - i\vec{\partial}\cdot\vec{H}. \tag{A.152}$$

Separating the scalar, vector, pseudovector and pseudoscalar part, we obtain Maxwell's equations:

$$\mathrm{j}_0 = \vec{\partial}\cdot\vec{E}, \tag{A.153}$$

$$\vec{\mathrm{j}} = \partial_0\vec{E} - \vec{\partial}\times\vec{H}, \tag{A.154}$$

$$0 = \partial_0\vec{H} + \vec{\partial}\times\vec{E}, \tag{A.155}$$

$$0 = \vec{\partial}\cdot\vec{H}. \tag{A.156}$$

We also have:

$$\begin{aligned}-\nabla\widehat{A}\nabla &= (\vec{E} + i\vec{H})(-\partial_0 + \vec{\partial}) \\ &= \vec{\partial}\cdot\vec{E} + (-\partial_0\vec{E} + \vec{\partial}\times\vec{H}) - i(\partial_0\vec{H} + \vec{\partial}\times\vec{E}) + i\vec{\partial}\cdot\vec{H} \\ &= \vec{\partial}\cdot\vec{E} - (\partial_0\vec{E} - \vec{\partial}\times\vec{H}) = \mathrm{j} = \nabla\widehat{F} = \Box A. \end{aligned}\tag{A.157}$$

Another form of electromagnetism exists when $A = A^\dagger$ (space-time vector) is replaced by $iB = -iB^\dagger$ (space-time pseu-dovector):

$$\begin{aligned}\vec{E}_m + i\vec{H}_m = F_m &= \nabla\widehat{iB} = -i\nabla\widehat{B} \\ &= -i[(\partial_0 - \vec{\partial})(B^0 - \vec{B})] = -i[\partial_0 B^0 - \partial_0\vec{B} - \vec{\partial}B^0 + \vec{\partial}\vec{B}] \\ &= -i(\partial_0 B^0 + \vec{\partial}\cdot\vec{B}) - i(-\partial_0\vec{B} - \vec{\partial}B^0) + \vec{\partial}\times\vec{B} \\ \vec{H}_m &= \partial_0\vec{B} + \vec{\partial}B^0;\ \vec{E}_m = \vec{\partial}\times\vec{B}. \end{aligned}\tag{A.158}$$

Thus we have also obtained a bivector F_m, the sum of a vector $\vec{E}_m$ and of a pseudovector $i\vec{H}_m$, if and only if the gauge condition $0 = \partial_0 B^0 + \vec{\partial}\cdot\vec{B}$ is satisfied. We also have:

$$\begin{aligned}-iB\widehat{\nabla} = \overline{F}_m &= -i(\partial_0 B^0 + \vec{\partial}\cdot\vec{B}) + i(-\partial_0\vec{B} - \vec{\partial}B^0) - \vec{\partial}\times\vec{B} \\ F_m - \overline{F}_m &= 2[i(\partial_0\vec{B} + \vec{\partial}B^0) + \vec{\partial}\times\vec{B}] = 2(\vec{E}_m + i\vec{H}_m) = 2F_m. \end{aligned}\tag{A.159}$$

We thus let:

$$F_m := \vec{E}_m + i\vec{H}_m := \frac{1}{2}(\nabla\widehat{iB} - \overline{\nabla\widehat{iB}}) = \frac{i}{2}(-\nabla\widehat{B} + B^\dagger\overline{\nabla}). \qquad (A.160)$$

That implies:

$$\widehat{F}_m = -F_m^\dagger = -\vec{E}_m + i\vec{H}_m = \frac{i}{2}(\widehat{\nabla}B - \widehat{B}\nabla). \qquad (A.161)$$

$$\nabla\widehat{F}_m = \frac{i}{2}(\nabla\widehat{\nabla}B - \nabla\widehat{B}\nabla) = \frac{i}{2}(\Box B - \nabla\widehat{B}\nabla). \qquad (A.162)$$

Moreover we get:

$$\nabla\widehat{F}_m = i\nabla(\widehat{\nabla}B) = i(\nabla\widehat{\nabla})B = i\Box B = iB\Box, \qquad (A.163)$$

$$(\nabla\widehat{F}_m)^\dagger = (iB\Box)^\dagger = -i\Box B = -\nabla\widehat{F}_m. \qquad (A.164)$$

Therefore $-i\mathrm{k} = \nabla\widehat{F}_m$ is also a covariant pseudo-vector in space-time, called "magnetic current". We have also:

$$\begin{aligned} -i\mathrm{k} &= -i\mathrm{k}^0 + i\vec{\mathrm{k}} = \nabla\widehat{F}_m = (\partial_0 - \vec{\partial})(-\vec{E}_m + i\vec{H}_m) \qquad (A.165)\\ &= \vec{\partial}\cdot\vec{E}_m + (-\partial_0\vec{E}_m + \vec{\partial}\times\vec{H}_m) + i(\partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m) - i\vec{\partial}\cdot\vec{H}_m. \end{aligned}$$

Separating the scalar, vector, pseudo-vector and pseudo-scalar part, we obtain Maxwell's magnetic equations (your attention please to the sign!):

$$0 = \vec{\partial}\cdot\vec{E}_m, \qquad (A.166)$$

$$0 = -\partial_0\vec{E}_m + \vec{\partial}\times\vec{H}_m, \qquad (A.167)$$

$$\vec{\mathrm{k}} = \partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m, \qquad (A.168)$$

$$\mathrm{k}_0 = \vec{\partial}\cdot\vec{H}_m. \qquad (A.169)$$

We also have:

$$\begin{aligned} -i\nabla\widehat{B}\nabla &= (\vec{E}_m + i\vec{H}_m)(\partial_0 - \vec{\partial}) \qquad (A.170)\\ &= -\vec{\partial}\cdot\vec{E}_m + (\partial_0\vec{E}_m - \vec{\partial}\times\vec{H}_m) + i(\partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m) - i\vec{\nabla}\cdot\vec{H}_m\\ &= -i\vec{\partial}\cdot\vec{H}_m + i(\partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m) = -i\mathrm{k}_0 + i\vec{\mathrm{k}} = \nabla\widehat{F}_m = \Box iB. \end{aligned}$$

The full electromagnetism, with both electric charges and magnetic monopoles, has thus the simple rules:

$$F = \nabla\widehat{A + iB};\ \mathrm{j} - i\mathrm{k} = \nabla\widehat{F};\ \Box(A + iB) = \mathrm{j} - i\mathrm{k}. \qquad (A.171)$$

A.4 Tensor densities

We use the electron wave as:

$$\phi = \sqrt{2}(\xi\ \ \widehat{\eta}) = \sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}, \qquad (A.172)$$

which gives:

$$\widehat{\phi} = \sqrt{2}(\eta \ \ \widehat{\xi}) = \sqrt{2}\begin{pmatrix} \eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^* \end{pmatrix}, \tag{A.173}$$

$$\phi^\dagger = \sqrt{2}\begin{pmatrix} \xi^\dagger \\ \widehat{\eta}^\dagger \end{pmatrix} \ ; \ \overline{\phi} = \sqrt{2}\begin{pmatrix} \eta^\dagger \\ \widehat{\xi}^\dagger \end{pmatrix} = \sqrt{2}\begin{pmatrix} \eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1 \end{pmatrix}. \tag{A.174}$$

A.4.1 Calculation of Ω_1 and Ω_2, and the determinant

We have with Dirac matrices:

$$\Omega_1 = \overline{\psi}\psi = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \eta^\dagger\xi + \xi^\dagger\eta = \eta_1^*\xi_1 + \eta_2^*\xi_2 + \xi_1^*\eta_1 + \xi_2^*\eta_2,$$

$$\Omega_2 = \overline{\psi}(-i\gamma_5)\psi = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix}\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = -i\eta^\dagger\xi + i\xi^\dagger\eta, \tag{A.175}$$

$$\Omega_1 + i\Omega_2 = 2\eta^\dagger\xi;\ \Omega_1 - i\Omega_2 = 2\xi^\dagger\eta;\ \Omega_2 = i(-\eta_1^*\xi_1 - \eta_2^*\xi_2 + \xi_1^*\eta_1 + \xi_2^*\eta_2).$$

Whereas with the Pauli algebra we obtain:

$$\phi\overline{\phi} = \overline{\phi}\phi = \det(\phi) = 2(\xi_1\eta_1^* + \xi_2\eta_2^*) = 2\eta^\dagger\xi = \Omega_1 + i\Omega_2, \tag{A.176}$$

$$\widehat{\phi}\phi^\dagger = \phi^\dagger\widehat{\phi} = \det(\phi)^* = 2(\eta_1\xi_1^* + \eta_2\xi_2^*) = 2\xi^\dagger\eta = \Omega_1 - i\Omega_2. \tag{A.177}$$

We also obtain, for any ϕ with value in Cl_3:

$$\phi[\det(\phi)^{-1}\overline{\phi}] = 1;\ \phi^{-1} = \det(\phi)^{-1}\overline{\phi}. \tag{A.178}$$

The second reason of our interest in Cl_3 comes from the subset Cl_3^* of the invertible elements, where any M satisfies $\det(M) \neq 0$. It forms a multiplicative Lie group. Moreover that Lie group has the Cl_3 algebra as Lie algebra. The Cl_3^* group is the invariance group used throughout this book. Most of the progress brought about by Clifford algebra in quantum physics comes from the use of this multiplication, which was not computable in the Dirac theory using a function with value in $\mathbb{C}^4$ (not a ring).

A.4.2 Calculation of D_μ^ν

This calculation also gives the R_μ^ν of 1.1.2. It is enough to replace ϕ by M, that means $\sqrt{2}\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We first calculate the components of the vector $\mathrm{D}_0 = \mathrm{J}$. With the Dirac matrices we have:

$$\mathrm{D}_0^\mu = \mathrm{J}^\mu = \overline{\psi}\gamma^\mu\psi = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \widehat{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \eta^\dagger\sigma^\mu\eta + \xi^\dagger\widehat{\sigma}^\mu\xi. \tag{A.179}$$

We see here that the J current is the sum of the D_R and D_L currents:

$$\mathrm{D}_0^\mu = \mathrm{J}^\mu = \mathrm{D}_R^\mu + \mathrm{D}_L^\mu;\ \mathrm{D}_R^\mu = \xi^\dagger\widehat{\sigma}^\mu\xi;\ \mathrm{D}_L^\mu = \eta^\dagger\sigma^\mu\eta, \tag{A.180}$$

That comes from:

$$\begin{aligned}\mathrm{D}_R &:= \phi\frac{1+\sigma_3}{2}\phi^\dagger = 2\begin{pmatrix}\xi_1 & -\eta_2^*\\ \xi_2 & \eta_1^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1^* & \xi_2^*\\ -\eta_2 & \eta_1\end{pmatrix} = 2\begin{pmatrix}\xi_1^*\xi_1 & \xi_2^*\xi_1\\ \xi_1^*\xi_2 & \xi_2^*\xi_2\end{pmatrix}\\ &= \xi_1^*\xi_1(1+\sigma_3) + \xi_2^*\xi_2(1-\sigma_3) + \xi_2^*\xi_1(\sigma_1+i\sigma_2) + \xi_1^*\xi_2(\sigma_1 - i\sigma_2)\\ &= \xi^\dagger\xi + (\xi^\dagger\sigma_3\xi\sigma_3 + (\xi^\dagger\sigma_1\xi)\sigma_1 + (\xi^\dagger\sigma_2\xi)\sigma_2 = (\xi^\dagger\widehat{\sigma}^\mu\xi)\sigma_\mu. \qquad (A.181)\end{aligned}$$

Similarly we have:

$$\begin{aligned}\mathrm{D}_L &:= \phi\frac{1-\sigma_3}{2}\phi^\dagger = 2\begin{pmatrix}\xi_1 & -\eta_2^*\\ \xi_2 & \eta_1^*\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi_1^* & \xi_2^*\\ -\eta_2 & \eta_1\end{pmatrix} = 2\begin{pmatrix}\eta_2^*\eta_2 & -\eta_2^*\eta_1\\ -\eta_1^*\eta_2 & \eta_1^*\eta_1\end{pmatrix}\\ &= \eta_2^*\eta_2(1+\sigma_3) + \eta_1^*\eta_1(1-\sigma_3) - \eta_2^*\eta_1(\sigma_1+i\sigma_2) - \eta_1^*\eta_2(\sigma_1 - i\sigma_2)\\ &= \eta^\dagger\eta + (\eta^\dagger\sigma^3\eta\sigma_3 + (\eta^\dagger\sigma^1\eta)\sigma_1 + (\eta^\dagger\sigma^2\eta)\sigma_2 = (\eta^\dagger\sigma^\mu\eta)\sigma_\mu. \qquad (A.182)\end{aligned}$$

We have:

$$\mathrm{D}_R + \mathrm{D}_L = \phi(\frac{1+\sigma_3}{2} + \frac{1-\sigma_3}{2})\phi^\dagger = \phi\phi^\dagger = \mathrm{D}_0 = \mathrm{J}, \qquad (A.183)$$

$$\mathrm{D}_R - \mathrm{D}_L = \phi(\frac{1+\sigma_3}{2} - \frac{1-\sigma_3}{2})\phi^\dagger = \phi\sigma_3\phi^\dagger = \mathrm{D}_3 = \mathrm{K}. \qquad (A.184)$$

That gives:

$$\mathrm{D}_0^0 = \mathrm{J}^0 = \xi^\dagger\sigma_0\xi + \eta^\dagger\sigma_0\eta = \xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*, \qquad (A.185)$$

$$\mathrm{D}_0^1 = \mathrm{J}^1 = \xi^\dagger\sigma_1\xi - \eta^\dagger\sigma_1\eta = \xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*, \qquad (A.186)$$

$$\mathrm{D}_0^2 = \mathrm{J}^2 = \xi^\dagger\sigma_2\xi - \eta^\dagger\sigma_2\eta = i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*), \qquad (A.187)$$

$$\mathrm{D}_0^3 = \mathrm{J}^3 = \xi^\dagger\sigma_3\xi - \eta^\dagger\sigma_3\eta = \xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*. \qquad (A.188)$$

Now beginning with the tensors known in the formalism of Dirac matrices, we use (more details in B.1.1):

$$\gamma^0\gamma_5 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} = \begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}, \qquad (A.189)$$

$$\gamma^j\gamma_5 = \begin{pmatrix}0 & -\sigma_j\\ \sigma_j & 0\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} = \begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix},\ j = 1,2,3, \qquad (A.190)$$

$$\mathrm{K} = \mathrm{K}^\mu\sigma_\mu = (\xi^\dagger\widehat{\sigma}^\mu\xi)\sigma_\mu - (\eta^\dagger\sigma^\mu\eta)\sigma_\mu = \mathrm{D}_R - \mathrm{D}_L = \mathrm{D}_3. \qquad (A.191)$$

We thus obtain:

$$\mathrm{D}_3^0 = \xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*, \qquad (A.192)$$

$$\mathrm{D}_3^3 = \xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*, \qquad (A.193)$$

$$\mathrm{D}_3^1 = \xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*, \qquad (A.194)$$

$$\mathrm{D}_3^2 = i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*). \qquad (A.195)$$

For the calculation of components of D_1 and D_2, which are unknown in the old formalism of Dirac matrices, we directly use the Pauli algebra:

$$\begin{aligned}
&D_1 + iD_2 = \phi(\sigma_1 + i\sigma_2)\phi^\dagger \\
&= 2\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}\begin{pmatrix}0 & 2 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1\end{pmatrix} = 4\begin{pmatrix}-\eta_2\xi_1 & \eta_1\xi_1 \\ -\eta_2\xi_2 & \eta_1\xi_2\end{pmatrix} \qquad (A.196)\\
&= 2[-\eta_2\xi_1(1+\sigma_3) + \eta_1\xi_2(1-\sigma_3) + \eta_1\xi_1(\sigma_1 + i\sigma_2) - \eta_2\xi_2(\sigma_1 - i\sigma_2)] \\
&= 2[\widehat{\eta}^\dagger\xi + (\widehat{\eta}^\dagger\sigma_3\xi)\sigma_3 + (\widehat{\eta}^\dagger\sigma_1\xi)\sigma_1 + (\widehat{\eta}^\dagger\sigma_2\xi)\sigma_2 \\
&D_1 + iD_2 = 2(\widehat{\eta}^\dagger\widehat{\sigma}^\mu\xi)\sigma_\mu;\ \widehat{\eta}^\dagger = (-\eta_2 \ \ \eta_1). \qquad (A.197)
\end{aligned}$$

Similarly we get:

$$\begin{aligned}
&D_1 - iD_2 = \phi(\sigma_1 - i\sigma_2)\phi^\dagger \\
&= 2\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}\begin{pmatrix}0 & 0 \\ 2 & 0\end{pmatrix}\begin{pmatrix}\xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1\end{pmatrix} = 4\begin{pmatrix}-\xi_1^*\eta_2^* & -\xi_2^*\eta_2^* \\ \xi_1^*\eta_1^* & \xi_2^*\eta_1^*\end{pmatrix} \qquad (A.198)\\
&= 2[-\xi_1^*\eta_2^*(1+\sigma_3) + \xi_2^*\eta_1^*(1-\sigma_3) - \xi_2^*\eta_2^*(\sigma_1 + i\sigma_2) + \xi_2^*\eta_1^*(\sigma_1 - i\sigma_2)] \\
&= 2[\xi^\dagger\widehat{\eta} + (\xi^\dagger\sigma_3\widehat{\eta})\sigma_3 + (\xi^\dagger\sigma_1\widehat{\eta})\sigma_1 + (\xi^\dagger\sigma_2\widehat{\eta})\sigma_2 \\
&D_1 - iD_2 = 2(\xi^\dagger\widehat{\sigma}^\mu\widehat{\eta})\sigma_\mu. \qquad (A.199)
\end{aligned}$$

Hence by adding and subtracting we obtain:

$$D_1^0 = -\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1, \qquad (A.200)$$
$$D_1^3 = -\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1, \qquad (A.201)$$
$$D_1^1 = \xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1, \qquad (A.202)$$
$$D_1^2 = i(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1). \qquad (A.203)$$

$$D_2^0 = i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1), \qquad (A.204)$$
$$D_2^3 = i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1), \qquad (A.205)$$
$$D_2^1 = i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1), \qquad (A.206)$$
$$D_2^2 = \xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1. \qquad (A.207)$$

A.4.3 Calculation of S_k

For the calculation of $S = S_3$, the formalism of Dirac matrices gives with $S^{\mu\nu} = \overline{\psi}i\gamma^\mu\gamma^\nu\psi$:

$$E_3^3 := S_3^{12} = \overline{\psi}i\gamma^1\gamma^2\psi. \qquad (A.208)$$

And we have:

$$i\gamma^1\gamma^2 = i\begin{pmatrix}0 & -\sigma_1 \\ \sigma_1 & 0\end{pmatrix}\begin{pmatrix}0 & -\sigma_2 \\ \sigma_2 & 0\end{pmatrix} = \begin{pmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{pmatrix}, \qquad (A.209)$$

and similarly we have:

$$i\gamma^2\gamma^3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \;;\; i\gamma^3\gamma^1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{A.210}$$

We then get:

$$E_3^3 := S_3^{12} = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \eta^\dagger\sigma_3\xi + \xi^\dagger\sigma_3\eta \tag{A.211}$$
$$= \eta_1^*\xi_1 - \eta_2^*\xi_2 + \xi_1^*\eta_1 - \xi_2^*\eta_2.$$

And similarly:

$$E_3^1 := S_3^{23} = \eta^\dagger\sigma_1\xi + \xi^\dagger\sigma_1\eta = \eta_1^*\xi_2 + \eta_2^*\xi_1 + \xi_1^*\eta_2 + \xi_2^*\eta_1, \tag{A.212}$$
$$E_3^2 := S_3^{31} = \eta^\dagger\sigma_2\xi + \xi^\dagger\sigma_2\eta = i(-\eta_1^*\xi_2 + \eta_2^*\xi_1 - \xi_1^*\eta_2 + \xi_2^*\eta_1). \tag{A.213}$$

Next we have:

$$\gamma^1\gamma^0 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{A.214}$$

which gives:

$$H_3^1 := S_3^{10} = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix} \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = -i\eta^\dagger\sigma_1\xi + i\xi^\dagger\sigma_1\eta \tag{A.215}$$
$$= i(-\eta_1^*\xi_2 - \eta_2^*\xi_1 + \xi_1^*\eta_2 + \xi_2^*\eta_1).$$

Similarly we have:

$$H_3^2 := S_3^{20} = -i\eta^\dagger\sigma_2\xi + i\xi^\dagger\sigma_2\eta = -\eta_1^*\xi_2 - \eta_2^*\xi_1 + \xi_1^*\eta_2 - \xi_2^*\eta_1, \tag{A.216}$$
$$H_3^3 := S_3^{30} = -i\eta^\dagger\sigma_3\xi + \xi^\dagger\sigma_3\eta = i(-\eta_1^*\xi_1 + \eta_2^*\xi_2 + \xi_1^*\eta_1 - \xi_2^*\eta_2). \tag{A.217}$$

We deduce that:

$$S_3^{12} + iS_3^{30} = 2\eta^\dagger\sigma_3\xi = 2(\xi_1\eta_1^* - \xi_2\eta_2^*), \tag{A.218}$$
$$S_3^{23} + iS_3^{10} = 2\eta^\dagger\sigma_1\xi = 2(\xi_2\eta_1^* - \xi_1\eta_2^*), \tag{A.219}$$
$$S_3^{31} + iS_3^{20} = 2\eta^\dagger\sigma_2\xi = 2i(-\xi_2\eta_1^* + \xi_1\eta_2^*), \tag{A.220}$$
$$S_3^{23} + iS_3^{10} + iS_3^{31} - S_3^{20} = 4\xi_2\eta_1^*, \tag{A.221}$$
$$S_3^{23} + iS_3^{10} - iS_3^{31} + S_3^{20} = 4\xi_1\eta_2^*. \tag{A.222}$$

And we have:

$$\begin{aligned}
&S_3^{23}\sigma_1 + S_3^{31}\sigma_2 + S_3^{12}\sigma_3 + S_3^{10}i\sigma_1 + S_3^{20}i\sigma_2 + S_3^{30}i\sigma_3 \\
&= \begin{pmatrix} S_3^{12} + iS_3^{30} & S_3^{23} + iS_3^{10} - iS_3^{31} + S_3^{20} \\ S_3^{23} + iS_3^{10} + iS_3^{31} - S_3^{20} & -(S_3^{12} + iS_3^{30}) \end{pmatrix} \\
&= 2\begin{pmatrix} \xi_1\eta_1^* - \xi_2\eta_2^* & 2\xi_1\eta_2^* \\ 2\xi_2\eta_1^* & -(\xi_1\eta_1^* - \xi_2\eta_2^*) \end{pmatrix} \\
&= 2\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1 \end{pmatrix} \\
&= \phi\sigma_3\overline{\phi} = S = S_3, \qquad \text{(A.223)}
\end{aligned}$$

For the calculation of the components of S_1 and S_2, which are unknown in the formalism of Dirac matrices, we start directly from the Pauli algebra. We use:

$$R := \phi\frac{1+\sigma_3}{2} = \sqrt{2}\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix};\ L := \phi\frac{1-\sigma_3}{2} = \sqrt{2}\begin{pmatrix} 0 & -\eta_2^* \\ 0 & \eta_1^* \end{pmatrix},$$

$$S_R := \frac{1}{2}(S_1 + iS_2) = \phi\frac{1}{2}(\sigma_1 + i\sigma_2)\overline{\phi} = R\frac{1}{2}(\sigma_1 + i\sigma_2)\overline{R}, \qquad \text{(A.224)}$$

$$S_L := \frac{1}{2}(S_1 - iS_2) = \phi\frac{1}{2}(\sigma_1 - i\sigma_2)\overline{\phi} = L\frac{1}{2}(\sigma_1 - i\sigma_2)\overline{L}. \qquad \text{(A.225)}$$

We let:

$$S_R := \vec{E}_R + i\vec{H}_R;\ \vec{E}_R := E_R^j\sigma_j;\ \vec{H}_R := H_R^j\sigma_j, \qquad \text{(A.226)}$$

$$E_R^1 := S_R^{23};\ E_R^2 := S_R^{31};\ E_R^3 := S_R^{12};\ H_R^j := S_R^{j0}, \qquad \text{(A.227)}$$

$$S_L := \vec{E}_L + i\vec{H}_L;\ \vec{E}_L := E_L^j\sigma_j;\ \vec{H}_L := H_L^j\sigma_j, \qquad \text{(A.228)}$$

$$E_L^1 := S_L^{23};\ E_L^2 := S_L^{31};\ E_L^3 := S_L^{12};\ H_L^j := S_L^{j0}. \qquad \text{(A.229)}$$

That gives:

$$E_R^1 = \frac{1}{2}(\xi_1^2 - \xi_2^2 + \bar{\xi}_1^2 - \bar{\xi}_2^2);\ H_R^1 = \frac{i}{2}(-\xi_1^2 + \xi_2^2 + \bar{\xi}_1^2 - \bar{\xi}_2^2), \qquad \text{(A.230)}$$

$$E_R^2 = \frac{i}{2}(\xi_1^2 + \xi_2^2 - \bar{\xi}_1^2 - \bar{\xi}_2^2);\ H_R^2 = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \bar{\xi}_1^2 + \bar{\xi}_2^2), \qquad \text{(A.231)}$$

$$E_R^3 = -\xi_1\xi_2 - \bar{\xi}_1\bar{\xi}_2;\ H_R^3 = i(\xi_1\xi_2 - \bar{\xi}_1\bar{\xi}_2), \qquad \text{(A.232)}$$

$$E_L^1 = \frac{1}{2}(\eta_1^2 - \eta_2^2 + \bar{\eta}_1^2 - \bar{\eta}_2^2);\ H_L^1 = \frac{i}{2}(\eta_1^2 - \eta_2^2 - \bar{\eta}_1^2 + \bar{\eta}_2^2), \qquad \text{(A.233)}$$

$$E_L^2 = \frac{i}{2}(\eta_1^2 + \eta_2^2 - \bar{\eta}_1^2 - \bar{\eta}_2^2);\ H_L^2 = \frac{1}{2}(-\eta_1^2 - \eta_2^2 - \bar{\eta}_1^2 - \bar{\eta}_2^2), \qquad \text{(A.234)}$$

$$E_L^3 = -\eta_1\eta_2 - \bar{\eta}_1\bar{\eta}_2;\ H_L^3 = i(-\eta_1\eta_2 + \bar{\eta}_1\bar{\eta}_2). \qquad \text{(A.235)}$$

The link with S_1 and S_2 is as follows:

$$S_1 = \vec{E}_1 + i\vec{H}_1 = S_R + S_L = \vec{E}_R + i\vec{H}_R + \vec{E}_L + i\vec{H}_L,$$
$$\vec{E}_1 = \vec{E}_R + \vec{E}_L;\ \vec{H}_1 = \vec{H}_R + \vec{H}_L, \tag{A.236}$$
$$S_2 = \vec{E}_2 + i\vec{H}_2 = -iS_R + iS_L = -i(\vec{E}_R + i\vec{H}_R) + i(\vec{E}_L + i\vec{H}_L),$$
$$\vec{E}_2 = \vec{H}_R - \vec{H}_L;\ \vec{H}_2 = \vec{E}_L - \vec{E}_R. \tag{A.237}$$

We then get:

$$S_1^{12} = -\xi_1\xi_2 - \eta_1\eta_2 - \xi_1^*\xi_2^* - \eta_1^*\eta_2^*, \tag{A.238}$$
$$S_1^{30} = i(\xi_1\xi_2 - \eta_1\eta_2 - \xi_1^*\xi_2^* + \eta_1^*\eta_2^*), \tag{A.239}$$
$$S_1^{23} = \frac{1}{2}(\xi_1^2 - \xi_2^2 + \eta_1^2 - \eta_2^2 + \xi_1^{*2} - \xi_2^{*2} + \eta_1^{*2} - \eta_2^{*2}), \tag{A.240}$$
$$S_1^{10} = \frac{i}{2}(-\xi_1^2 + \xi_2^2 + \eta_1^2 - \eta_2^2 + \xi_1^{*2} - \xi_2^{*2} - \eta_1^{*2} + \eta_2^{*2}), \tag{A.241}$$
$$S_1^{20} = \frac{1}{2}(\xi_1^2 + \xi_2^2 - \eta_1^2 - \eta_2^2 + \xi_1^{*2} + \xi_2^{*2} - \eta_1^{*2} - \eta_2^{*2}), \tag{A.242}$$
$$S_1^{31} = \frac{i}{2}(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 - \xi_1^{*2} - \xi_2^{*2} - \eta_1^{*2} - \eta_2^{*2}). \tag{A.243}$$

Similarly we have:

$$S_2^{12} = i(\xi_1\xi_2 + \eta_1\eta_2 - \xi_1^*\xi_2^* - \eta_1^*\eta_2^*), \tag{A.244}$$
$$S_2^{30} = \xi_1\xi_2 - \eta_1\eta_2 + \xi_1^*\xi_2^* - \eta_1^*\eta_2^*, \tag{A.245}$$
$$S_2^{23} = \frac{i}{2}(-\xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2 + \xi_1^{*2} - \xi_2^{*2} + \eta_1^{*2} - \eta_2^{*2}), \tag{A.246}$$
$$S_2^{10} = \frac{1}{2}(-\xi_1^2 + \xi_2^2 + \eta_1^2 - \eta_2^2 - \xi_1^{*2} + \xi_2^{*2} + \eta_1^{*2} - \eta_2^{*2}), \tag{A.247}$$
$$S_2^{20} = \frac{i}{2}(-\xi_1^2 - \xi_2^2 + \eta_1^2 + \eta_2^2 + \xi_1^{*2} + \xi_2^{*2} - \eta_1^{*2} - \eta_2^{*2}), \tag{A.248}$$
$$S_2^{31} = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 + \xi_1^{*2} + \xi_2^{*2} + \eta_1^{*2} + \eta_2^{*2}). \tag{A.249}$$

We also obtain the number of 36 densities by remarking that there are 8 squares and $28 = 8 \times 7/2$ pairs.

A.4.4 Calculation of $\overline{D}_\mu^\nu$

Let ϕ be an invertible element in Cl_3^*, with determinant $\rho e^{i\beta}$. Let D and $\overline{D}$ be two similitudes satisfying:

$$D : \mathrm{x} \mapsto \mathrm{x}' = D(\mathrm{x}) = \phi\mathrm{x}\phi^\dagger\ ;\ \ \overline{D} : \mathrm{x} \mapsto \mathrm{x}' = \overline{D}(\mathrm{x}) = \overline{\phi}\mathrm{x}\widehat{\phi}. \tag{A.250}$$

Let P such that:

$$\phi = \sqrt{\rho}e^{i\frac{\beta}{2}}P, \tag{A.251}$$

and let Lo and $\overline{L}o$ two similitudes such that:

$$Lo : \mathrm{x} \mapsto \mathrm{x}' = Lo(\mathrm{x}) = P\mathrm{x}P^\dagger \ ; \ \ \overline{L}o : \mathrm{x} \mapsto \mathrm{x}' = \overline{L}o(\mathrm{x}) = \overline{P}\mathrm{x}\widehat{P}. \tag{A.252}$$

We have:

$$\rho e^{i\beta} = \det(\phi) = \phi\overline{\phi} = \sqrt{\rho}e^{i\frac{\beta}{2}}P\sqrt{\rho}e^{i\frac{\beta}{2}}\overline{P} = \rho e^{i\beta}P\overline{P}, \tag{A.253}$$

then we get:

$$P\overline{P} = 1 \ ; \ \ \overline{P} = P^{-1} \ ; \ \ \overline{L}o = Lo^{-1}. \tag{A.254}$$

P is thus an element in $SL(2,\mathbb{C})$ and Lo is a Lorentz transformation. We know that, for such a transformation, if we denote by (Lo) the matrix of Lo in an orthonormal basis and g the signature-matrix:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.255}$$

we have the following, M^t being the transposed matrix [8] of M:

$$(Lo)^{-1} = g(Lo)^t g \ ; \ \ (\overline{L}o)g = g(Lo)^t. \tag{A.256}$$

We also have:

$$D(\mathrm{x}) = \phi \mathrm{x}\phi^\dagger = \sqrt{\rho}e^{i\frac{\beta}{2}}P\mathrm{x}\sqrt{\rho}e^{-i\frac{\beta}{2}}P^\dagger = \rho P\mathrm{x}P^\dagger = \rho Lo(\mathrm{x}), \tag{A.257}$$

Hence:

$$D = \rho Lo \ ; \ \ (D) = \rho(Lo). \tag{A.258}$$

Similarly we have:

$$\overline{D}(\mathrm{x}) = \overline{M}\mathrm{x}\widehat{M} = \sqrt{\rho}e^{i\frac{\beta}{2}}\overline{P}\mathrm{x}\sqrt{\rho}e^{-i\frac{\beta}{2}}\widehat{P} = \rho\overline{P}\mathrm{x}\widehat{P} = \rho\overline{L}o(\mathrm{x}), \tag{A.259}$$

$$\overline{D} = \rho\overline{L}o \ ; \ \ (\overline{D}) = \rho(\overline{L}o). \tag{A.260}$$

Multiplying (A.256) by ρ, we get:

$$(\overline{D})g = g(D)^t \ ; \ \ (\overline{D}) = g(D)^t g. \tag{A.261}$$

That gives, for $j = 1,\ 2,\ 3$ and $k = 1,\ 2,\ 3$:

$$\overline{D}^0_0 = D^0_0 \ ; \ \ \overline{D}^j_0 = -D^0_j \ ; \ \ \overline{D}^0_j = -D^j_0 \ ; \ \ \overline{D}^k_j = D^j_k \tag{A.262}$$

The result is: rows, like columns, of the matrix (D^ν_μ), are orthogonal, because we have for D and $\overline{D}$:

$$D_\mu = \phi\sigma_\mu\phi^\dagger = D^\nu_\mu\sigma_\nu \ ; \ \ \overline{D}_\mu = \overline{\phi}\sigma_\mu\widehat{\phi} = \overline{D}^\nu_\mu\sigma_\nu, \tag{A.263}$$

$$D_\mu \cdot D_\nu = \overline{D}_\mu \cdot \overline{D}_\nu = \delta_{\mu\nu}\rho^2. \tag{A.264}$$

8. The transposition exchanges the rows and columns: if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then: $M^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. We have, for any matrices A and B, $(AB)^t = B^t A^t$ and $\det(A^t) = \det(A)$.

A.4.5 Proof of $\nabla = \overline{M}\nabla'\widehat{M}$

Since ϕ has the same structure as M, we use same notation:

$$M = \sqrt{2}(\xi \ \ \widehat{\eta}) = \sqrt{2}\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}, \tag{A.265}$$

and that gives:

$$\widehat{M} = \sqrt{2}(\eta \ \ \widehat{\xi}) = \sqrt{2}\begin{pmatrix} \eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^* \end{pmatrix}, \tag{A.266}$$

$$M^\dagger = \sqrt{2}\begin{pmatrix} \xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1 \end{pmatrix} \ ; \ \overline{M} = \sqrt{2}\begin{pmatrix} \eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1 \end{pmatrix}. \tag{A.267}$$

We get:

$$\overline{M}\nabla'\widehat{M} = 2\begin{pmatrix} \eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1 \end{pmatrix}\begin{pmatrix} \partial_0' - \partial_3' & -\partial_1' + i\partial_2' \\ -\partial_1' - i\partial_2' & \partial_0' + \partial_3' \end{pmatrix}\begin{pmatrix} \eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^* \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.268}$$

The R_μ^ν are obtained by (A.185) to (A.195) giving the D_μ^ν, we have:

$$\begin{aligned} A &= 2[(\eta_1\eta_1^* + \eta_2\eta_2^*)\partial_0' + (-\eta_1\eta_2^* - \eta_2\eta_1^*)\partial_1' \\ &\quad + i(\eta_2\eta_1^* - \eta_1\eta_2^*)\partial_2' + (-\eta_1\eta_1^* + \eta_2\eta_2^*)\partial_3'] \\ &= (R_0^0 - R_3^0)\partial_0' + (R_0^1 - R_3^1)\partial_1' + (R_0^2 - R_3^2)\partial_2' + (R_0^3 - R_3^3)\partial_3' \\ &= R_0^\mu\partial_\mu' - R_3^\mu\partial_\mu' = \partial_0 - \partial_3. \end{aligned} \tag{A.269}$$

$$\begin{aligned} C &= 2[(\xi_1\eta_2 - \xi_2\eta_1)\partial_0' + (-\xi_1\eta_1 + \xi_2\eta_2)\partial_1' \\ &\quad - i(\xi_1\eta_1 + \xi_2\eta_2)\partial_2' + (\xi_1\eta_2 + \xi_2\eta_1)\partial_3'] \\ &= (-R_1^0 - iR_2^0)\partial_0' + (-R_1^1 - iR_2^1)\partial_1' + (-R_1^2 - iR_2^2)\partial_2' + (-R_1^3 - iR_2^3)\partial_3' \\ &= -R_1^\mu\partial_\mu' - iR_2^\mu\partial_\mu' = -\partial_1 - i\partial_2. \end{aligned} \tag{A.270}$$

$$\begin{aligned} B &= 2[(\xi_1^*\eta_2^* - \xi_2^*\eta_1^*)\partial_0' + (-\xi_1^*\eta_1^* + \xi_2^*\eta_2^*)\partial_1' \\ &\quad + i(\xi_1^*\eta_1^* + \xi_2^*\eta_2^*)\partial_2' + (\xi_1^*\eta_2^* + \xi_2^*\eta_1^*)\partial_3'] \\ &= (-R_1^0 + iR_2^0)\partial_0' + (-R_1^1 + iR_2^1)\partial_1' + (-R_1^2 + iR_2^2)\partial_2' + (-R_1^3 + iR_2^3)\partial_3' \\ &= -R_1^\mu\partial_\mu' + iR_2^\mu\partial_\mu' = -\partial_1 + i\partial_2. \end{aligned} \tag{A.271}$$

$$\begin{aligned} D &= 2[(\xi_1\xi_1^* + \xi_2\xi_2^*)\partial_0' + (\xi_1\xi_2^* + \xi_2\xi_1^*)\partial_1' \\ &\quad + i(\xi_1\xi_2^* - \xi_2\xi_1^*)\partial_2' + (\xi_1\xi_1^* - \xi_2\xi_2^*)\partial_3'] \\ &= (R_0^0 + R_3^0)\partial_0' + (R_0^1 + R_3^1)\partial_1' + (R_0^2 + R_3^2)\partial_2' + (R_0^3 + R_3^3)\partial_3' \\ &= R_0^\mu\partial_\mu' + R_3^\mu\partial_\mu' = \partial_0 + \partial_3. \end{aligned} \tag{A.272}$$

We hence obtain:

$$\overline{M}\nabla'\widehat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} = \nabla. \tag{A.273}$$

A.4.6 Proof of $\det(R_\mu^\nu) = r^4$

We let:

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} := \begin{pmatrix} \mathrm{x}^0+\mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1+i\mathrm{x}^2 & \mathrm{x}^0-\mathrm{x}^3 \end{pmatrix} ; \begin{pmatrix} y_1' & y_2' \\ y_3' & y_4' \end{pmatrix} := \begin{pmatrix} \mathrm{x}^{0\prime}+\mathrm{x}^{3\prime} & \mathrm{x}^{1\prime} - i\mathrm{x}^{2\prime} \\ \mathrm{x}^{1\prime}+i\mathrm{x}^{2\prime} & \mathrm{x}^{0\prime}-\mathrm{x}^{3\prime} \end{pmatrix}, \tag{A.274}$$

$$Y := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{x}^0 \\ \mathrm{x}^1 \\ \mathrm{x}^2 \\ \mathrm{x}^3 \end{pmatrix} = NX,$$

$$Y' := \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{x}^{0\prime} \\ \mathrm{x}^{1\prime} \\ \mathrm{x}^{2\prime} \\ \mathrm{x}^{3\prime} \end{pmatrix} = NX'. \tag{A.275}$$

We then have:

$$X = N^{-1}Y \ ; \quad X' = N^{-1}Y'. \tag{A.276}$$

We also let:

$$Y' = PY \ ; \quad X' = DX. \tag{A.277}$$

We get:

$$PNX = PY = Y' = NX' = NDX \ ; \quad PN = ND \ ; \quad D = N^{-1}PN, \tag{A.278}$$

which implies:

$$\det(R_\mu^\nu) = \det(N^{-1}PN) = \det(N^{-1})\det(P)\det(N) = \det(P). \tag{A.279}$$

We have:

$$\begin{pmatrix} y_1' & y_2' \\ y_3' & y_4' \end{pmatrix} = \mathrm{x}' = M\mathrm{x}M^\dagger = 2\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}\begin{pmatrix} y_1 & y_2^* \\ y_3 & y_4 \end{pmatrix}\begin{pmatrix} \xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1 \end{pmatrix} \tag{A.280}$$

$$= 2\begin{pmatrix} \xi_1\xi_1^*y_1 - \eta_2^*\xi_1^*y_3 & \xi_1\xi_2^*y_1 - \eta_2^*\xi_2^*y_3 \\ -\xi_1\eta_2y_2 + \eta_2^*\eta_2y_4 & +\xi_1\eta_1y_2 - \eta_2^*\eta_1y_4 \\ & \\ \xi_2\xi_1^*y_1 + \eta_1^*\xi_1^*y_3 & \xi_2\xi_2^*y_1 + \xi_2^*\eta_1^*y_3 \\ -\xi_2\eta_2y_2 - \eta_1^*\eta_2y_4 & +\xi_2\eta_1y_2 + \eta_1^*\eta_1y_4 \end{pmatrix},$$

that gives:

$$Y' = PY \ ; \quad P = 2\begin{pmatrix} \xi_1\xi_1^* & -\xi_1\eta_2 & -\xi_1^*\eta_2^* & \eta_2\eta_2^* \\ \xi_1\xi_2^* & \xi_1\eta_1 & -\xi_2^*\eta_2^* & -\eta_1\eta_2^* \\ \xi_2\xi_1^* & -\xi_2\eta_2 & \xi_1^*\eta_1^* & -\eta_2\eta_1^* \\ \xi_2\xi_2^* & \xi_2\eta_1 & \xi_2^*\eta_1^* & \eta_1\eta_1^* \end{pmatrix}. \tag{A.281}$$

The calculation of the determinant of P thus gives:

$$\begin{aligned}\det(P) &= 16(\xi_1^2\xi_1^{*2}\eta_1^2\eta_1^{*2} + \xi_1^2\xi_2^{*2}\eta_2^2\eta_1^{*2} + \xi_2^2\xi_1^{*2}\eta_1^2\eta_2^{*2} + \xi_2^2\xi_2^{*2}\eta_2^2\eta_2^{*2} \\ &+ 2\xi_1^2\xi_1^*\xi_2^*\eta_1\eta_1^{*2}\eta_2 + 4\xi_1\xi_1^*\xi_2\xi_2^*\eta_1\eta_1^*\eta_2\eta_2^* \\ &+ 2\xi_1\xi_1^{*2}\xi_2\eta_1^2\eta_1^*\eta_2^* + 2\xi_1\xi_2\xi_2^{*2}\eta_1^*\eta_2^2\eta_2^* + 2\xi_1^*\xi_2^2\xi_2^*\eta_1\eta_2\eta_2^{*2}) \\ &= 16(\xi_1\xi_1^*\eta_1\eta_1^* + \xi_1\xi_2^*\eta_1^*\eta_2 + \xi_1^*\xi_2\eta_1\eta_2^* + \xi_2\xi_2^*\eta_2\eta_2^*)^2 \\ &= 16[(\xi_1\eta_1^* + \xi_2\eta_2^*)(\xi_1^*\eta_1 + \xi_2^*\eta_2)]^2. \end{aligned} \quad (A.282)$$

We thus get:

$$\begin{aligned}\det(R_\mu^\nu) &= [2(\xi_1\eta_1^* + \xi_2\eta_2^*)2(\xi_1^*\eta_1 + \xi_2^*\eta_2)]^2 \\ &= [re^{i\theta}re^{-i\theta}]^2 = (r^2)^2 = r^4. \end{aligned} \quad (A.283)$$

A.4.7 Relations between tensors

We have:

$$\mathrm{D}_\mu\widehat{\mathrm{D}}_\nu = \phi\sigma_\mu\phi^\dagger\widehat{\phi\sigma_\nu\phi^\dagger} = \phi\sigma_\mu\phi^\dagger\widehat{\phi}\widehat{\sigma}_\nu\widehat{\phi}^\dagger \quad (A.284)$$

$$= \phi\sigma_\mu(\Omega_1 - i\Omega_2)\widehat{\sigma}_\nu\overline{\phi} = (\Omega_1 - i\Omega_2)\phi\sigma_\mu\widehat{\sigma}_\nu\overline{\phi}. \quad (A.285)$$

For $j = 1,\ 2,\ 3$ that gives:

$$\mathrm{D}_0\widehat{\mathrm{D}}_j = (\Omega_1 - i\Omega_2)\phi\widehat{\sigma}_j\overline{\phi} = -(\Omega_1 - i\Omega_2)S_j, \quad (A.286)$$

$$\mathrm{D}_j\widehat{\mathrm{D}}_0 = (\Omega_1 - i\Omega_2)\phi\sigma_j\overline{\phi} = (\Omega_1 - i\Omega_2)S_j, \quad (A.287)$$

$$\mathrm{D}_1\widehat{\mathrm{D}}_2 = (\Omega_1 - i\Omega_2)\phi\sigma_1\widehat{\sigma}_2\overline{\phi} = (\Omega_1 - i\Omega_2)\phi(-i)\sigma_3\overline{\phi} = -(\Omega_2 + i\Omega_1)S_3, \quad (A.288)$$

$$\mathrm{D}_2\widehat{\mathrm{D}}_1 = (\Omega_1 - i\Omega_2)\phi\sigma_2\widehat{\sigma}_1\overline{\phi} = (\Omega_1 - i\Omega_2)\phi i\sigma_3\overline{\phi} = (\Omega_2 + i\Omega_1)S_3. \quad (A.289)$$

And similarly we get:

$$\mathrm{D}_2\widehat{\mathrm{D}}_3 = -\mathrm{D}_3\widehat{\mathrm{D}}_2 = -(\Omega_2 + i\Omega_1)S_1, \quad (A.290)$$

$$\mathrm{D}_3\widehat{\mathrm{D}}_1 = -\mathrm{D}_1\widehat{\mathrm{D}}_3 = -(\Omega_2 + i\Omega_1)S_2. \quad (A.291)$$

For $j = 1,\ 2,\ 3$ and for $k = 1,\ 2,\ 3$, we have:

$$\mathrm{D}_j\widehat{S}_k = \phi\sigma_j\phi^\dagger\widehat{\phi\sigma_k\overline{\phi}} = \phi\sigma_j\phi^\dagger\widehat{\phi}\widehat{\sigma}_k\phi^\dagger = -(\Omega_1 - i\Omega_2)\phi\sigma_j\sigma_k\phi^\dagger, \quad (A.292)$$

$$S_j\mathrm{D}_k = \phi\sigma_j\overline{\phi}\phi\sigma_k\phi^\dagger = (\Omega_1 + i\Omega_2)\phi\sigma_j\sigma_k\phi^\dagger. \quad (A.293)$$

Hence for $j = 1,\ 2,\ 3$ we obtain:

$$\mathrm{D}_j\widehat{S}_j = -(\Omega_1 - i\Omega_2)\phi\phi^\dagger = (-\Omega_1 + i\Omega_2)\mathrm{D}_0, \quad (A.294)$$

$$S_j\mathrm{D}_j = (\Omega_1 + i\Omega_2)\phi\phi^\dagger = (\Omega_1 + i\Omega_2)\mathrm{D}_0. \quad (A.295)$$

And for $k \neq j$ we have:

$$D_1\widehat{S}_2 = -i(\Omega_1 - i\Omega_2)\phi\sigma_3\phi^\dagger = -(\Omega_2 + i\Omega_1)D_3 = -D_2\widehat{S}_1, \quad (A.296)$$
$$S_1D_2 = i(\Omega_1 + i\Omega_2)\phi\sigma_3\phi^\dagger = (-\Omega_2 + i\Omega_1)D_3, = -S_2D_1, \quad (A.297)$$
$$D_2\widehat{S}_3 = -i(\Omega_1 - i\Omega_2)\phi\sigma_1\phi^\dagger = -(\Omega_2 + i\Omega_1)D_1 = -D_3\widehat{S}_2, \quad (A.298)$$
$$S_2D_3 = i(\Omega_1 + i\Omega_2)\phi\sigma_1\phi^\dagger = (-\Omega_2 + i\Omega_1)D_1, = -S_3D_2, \quad (A.299)$$
$$D_3\widehat{S}_1 = -i(\Omega_1 - i\Omega_2)\phi\sigma_2\phi^\dagger = -(\Omega_2 + i\Omega_1)D_2 = -D_1\widehat{S}_3, \quad (A.300)$$
$$S_3D_1 = i(\Omega_1 + i\Omega_2)\phi\sigma_2\phi^\dagger = (-\Omega_2 + i\Omega_1)D_2 = -S_2D_1. \quad (A.301)$$

For $j = 1,\ 2,\ 3$, we also have:

$$D_0\widehat{S}_j = \phi\phi^\dagger\widehat{\phi\sigma_j\overline{\phi}} = \phi\phi^\dagger\widehat{\overline{\phi}}\widehat{\sigma}_j\phi^\dagger = (-\Omega_1 + i\Omega_2)\phi\sigma_j\phi^\dagger$$
$$= (-\Omega_1 + i\Omega_2)D_j, \quad (A.302)$$
$$S_jD_0 = \phi\sigma_j\overline{\phi}\phi\phi^\dagger = (\Omega_1 + i\Omega_2)\phi\sigma_j\phi^\dagger = (\Omega_1 + i\Omega_2)D_j. \quad (A.303)$$

Finally we get for $j = 1,\ 2,\ 3$ and for $k = 1,\ 2,\ 3$:

$$S_jS_k = \phi\sigma_j\overline{\phi}\phi\sigma_k\overline{\phi} = (\Omega_1 + i\Omega_2)\phi\sigma_j\sigma_k\overline{\phi}, \quad (A.304)$$
$$S_jS_j = (\Omega_1 + i\Omega_2)\phi\overline{\phi} = (\Omega_1 + i\Omega_2)^2, \quad (A.305)$$

while for $k \neq j$, we get:

$$S_1S_2 = -S_2S_1 = (-\Omega_2 + i\Omega_1)S_3, \quad (A.306)$$
$$S_2S_3 = -S_3S_2 = (-\Omega_2 + i\Omega_1)S_1, \quad (A.307)$$
$$S_3S_1 = -S_1S_3 = (-\Omega_2 + i\Omega_1)S_2. \quad (A.308)$$

Appendix B

Other Clifford algebras

We present two space-time algebras, $Cl_{1,3}$ and $Cl_{3,1}$, sub-algebras of the algebra of Dirac matrices, $M_4(\mathbb{C})$. We study first $Cl_{1,3}$, its link with the Pauli algebra and the link between the invariant wave equation and the Lagrangian density. We study the same link with space-time algebra. We calculate Tétrode's tensor. We present next $Cl_{3,1}$ and the representations of Clifford algebras allowed by real 4×4 matrices. Then we study the reversion in the Clifford algebra $Cl_{3,3} = \mathrm{End}(Cl_3)$ that we need for the study of weak and strong interactions and gravitation. Finally we present matrix representations of Clifford algebras with real 8×8 matrices.

B.1 Clifford algebras of space-time

Space-time is formed by time, we note its variable $\mathrm{x}^0 = ct$, where c is light speed, and by space, 3-dimensional, where each point is located by a triple of real numbers x^1, x^2, x^3. Space-time is pseudo-Euclidean, with signature $+,-,-,-$ or $-,+,+,+$. Hence two different Clifford algebras exist, noted respectively $Cl_{1,3}$ and $Cl_{3,1}$. Proponents of Clifford algebra can generally be divided into two sides: those who put a $+$ sign for time (Hestenes [77][82]), and those who put a $-$ sign for time (Deheuvels [63]). We will see in B.2 that both sides give two subalgebras of $Cl_{3,3}$. Here we most often use a $+$ sign for time, which corresponds to Hestenes' choice. It is useful because the metric of space-time is given by the determinant (A.178).

B.1.1 The $Cl_{1,3}$ Clifford algebra

Here we use a + sign for time, Hestenes' choice. The Clifford algebra $Cl_{1,3}$ contains all real numbers and all space-time vectors $\mathbf{x}$ such that:

$$\mathbf{x} = \mathrm{x}^0\gamma_0 + \mathrm{x}^1\gamma_1 + \mathrm{x}^2\gamma_2 + \mathrm{x}^3\gamma_3 = \mathrm{x}^\mu\gamma_\mu. \tag{B.1}$$

These four γ_μ form an orthonormal basis of space-time:

$$(\gamma_0)^2 = 1 \quad ; \quad (\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -1 \quad ; \quad \gamma_\mu \cdot \gamma_\nu = 0 \ , \ \mu \neq \nu. \tag{B.2}$$

The general term in $Cl_{1,3}$ is a sum:

$$N = s + v + B + p_v + p_s \tag{B.3}$$

where s is a real number, v is a vector in space-time, B is a 2-vector, p_v is a 3-vector (or pseudovector) and p_s is a pseudoscalar. With the definition of (1.4) :

$$\begin{aligned}
&\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \gamma^j := \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \\
&\sigma^j = -\widehat{\sigma}^j = \widehat{\sigma}_j := -\sigma_j;\ \gamma^j = -\gamma_j,\ j = 1,2,3, \\
&\gamma_0 = \gamma^0 := \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \ ; \ I_2 = \sigma_0 = \sigma^0 = \widehat{\sigma}^0 = \widehat{\sigma}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned} \tag{B.4}$$

And with:

$$v = v^\mu\gamma_n u;\ \vec{v} := v^j\sigma_j;\ p_v = ip_v^\mu\gamma_\mu;\ p_s =: p\mathbf{i};\ \vec{p_v} := p_v^j\sigma_j, \tag{B.5}$$

$$B =: \begin{pmatrix} \vec{E} + i\vec{H} & 0 \\ 0 & -\vec{E} + i\vec{H} \end{pmatrix} =: \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix};\ \vec{E} = E^j\sigma_j;\ \vec{H} = H^j\sigma_j, \tag{B.6}$$

The general element N of the algebra reads:

$$N = \begin{pmatrix} s + F + ip & v^0 + \vec{v} + i(p_v^0 + \vec{p_v}) \\ v^0 - \vec{v} - i(p_v^0 - \vec{p_v}) & s + \widehat{F} - ip \end{pmatrix} =: \begin{pmatrix} M_e & M_o \\ \widehat{M_o} & \widehat{M_e} \end{pmatrix}, \tag{B.7}$$

$$M_e = s + F + ip = s + E^j\sigma_j + iH^j\sigma_j + ip;\ M_o = v^0 + \vec{v} + i(p_v^0 + \vec{p_v}).$$

There are $1 + 4 + 6 + 4 + 1 = 16 = 2^4$ dimensions (fourth line of the arithmetical triangle) on the real field because: There are 6 independent 2-vectors $\gamma_{01} = \gamma_0\gamma_1$, γ_{02}, γ_{03}, γ_{12}, γ_{23} and γ_{31}, where $\gamma_{ji} = -\gamma_{ij}$, $j \neq i$, and 4 3-vectors γ_{012}, γ_{023}, γ_{031} and γ_{123} and a single pseudoscalar:

$$\gamma_{0j} = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix};\ \gamma_{12} = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix};\ \gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{B.8}$$

$$p_s = p\gamma_{0123} \quad ; \quad \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3 = \mathbf{i} = i\gamma_5, \tag{B.9}$$

where b is a real number. The even part of N is $N^+ = s + B + p_s$, while the odd part is $N^- = v + p_v$. The main automorphism satisfies $N \mapsto \widehat{N} = s - v + B - p_v + p_s$. The reversion satisfies:

$$N \mapsto \widetilde{N} = s + v - B - p_v + p_s = \begin{pmatrix} \overline{M}_e & M_o^\dagger \\ \overline{M}_o & M_e^\dagger \end{pmatrix}. \tag{B.10}$$

Among the $16 = 2^4 = 1 + 4 + 6 + 4 + 1$ generators of $Cl_{1,3}$, $10 = 5 \times 4/2$ have a square equal to -1 and $6 = 4 \times 3/2$ have a square equal to 1. The privileged differential operator in $Cl_{1,3}$ is:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu \;\; ; \;\; \gamma^0 = \gamma_0 \;\; ; \;\; \gamma^j = -\gamma_j \; , \; j = 1,\ 2,\ 3. \tag{B.11}$$

It satisfies:

$$\boldsymbol{\partial\partial} = \Box = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2. \tag{B.12}$$

Most physicists do not directly use the Clifford algebra of space-time, but use instead the matrix algebra $M_4(\mathbb{C})$, an algebra on the complex field. This algebra is 16-dimensional on the complex field, and thus 32-dimensional on the real field. Therefore $M_4(\mathbb{C}) \neq Cl_{1,3}$. The Dirac matrices are not uniquely defined. The easiest way to link $Cl_{1,3}$ to Cl_3 makes use of[1] (1.4), the Pauli and Dirac matrices reminded in (B.4). Moreover we identify scalar matrices and complex numbers, writing:

$$1 = I_2; \quad z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \; z = x + iy \in \mathbb{C}. \tag{B.13}$$

We then have:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}; \quad \nabla = \sigma^\mu \partial_\mu. \tag{B.14}$$

Easily we establish that:

$$\gamma_{0j} = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}; \; \gamma_{23} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}; \; \mathbf{i} = \gamma_{0123} = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}$$

$$N = \begin{pmatrix} M_e & M_o \\ \widehat{M_o} & \widehat{M_e} \end{pmatrix}, \; M_e \in Cl_3; \; M_o \in Cl_3. \tag{B.15}$$

Isomorphism between $Cl_{1,3}^+$ and Cl_3 : let N^+ be any even element. With:

$$N^+ = a + B + p_s \;\; ; \;\; B = u_1\gamma_{10} + u_2\gamma_{20} + u_3\gamma_{30} + v_1\gamma_{32} + v_2\gamma_{13} + v_3\gamma_{21},$$
$$p_s = b\gamma_{0123} = b\mathbf{i}, \tag{B.16}$$

$$M_e = a + \vec{u} + i\vec{v} + ib \; ; \;\; \vec{u} = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3,$$
$$\vec{v} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3, \tag{B.17}$$

1. This choice of Dirac matrices is not the choice used in Dirac theory to calculate the solutions for the hydrogen atom. But it is the choice made for high velocities and when special relativity is required (Weyl matrices). It is also the choice of electroweak theory. We will see in Appendix C that this choice also allows us to solve the equation in the case of the H atom by separation of variables in spherical coordinates. This also proves that it was possible to completely avoid the choice made in 1928 to be reduced to the Pauli equation.

B is a bivector and p_s is a pseudoscalar in space-time. Similarly each odd element N^- reads:

$$N^- = \begin{pmatrix} 0 & M_o \\ \widehat{M}_o & 0 \end{pmatrix}; \; M_o = \mathrm{v} + i\mathrm{w} = (\mathrm{v}_\mu + i\mathrm{w}_\mu)\sigma^\mu. \tag{B.18}$$

With (B.4) we have:

$$N^+ = \begin{pmatrix} M_e & 0 \\ 0 & \widehat{M_e} \end{pmatrix} \; ; \; \widetilde{N}^+ = \begin{pmatrix} \overline{M}_e & 0 \\ 0 & M_e^\dagger \end{pmatrix}. \tag{B.19}$$

Since the parity conjugation $P : M \mapsto \widehat{M}$ is compatible with the addition and the multiplication, the algebra of M is isomorphic to the algebra of N. Since N contains both M and $\widehat{M}$, the Dirac matrices use the two nonequivalent representations of the sub-group $SL(2, \mathbb{C})$ of Cl_3^* (this is well known in Lie group theory [1]).

The Dirac operator is:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \partial_0 - \vec{\partial} \\ \partial_0 + \vec{\partial} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}. \tag{B.20}$$

Similarly, the electromagnetic potential is the vector:

$$\mathbf{A} := \gamma_\mu A^\mu = \begin{pmatrix} 0 & A^0 + \vec{A} \\ A^0 - \vec{A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix}. \tag{B.21}$$

The electromagnetic field, with no magnetic monopole, is the bivector:

$$\begin{aligned} \mathbf{F} &:= \boldsymbol{\partial} \wedge \mathbf{A} = (\boldsymbol{\partial}\mathbf{A} - \mathbf{A}\boldsymbol{\partial})/2, \qquad \text{(B.22)} \\ 2\mathbf{F} &= \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} - \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix}\begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \nabla\widehat{A} - A\widehat{\nabla} & 0 \\ 0 & \widehat{\nabla}A - \widehat{A}\nabla \end{pmatrix} = 2\begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix}. \end{aligned}$$

The electric current satisfies:

$$\mathbf{j} = \boldsymbol{\partial}\mathbf{F} = \boldsymbol{\partial}\boldsymbol{\partial}\mathbf{A} = \Box\mathbf{A} = \begin{pmatrix} 0 & \nabla\widehat{F} \\ \widehat{\nabla}F & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{j} \\ \widehat{\mathrm{j}} & 0 \end{pmatrix}. \tag{B.23}$$

B.1.2 $Cl_{1,3}$ as Cartesian product $Cl_3 \times Cl_3$

We can shorten the calculations in $Cl_{1,3}$ by considering only the first row of the Dirac matrices, when using the blocks made of Pauli matrices. This is possible because the second row is obtained from the first one by using the P automorphism on Cl_3. The general element of $Cl_{1,3}$ may thus be expressed as a couple of elements of Cl_3:

$$M = (A \;\; B);\; N = (C \;\; D);\; M + N = (A + C \;\; B + D), \tag{B.24}$$

$$MN = (AC + B\widehat{D} \;\; AD + B\widehat{C}) \tag{B.25}$$

$$\gamma^\mu = (0 \;\; \sigma^\mu);\; \mathbf{x} = \mathrm{x}^\mu\gamma_\mu = (0 \;\; \mathrm{x}) = (0 \;\; \mathrm{x}^\mu\sigma_\mu), \tag{B.26}$$

$$\boldsymbol{\partial} = \gamma^\mu\partial_\mu = (0 \;\; \nabla) = (0 \;\; \sigma^\mu\partial_\mu);\; \boldsymbol{\partial}(A \;\; B) = (\nabla\widehat{B} \;\; \nabla\widehat{A}). \tag{B.27}$$

With these notations, and for any $\mathbf{u}$ and $\mathbf{v}$ in space-time, we have:

$$\mathbf{uv} + \mathbf{vu} = (0 \;\; \mathrm{u})(0 \;\; \mathrm{v}) + (0 \;\; \mathrm{v})(0 \;\; \mathrm{u}) = (\mathrm{u}\widehat{\mathrm{v}} + \mathrm{v}\widehat{\mathrm{u}} \;\; 0). \tag{B.28}$$

Identifying A and $(A \;\; 0)$, we then have:

$$\mathbf{uv} + \mathbf{vu} = \mathrm{u}\widehat{\mathrm{v}} + \mathrm{v}\widehat{\mathrm{u}} = \mathrm{u}^0\mathrm{v}^0 - \mathrm{u}^1\mathrm{v}^1 - \mathrm{u}^2\mathrm{v}^2 - \mathrm{u}^3\mathrm{v}^3. \tag{B.29}$$

This identification allows us to consider $Cl_{1,3}$ as a Cl_3-module:

$$X(A \;\; B) = (X \;\; 0)(A \;\; B) = (XA \;\; XB), \tag{B.30}$$

for any X, A and B in Cl_3. This is what allows us the use in $Cl_{1,3}$ of the complex field, which is the center (set of the elements that commute with any other) of Cl_3, nevertheless the fact that the center of $Cl_{1,3}$ is only the real field.

B.1.3 Proof of $R^\mu_\nu\gamma^\nu = \widetilde{N}\gamma^\mu N$

Using the aforementioned notation we have:

$$N = \begin{pmatrix} M & 0 \end{pmatrix} \;;\; \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \end{pmatrix} \;;\; \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \end{pmatrix}, \tag{B.31}$$

the equation $R^\mu_\nu\gamma^\nu = \widetilde{N}\gamma^\mu N$ is equivalent to:

$$\begin{pmatrix} 0 & R^\mu_\nu\sigma^\nu \end{pmatrix} = \begin{pmatrix} 0 & \overline{M}\sigma^\mu\widehat{M} \end{pmatrix}. \tag{B.32}$$

And A.4.5: $\nabla = \overline{M}\nabla'\widehat{M}$, may also be expressed as:

$$\partial_\nu\sigma^\nu = \overline{M}\partial'_\mu\sigma^\mu\widehat{M}, \tag{B.33}$$

which means:

$$R^\mu_\nu\partial'_\mu\sigma^\nu = \overline{M}\sigma^\mu\widehat{M}\partial'_\mu. \tag{B.34}$$

And thus we get:

$$R^\mu_\nu\sigma^\nu = \overline{M}\sigma^\mu\widehat{M};\; R^\mu_\nu\gamma^\nu = \widetilde{N}\gamma^\mu N. \tag{B.35}$$

B.1.4 Invariant equation and Lagrangian density

We will now prove that the Lagrangian density of the Dirac theory is the real part, (in the sense where the set of real numbers is a sub-linear space with dimension 1 of the Clifford algebra) of the wave equation in its invariant form (1.115). Then noting $\langle M\rangle_n$ the n-vector part of M, we must prove that:

$$L = \langle \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi\rangle_0. \tag{B.36}$$

And we have:

$$\begin{aligned}\overline{\phi}A\widehat{\phi} &= A^\mu\overline{\phi}\sigma_\mu\widehat{\phi} = A_0\overline{D}_0 - \sum_{j=1}^{j=3}A_j\overline{D}_j = A_0(\overline{D}_0^\mu\sigma_\mu) - \sum_{j=1}^{j=3}A_j(\overline{D}_j^\mu\sigma_\mu)\\ &= A_0(\overline{D}_0^0 + \sum_{j=1}^{j=3}\overline{D}_0^j\sigma_j) - \sum_{j=1}^{j=3}A_j(\overline{D}_j^0 + \sum_{k=1}^{k=3}\overline{D}_j^k\sigma_k).\end{aligned} \tag{B.37}$$

We established with the calculation of the similitude $\overline{D}$ (A.262) that we have:

$$\overline{\phi}A\widehat{\phi} = A_0(D_0^0 - \sum_{j=1}^{j=3}D_j^0\sigma_j) - \sum_{j=1}^{j=3}A_j(-D_0^j + \sum_{k=1}^{k=3}D_k^j\sigma_k) = A_\nu D_\mu^\nu\sigma^\mu. \tag{B.38}$$

The scalar part is then:

$$\langle\overline{\phi}A\widehat{\phi}\rangle_0 = D_0^\nu A_\nu = A_\nu \mathrm{J}^\nu = A_\mu\overline{\psi}\gamma^\mu\psi. \tag{B.39}$$

We then have $\overline{\phi}\phi = \Omega_1 + i\Omega_2$, and thus:

$$\langle m\overline{\phi}\phi\rangle_0 = m\Omega_1 = m\overline{\psi}\psi. \tag{B.40}$$

We next have:

$$\begin{aligned}&\frac{1}{2}[(\overline{\psi}\gamma^\mu(-i)\partial_\mu\psi) + (\overline{\psi}\gamma^\mu(-i)\partial_\mu\psi)^\dagger] = \frac{i}{2}(-\overline{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\overline{\psi}\gamma^\mu\psi)\\ &= -\frac{i}{2}[\xi^\dagger(\widehat{\nabla}\xi) - (\xi^\dagger\widehat{\nabla})\xi + \eta^\dagger(\nabla\eta) - (\eta^\dagger\nabla)\eta].\end{aligned} \tag{B.41}$$

With Cl_3 we have:

$$\begin{aligned}&\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}]\sigma_{21} = -\frac{i}{2}[\overline{\phi}(\nabla\widehat{\phi}\sigma_3) - (\overline{\phi}\nabla)\widehat{\phi}\sigma_3]\\ &= -i\begin{pmatrix}\eta^\dagger(\nabla\eta) - (\eta^\dagger\nabla)\eta & -\eta^\dagger(\nabla\widehat{\xi}) + (\eta^\dagger\nabla)\widehat{\xi}\\ \widehat{\xi}^\dagger(\nabla\eta) - (\widehat{\xi}^\dagger\nabla)\eta & -\widehat{\xi}^\dagger(\nabla\widehat{\xi}) + (\widehat{\xi}^\dagger\nabla)\widehat{\xi}\end{pmatrix},\\ &\langle\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}]\sigma_{21}\rangle_0 = -\frac{i}{2}[\eta^\dagger(\nabla\eta) - (\eta^\dagger\nabla)\eta - \widehat{\xi}^\dagger(\nabla\widehat{\xi}) + (\widehat{\xi}^\dagger\nabla)\widehat{\xi}].\end{aligned} \tag{B.42}$$

And we have:

$$-\widehat{\xi}^{\dagger}(\nabla\widehat{\xi}) + (\widehat{\xi}^{\dagger}\nabla)\widehat{\xi} = \overline{(\widehat{\xi}^{\dagger}\nabla)\widehat{\xi}} - \overline{\widehat{\xi}^{\dagger}(\nabla\widehat{\xi})} = \xi^{\dagger}(\widehat{\nabla}\xi) - (\xi^{\dagger}\widehat{\nabla})\xi. \quad \text{(B.43)}$$

We thus get:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[(\overline{\psi}\gamma^{\mu}(-i)\partial_{\mu}\psi) + (\overline{\psi}\gamma^{\mu}(-i)\partial_{\mu}\psi)^{\dagger}] + qA_{\mu}\overline{\psi}\gamma^{\mu}\psi + m\overline{\psi}\psi \\ &= \langle\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi\rangle_0. \end{aligned} \quad \text{(B.44)}$$

B.1.5 The $Cl_{3,1}$ algebra

To bring together $Cl_{3,1}$ and $Cl_{1,3}$, and since they are two sub-algebra of $Cl_{3,3}$, we may use the following generators:

$$\begin{aligned}\lambda_0 = -\lambda^0 &:= \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix};\ \lambda_0^2 = -I_4, \\ \lambda_j = \lambda^j &:= \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix},\ j = 1,2,3;\ \lambda_j^2 = I_4. \end{aligned} \quad \text{(B.45)}$$

The Clifford algebra of space-time $Cl_{3,1}$ contains the real field $\mathbb{R}$ and the 4-dimensional linear space of space-time such as:

$$\begin{aligned}\mathbf{x} &= \mathrm{x}^{\mu}\lambda_{\mu};\ \vec{\mathrm{x}} = \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3 \\ \mathbf{x} &= \begin{pmatrix} 0 & \mathrm{x}^0 + \vec{\mathrm{x}} \\ -(\mathrm{x}^0 - \vec{\mathrm{x}}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{x} \\ -\widehat{\mathrm{x}} & 0 \end{pmatrix}. \end{aligned} \quad \text{(B.46)}$$

The four λ_{μ} form an orthonormal basis of space-time with signature $-,+,+,+$:

$$(\lambda_0)^2 = -I_4;\quad (\lambda_1)^2 = (\lambda_2)^2 = (\lambda_3)^2 = I_4;\quad \lambda_{\mu}\cdot\lambda_{\nu} = 0\ ,\ \mu \neq \nu, \quad \text{(B.47)}$$

$$\mathbf{x}\cdot\mathbf{x} = -(\mathrm{x}^0)^2 + (\mathrm{x}^1)^2 + (\mathrm{x}^2)^2 + (\mathrm{x}^3)^2. \quad \text{(B.48)}$$

We also have:

$$\mathbf{d} := \lambda^{\mu}\partial_{\mu} = \lambda^0\partial_0 + \lambda^j\partial_j = \begin{pmatrix} 0 & \partial_0 + \vec{\partial} \\ -\partial_0 + \vec{\partial} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla \\ -\widehat{\nabla} & 0 \end{pmatrix}. \quad \text{(B.49)}$$

With the following notations:

$$\lambda_{ab} := \lambda_a\lambda_b;\ \lambda_{abc} = \lambda_a\lambda_b\lambda_c;\ \lambda_{0123} = \lambda_0\lambda_1\lambda_2\lambda_3, \quad \text{(B.50)}$$

It is easy to establish that:

$$\lambda_{0j} = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}; \lambda_{23} = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}; \mathbf{i} = \lambda_{0123} = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}. \quad \text{(B.51)}$$

With the even part M_e and M_o of (B.15), the general term of $Cl_{3,1}$ satisfies:

$$N_{3,1} = \begin{pmatrix} \overline{M}_e & iM_o \\ i\widehat{M}_o & M_e^\dagger \end{pmatrix}.$$

The even part of $N_{3,1}$ is $N_{3,1}^+ = s - B + p_s$, while the odd part is $N_{3,1}^- = i\mathrm{v} - \mathrm{w}$. The reversion satisfies:

$$N_{3,1} \mapsto \widetilde{N}_{3,1} = s + i\mathrm{v} + B + \mathrm{w} + p_s. \tag{B.52}$$

Among the $16 = 2^4 = 1 + 4 + 6 + 4 + 1$ generators of $Cl_{3,1}$, $10 = 5 \times 4/2$ have a square equal to 1 and $6 = 4 \times 3/2$ have a square equal to -1:

$$\begin{aligned} {\lambda_{0123}}^2 &= {\lambda_{01}}^2 = {\lambda_{02}}^2 = {\lambda_{03}}^2 = {\lambda_0}^2 = {\lambda_{123}}^2 = 1, \\ {\lambda_1}^2 &= {\lambda_2}^2 = {\lambda_3}^2 = {\lambda_{12}}^2 = {\lambda_{23}}^2 = {\lambda_{31}}^2 \\ &= {\lambda_{012}}^2 = {\lambda_{023}}^2 = {\lambda_{031}}^2 = 1^2 = 1. \end{aligned} \tag{B.53}$$

The even subalgebra $Cl_{3,1}^+$, made of any element $\overline{N} = s - B + p_s$ is 8-dimensional and is isomorphic to Cl_3. The privileged differential operator in $Cl_{3,1}$ satisfies:

$$\mathbf{d} = \lambda^\mu \partial_\mu;\ \lambda^0 = -\lambda_0;\ \lambda^j = \lambda_j\ ,\ j = 1,\ 2,\ 3. \tag{B.54}$$

It satisfies:

$$\mathbf{d}^2 = -\Box = -(\partial_0)^2 + (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2. \tag{B.55}$$

The $M_4(\mathbb{C})$ algebra generated by Dirac matrices is, as linear space on $\mathbb{R}$, 32-dimensional. Even if $Cl_{1,3}$ and $Cl_{3,1}$ are both 16-dimensional, the union of their matrix representations is only 24-dimensional on $\mathbb{R}$, because those algebras have together their even sub-algebra, isomorphic to Cl_3.

B.1.6 Real matrix representations

In the 4×4 complex matrix algebra, the subalgebra of real matrices is 16-dimensional on $\mathbb{R}$. A basis of that algebra is made of the following matrices: I_4, γ_0, γ_1, γ_3, γ_5, γ_{05}, γ_{10}, γ_{51}, γ_{53}, γ_{30}, γ_{13}, γ_{105}, γ_{305}, γ_{013}, γ_{135} and γ_{0135}. This algebra is both the Clifford algebra $Cl_{2,2}$ and the Clifford algebra $Cl_{3,1}$. As Clifford algebra $Cl_{2,2}$, the sub-space of scalars is the set of sI_4. The sub-space of vectors, 4-dimensional, has a basis $(\gamma_0, \gamma_5, \gamma_1, \gamma_3)$:

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};\ \gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \gamma_0^2 = \gamma_5^2 = I_4 \tag{B.56}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};\ \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};\ \gamma_1^2 = \gamma_3^2 = -I_4,$$

$$\gamma_\mu \gamma_\lambda + \gamma_\lambda \gamma_\mu = 0,\ \mu,\ \nu = 0, 5, 1, 3,\ \mu \neq \nu. \tag{B.57}$$

The sub-space of 2-vectors, 6-dimensional, has a basis made of γ_{05}, γ_{10}, γ_{51}, γ_{53}, γ_{30}, γ_{13}. We have:

$$\gamma_{05} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \ \gamma_{13} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{B.58}$$

$$\gamma_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; \ \gamma_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.59}$$

$$\gamma_{51} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : \ \gamma_{53} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{B.60}$$

$$\gamma_{05}^2 = \gamma_{13}^2 = -I_4; \ \gamma_{10}^2 = \gamma_{30}^2 = \gamma_{51}^2 = \gamma_{53}^2 = I_4. \tag{B.61}$$

Remark that instead the symmetrical signature $++--$ in $Cl_{2,2}$, an asymmetry appears between the number of $+1$ and the number of -1 for the square of 2-vectors. The sub-space of pseudo-vectors, 4 dimensional, has a basis made of γ_{105}, γ_{305}, γ_{013}, γ_{135}:

$$\gamma_{305} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \ \gamma_{105} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{B.62}$$

$$\gamma_{013} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \ \gamma_{135} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{B.63}$$

$$\gamma_{105}^2 = \gamma_{305}^2 = I_4; \ \gamma_{013}^2 = \gamma_{135}^2 = -I_4. \tag{B.64}$$

And the set of pseudo-scalars is 1-dimensional, with generator γ_{0135}:

$$\gamma_{0135} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \ \gamma_{0135}^2 = I_4 \tag{B.65}$$

Finally among 16 basis elements of $M_4(\mathbb{R}) = Cl_{2,2}$, 6 have a square $-I_4$ while 10 have a square I_4. Thus $M_4(\mathbb{R})$ cannot be the Clifford algebra $Cl_{1,3}$. That is however the Clifford algebra $Cl_{3,1}$, with a basis of the sub-space of vectors pour générateurs les $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ satisfying:

$$\lambda_0 := \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix};\ \lambda_1 := \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix};\ \lambda_2 := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix};\ \lambda_3 := \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},$$
$$\lambda_0^2 = -I_4;\ \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = I_4;\ \lambda_\mu\lambda_\nu = -\lambda_\nu\lambda_\mu,\ \mu \neq \nu. \tag{B.66}$$

B.2 Reversion in $Cl_{3,3}$

After Cl_3 and $Cl_{1,3}$, the third most used Clifford algebra in that work is $Cl_{3,3}$, an algebra isomorphic, as real algebra, to the algebra of endomorphisms (bijective linear applications) on the linear space Cl_3, noted $\text{End}(Cl_3)$. We use that Clifford algebra both as a natural generalization of $Cl_{1,3}$ used by Hestenes [77]–[82], and as a natural generalization of Cl_3 as algebra $\text{End}(\mathbb{A})$ of all endomorphisms on the sub-ring $\mathbb{A}$ of Cl_3 made of all complex diagonal matrices. Therefore we do not use the $M_8(\mathbb{R})$ algebra of real 8×8 matrices, but the sub-algebra of $M_8(\mathbb{C})$ generated by the following matrices:

$$\Gamma_\mu := \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix},\ \mu = 0,1,2,3, \tag{B.67}$$
$$\Gamma_4 := \begin{pmatrix} 0 & -iI_4 \\ iI_4 & 0 \end{pmatrix}\ ;\ \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \tag{B.68}$$
$$\mathbf{i} = \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma_5. \tag{B.69}$$

We use here without any change the (1.4) Weyl matrices:

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix};\ \gamma^j = -\gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}; \gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where the σ_j are Pauli matrices. Indices $\mu, \nu, \rho \dots$ run on $0,1,2,3$ and indices a, b, c, d, e run on $0,1,2,3,4,5$. We have:

$$\Gamma_a\Gamma_b = -\Gamma_b\Gamma_a,\ a \neq b, \tag{B.70}$$
$$\Gamma_0^2 = \Gamma_4^2 = \Gamma_5^2 = 1;\ \ \Gamma_1^2 = \Gamma_2^2 = \Gamma_3^2 = -1, \tag{B.71}$$
$$\Gamma_{\mu\nu} = \Gamma_\mu\Gamma_\nu = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \gamma_{\mu\nu} \end{pmatrix};\ \Gamma_{\mu\nu\rho} = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho} \\ \gamma_{\mu\nu\rho} & 0 \end{pmatrix}, \tag{B.72}$$
$$\Gamma_{0123} = \Gamma_{01}\Gamma_{23} = \begin{pmatrix} \gamma_{0123} & 0 \\ 0 & \gamma_{0123} \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \tag{B.73}$$
$$\Gamma_{45} = \Gamma_4\Gamma_5 = \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix};\ \Gamma_{012345} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}. \tag{B.74}$$

We also have:

$$\Gamma_{\mu 4} = \begin{pmatrix} i\gamma_\mu & 0 \\ 0 & -i\gamma_\mu \end{pmatrix};\ \Gamma_{\mu 5} = \begin{pmatrix} \gamma_\mu\gamma_5 & 0 \\ 0 & \gamma_\mu\gamma_5 \end{pmatrix}, \tag{B.75}$$

$$\Gamma_{\mu 45} = \begin{pmatrix} 0 & \gamma_\mu \mathbf{i} \\ -\gamma_\mu \mathbf{i} & 0 \end{pmatrix};\ \Gamma_{\mu\nu 4} = \begin{pmatrix} 0 & -i\gamma_{\mu\nu} \\ i\gamma_{\mu\nu} & 0 \end{pmatrix}, \tag{B.76}$$

$$\Gamma_{\mu\nu 5} = \begin{pmatrix} 0 & \gamma_{\mu\nu 5} \\ \gamma_{\mu\nu 5} & 0 \end{pmatrix};\ \Gamma_{\mu\nu 45} = \begin{pmatrix} -\gamma_{\mu\nu}\mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu}\mathbf{i} \end{pmatrix}, \tag{B.77}$$

$$\Gamma_{\mu\nu\rho 4} = \begin{pmatrix} i\gamma_{\mu\nu\rho} & 0 \\ 0 & -i\gamma_{\mu\nu\rho} & 0 \end{pmatrix};\ \Gamma_{\mu\nu\rho 5} = \begin{pmatrix} \gamma_{\mu\nu\rho 5} & 0 \\ 0 & \gamma_{\mu\nu\rho 5} \end{pmatrix}, \tag{B.78}$$

$$\Gamma_{\mu\nu\rho 45} = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho}\mathbf{i} \\ -\gamma_{\mu\nu\rho}\mathbf{i} & 0 \end{pmatrix};\ \Gamma_{01234} = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}, \tag{B.79}$$

$$\Gamma_{01235} = \begin{pmatrix} 0 & iI_4 \\ iI_4 & 0 \end{pmatrix};\ \Gamma_{012345} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}. \tag{B.80}$$

The general term in $Cl_{3,3}$ reads:

$$\Psi^{3,3} = \Psi_0^{3,3} + \Psi_1^{3,3} + \Psi_2^{3,3} + \Psi_3^{3,3} + \Psi_4^{3,3} + \Psi_5^{3,3} + \Psi_6^{3,3}, \tag{B.81}$$

$$\Psi_1^{3,3} = \sum_{a=0}^{a=5} N^a\Gamma_a,\ \Psi_2^{3,3} = \sum_{0\leqslant a<b\leqslant 5} N^{ab}\Gamma_{ab},\ \Psi_3^{3,3} = \sum_{0\leqslant a<b<c\leqslant 5} N^{abc}\Gamma_{abc},$$

$$\Psi_4^{3,3} = \sum_{0\leqslant a<b<c<d\leqslant 5} N^{abcd}\Gamma_{abcd},\ \Psi_5^{3,3} = \sum_{0\leqslant a<b<c<d<e\leqslant 5} N^{abcde}\Gamma_{abcde},$$

$$\Psi_0^{3,3} = \alpha I_8, \alpha\in\mathbb{R}\ ;\ \ \Psi_6^{3,3} = \omega\Gamma_{012345},\ \omega\in\mathbb{R}. \tag{B.82}$$

Scalar and pseudo-scalar terms read:

$$\alpha I_8 + \omega\Gamma_{012345} = \begin{pmatrix} \alpha+\omega & 0 \\ 0 & \alpha-\omega \end{pmatrix}. \tag{B.83}$$

For the calculation of the 1-vector term

$$N^a\Gamma_a = N^4\Gamma_4 + N^5\Gamma_5 + N^\mu\Gamma_\mu,$$

we let:

$$\beta := N^4\ ;\ \ \delta := N^5\ ;\ \ \mathbf{a} := N^\mu\gamma_\mu. \tag{B.84}$$

That gives:

$$\Psi_1^{3,3} = \begin{pmatrix} 0 & \mathbf{a} - i(\beta+\delta\mathbf{i}) \\ \mathbf{a} + i(\beta-\delta\mathbf{i}) & 0 \end{pmatrix}. \tag{B.85}$$

For the calculation of the 2-vector term:

$$N^{ab}\Gamma_{ab} = N^{45}\Gamma_{45} + N^{\mu 4}\Gamma_{\mu 4} + N^{\mu 5}\Gamma_{\mu 5} + N^{\mu\nu}\Gamma_{\mu\nu},$$

we let:

$$\epsilon := N^{45} \;;\quad \mathbf{b} := N^{\mu 4}\gamma_\mu \;;\quad \mathbf{c} := N^{\mu 5}\gamma_\mu \;;\quad \mathbf{A} := N^{\mu\nu}\gamma_{\mu\nu}. \tag{B.86}$$

That gives:

$$\Psi_2^{3,3} = \begin{pmatrix} -\epsilon\mathbf{i} + \mathbf{A} + i(\mathbf{b} - \mathbf{ci}) & 0 \\ 0 & \mathbf{A} + \epsilon\mathbf{i} - i(\mathbf{b} + \mathbf{ci}) \end{pmatrix}. \tag{B.87}$$

For the calculation of the 3-vector term:

$$N^{abc}\Gamma_{abc} = N^{\mu 45}\Gamma_{\mu 45} + N^{\mu\nu 4}\Gamma_{\mu\nu 4} + N^{\mu\nu 5}\Gamma_{\mu\nu 5} + N^{\mu\nu\rho}\Gamma_{\mu\nu\rho},$$

we let:

$$\mathbf{d} := N^{\mu 45}\gamma_\mu \;;\quad \mathbf{B} := N^{\mu\nu 4}\gamma_{\mu\nu} \;;\quad \mathbf{C} := N^{\mu\nu 5}\gamma_{\mu\nu} \;;\quad \mathbf{ie} := N^{\mu\nu\rho}\gamma_{\mu\nu\rho}. \tag{B.88}$$

That gives:

$$\Psi_3^{3,3} = \begin{pmatrix} 0 & (\mathbf{d} - \mathbf{e})\mathbf{i} - i(\mathbf{B} + \mathbf{Ci}) \\ -(\mathbf{d} + \mathbf{e})\mathbf{i} + i(\mathbf{B} - \mathbf{Ci}) & 0 \end{pmatrix}. \tag{B.89}$$

For the calculation of the 4-vector term:

$$N^{abcd}\Gamma_{abcd} = N^{\mu\nu 45}\Gamma_{\mu\nu 45} + N^{\mu\nu\rho 4}\Gamma_{\mu\nu\rho 4} + N^{\mu\nu\rho 5}\Gamma_{\mu\nu\rho 5} + N^{0123}\Gamma_{0123},$$

we let:

$$\mathbf{D} := N^{\mu\nu 45}\gamma_{\mu\nu} \;;\quad \mathbf{if} := N^{\mu\nu\rho 4}\gamma_{\mu\nu\rho} \;;\quad \mathbf{ig} := N^{\mu\nu\rho 5}\gamma_{\mu\nu\rho} \;;\quad \zeta := N^{0123}. \tag{B.90}$$

That gives:

$$\Psi_4^{3,3} = \begin{pmatrix} -\mathbf{Di} + \zeta\mathbf{i} - i(\mathbf{fi} + \mathbf{g}) & 0 \\ 0 & i(\mathbf{fi} - \mathbf{g}) + \mathbf{Di} + \zeta\mathbf{i} \end{pmatrix}. \tag{B.91}$$

For the calculation of the 5-vector term:

$$N^{abcde}\Gamma_{abcde} = N^{\mu\nu\rho 45}\Gamma_{\mu\nu\rho 45} + N^{01234}\Gamma_{01234} + N^{01235}\Gamma_{01235},$$

we let:

$$\mathbf{ih} := N^{\mu\nu\rho 45}\gamma_{\mu\nu\rho} \;;\quad \eta := N^{01234} \;;\quad \theta := N^{01235}. \tag{B.92}$$

That gives:

$$\Psi_5^{3,3} = \begin{pmatrix} 0 & \mathbf{h} + i(\theta - \eta\mathbf{i}) \\ -\mathbf{h} + i(\theta + \eta\mathbf{i}) & 0 \end{pmatrix}. \tag{B.93}$$

We then get:

$$\Psi^{3,3} = \begin{pmatrix} \Psi_l + i\Psi_b & \Psi_r + \Psi_g \\ \Psi_r - \Psi_g & \Psi_l - i\Psi_b \end{pmatrix}.. \tag{B.94}$$

That gives:

$$\Psi_l = \alpha + \mathbf{A} + \zeta\mathbf{i} - i(\mathbf{g} + \mathbf{ci}) = \mathcal{P}_1 - i\mathcal{I}_1, \tag{B.95}$$

$$\mathcal{P}_1 = \begin{pmatrix} \phi_e & 0 \\ 0 & \widehat{\phi}_e \end{pmatrix} = \alpha + \mathbf{A} + \zeta\mathbf{i};\ \mathcal{I}_1 = \begin{pmatrix} 0 & \phi_n \\ \widehat{\phi}_n & 0 \end{pmatrix} = \mathbf{g} + \mathbf{ci},$$

$$\Psi_b = \mathbf{b} - \mathbf{fi} + i(-\omega + \mathbf{Di} + \epsilon\mathbf{i}) = \mathcal{I}_4 - i\mathcal{P}_4, \tag{B.96}$$

$$\mathcal{P}_4 = \begin{pmatrix} \phi_{db} & 0 \\ 0 & \widehat{\phi}_{db} \end{pmatrix} = \omega - \mathbf{Di} - \epsilon\mathbf{i};\ \mathcal{I}_4 = \begin{pmatrix} 0 & \phi_{ub} \\ \widehat{\phi}_{ub} & 0 \end{pmatrix} = \mathbf{b} - \mathbf{fi},$$

$$\Psi_r = \mathbf{a} - \mathbf{ei} - i(-\theta + \mathbf{Ci} + \delta\mathbf{i}) = \mathcal{I}_2 - i\mathcal{P}_2, \tag{B.97}$$

$$\mathcal{P}_2 = \begin{pmatrix} \phi_{dr} & 0 \\ 0 & \widehat{\phi}_{dr} \end{pmatrix} = -\theta + \mathbf{Ci} + \delta\mathbf{i};\ \mathcal{I}_2 = \begin{pmatrix} 0 & \phi_{ur} \\ \widehat{\phi}_{ur} & 0 \end{pmatrix} = \mathbf{a} - \mathbf{ei},$$

$$\Psi_g = \mathbf{h} + \mathbf{di} - i(\beta + \mathbf{B} + \eta\mathbf{i}) = \mathcal{I}_3 - i\mathcal{P}_3, \tag{B.98}$$

$$\mathcal{P}_3 = \begin{pmatrix} \phi_{ug} & 0 \\ 0 & \widehat{\phi}_{ur} \end{pmatrix} = \beta + \mathbf{B} + \eta\mathbf{i};\ \mathcal{I}_3 = \begin{pmatrix} 0 & \phi_{ug} \\ \widehat{\phi}_{ug} & 0 \end{pmatrix} = \mathbf{h} + \mathbf{di},$$

All $\mathcal{P}_a$ terms, $a = 1, 2, 3, 4$, are even terms in space-time algebra. All $\mathcal{I}_a$ terms, $a = 1, 2, 3, 4$, are odd terms in space-time algebra. Remark that Ψ_l is alone while Ψ_r, Ψ_g, Ψ_b have exactly the same structure, different from Ψ_l. This is the geometric reason of the separation between leptons and quarks:

$$\begin{aligned} \Psi_l &= \mathcal{P}_1 - i\mathcal{I}_1, \\ i\Psi_r &= \mathcal{P}_2 + i\mathcal{I}_2, \\ i\Psi_g &= \mathcal{P}_3 + i\mathcal{I}_3, \\ i\Psi_b &= \mathcal{P}_4 + i\mathcal{I}_4. \end{aligned} \tag{B.99}$$

With $Cl_{1,3}$ the reverse of $\Psi = \alpha + \mathbf{A} + \zeta\mathbf{i} + (\mathbf{g} + \mathbf{ci})$ is:

$$\widetilde{\Psi} = \alpha - \mathbf{A} + \zeta\mathbf{i} + (\mathbf{g} - \mathbf{ci}), \tag{B.100}$$

while the reverse $Cl_{3,3}$ of:

$$\Psi^{3,3} = \Psi_0^{3,3} + \Psi_1^{3,3} + \Psi_2^{3,3} + \Psi_3^{3,3} + \Psi_4^{3,3} + \Psi_5^{3,3} + \Psi_6^{3,3}$$

is:

$$\widetilde{\Psi}^{3,3} = \Psi_0^{3,3} + \Psi_1^{3,3} - \Psi_2^{3,3} - \Psi_3^{3,3} + \Psi_4^{3,3} + \Psi_5^{3,3} - \Psi_6^{3,3}. \tag{B.101}$$

Thus the reversion in $Cl_{3,3}$ changes the sign of $\Psi_2^{3,3}$, $\Psi_3^{3,3}$, $\Psi_6^{3,3}$. That reversion thus changes the sign of ϵ, $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{e}$ and ω, and changes only these signs. Hence we see that Ψ_b is here set apart of the three other parts, each part transforms by $\Psi \mapsto \widetilde{\Psi}$ while Ψ_b transforms differently: $\phi_{db} \mapsto -\overline{\phi}_{db}$ that means by changing position in the diagonal. The only terms that changes sign are the scalar ϵ and ω, the vector $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ and the 2-vector $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Those changes of sign are different in $Cl_{3,3}$ compared to $Cl_{1,3}$. All differences are compensated by the change of position for ϕ_{db}.

B.2.1 Real 8×8 matrices

The $M_8(\mathbb{R})$ algebra made of real 8×8 matrices is 64-dimensional. That algebra is also the $Cl_{3,3}$ algebra. We may take as generators the Γ_a matrices (B.67) to (B.69), $a = 0, 1, 2, 3, 4, 5$. We also may consider no i and use as generators:

$$\Lambda_0 := \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} : \ \Lambda_1 := \begin{pmatrix} 0 & \gamma_0 \\ -\gamma_0 & 0 \end{pmatrix} : \ \Lambda_2 := \begin{pmatrix} 0 & -\gamma_{0135} \\ \gamma_{0135} & 0 \end{pmatrix},$$
$$\Lambda_3 := \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}; \ \Lambda_4 := \begin{pmatrix} 0 & \gamma_1 \\ -\gamma_1 & 0 \end{pmatrix}; \ \Lambda_5 := \begin{pmatrix} 0 & \gamma_3 \\ -\gamma_3 & 0 \end{pmatrix}. \tag{B.102}$$

We then have:

$$\Lambda_0^2 = \Lambda_4^2 = \Lambda_5^2 = I_8; \ \Lambda_1^2 = \Lambda_2^2 = \Lambda_3^2 = -I_8, \tag{B.103}$$

that means 3 vectors with square I_8 and 3 vectors with square $-I_8$. For the 2-vectors, we have:

$$\Lambda_{01} := \Lambda_0\Lambda_1 = \begin{pmatrix} -\gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} = -\Lambda_{10}; \ \Lambda_{01}^2 = I_8, \tag{B.104}$$

$$\Lambda_{02} = \begin{pmatrix} \gamma_{0135} & 0 \\ 0 & 2\gamma_{0135} \end{pmatrix} = -\Lambda_{20}; \ \Lambda_{02}^2 = I_8, \tag{B.105}$$

$$\Lambda_{03} = \begin{pmatrix} -\gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix} = -\Lambda_{30}; \ \Lambda_{03}^2 = I_8, \tag{B.106}$$

$$\Lambda_{04} = \begin{pmatrix} -\gamma_1 & 0 \\ 0 & \gamma_1 \end{pmatrix} = -\Lambda_{40}; \ \Lambda_{04}^2 = -I_8, \tag{B.107}$$

$$\Lambda_{05} = \begin{pmatrix} -\gamma_3 & 0 \\ 0 & \gamma_3 \end{pmatrix} = -\Lambda_{50}; \ \lambda_{05}^2 = -I_8, \tag{B.108}$$

$$\Lambda_{12} = \begin{pmatrix} \gamma_{135} & 0 \\ 0 & \gamma_{135} \end{pmatrix} = -\Lambda_{21}; \ \Lambda_{02}^2 = -I_8, \tag{B.109}$$

$$\Lambda_{13} = \begin{pmatrix} \gamma_{50} & 0 \\ 0 & \gamma_{50} \end{pmatrix} = -\Lambda_{31}; \ \Lambda_{13}^2 = -I_8, \tag{B.110}$$

$$\Lambda_{14} = \begin{pmatrix} \gamma_{10} & 0 \\ 0 & \gamma_{10} \end{pmatrix} = -\Lambda_{41}; \ \Lambda_{14}^2 = I_8, \tag{B.111}$$

$$\Lambda_{15} = \begin{pmatrix} \gamma_{30} & 0 \\ 0 & \gamma_{30} \end{pmatrix} = -\Lambda_{51}; \ \lambda_{15}^2 = I_8, \tag{B.112}$$

$$\Lambda_{23} = \begin{pmatrix} \gamma_{013} & 0 \\ 0 & \gamma_{013} \end{pmatrix} = -\Lambda_{32}; \ \Lambda_{23}^2 = -I_8, \tag{B.113}$$

$$\Lambda_{24} = \begin{pmatrix} \gamma_{053} & 0 \\ 0 & \gamma_{053} \end{pmatrix} = -\Lambda_{42}; \ \Lambda_{24}^2 = I_8, \tag{B.114}$$

$$\Lambda_{25} = \begin{pmatrix} \gamma_{015} & 0 \\ 0 & \gamma_{015} \end{pmatrix} = -\Lambda_{52}; \ \lambda_{25}^2 = I_8, \tag{B.115}$$

$$\Lambda_{34} = \begin{pmatrix} \gamma_{15} & 0 \\ 0 & \gamma_{15} \end{pmatrix} = -\Lambda_{43};\ \Lambda_{34}^2 = I_8, \tag{B.116}$$

$$\Lambda_{35} = \begin{pmatrix} \gamma_{35} & 0 \\ 0 & \gamma_{35} \end{pmatrix} = -\Lambda_{53};\ \lambda_{35}^2 = I_8, \tag{B.117}$$

$$\Lambda_{45} = \begin{pmatrix} \gamma_{31} & 0 \\ 0 & \gamma_{31} \end{pmatrix} = -\Lambda_{54};\ \lambda_{45}^2 = -I_8, \tag{B.118}$$

which means nine 2-vectors with square I_8, and only six 2-vectors with square $-I_8$. For the 3-vectors, we have:

$$\Lambda_{012} = \begin{pmatrix} 0 & \gamma_{135} \\ \gamma_{135} & 0 \end{pmatrix} = -\Lambda_{021};\ \Lambda_{012}^2 = -I_8, \tag{B.119}$$

$$\Lambda_{013} = \begin{pmatrix} 0 & \gamma_{50} \\ \gamma_{50} & 0 \end{pmatrix} = -\Lambda_{031};\ \Lambda_{013}^2 = -I_8, \tag{B.120}$$

$$\Lambda_{014} = \begin{pmatrix} 0 & \gamma_{10} \\ \gamma_{10} & 0 \end{pmatrix} = -\Lambda_{041};\ \Lambda_{014}^2 = I_8, \tag{B.121}$$

$$\Lambda_{015} = \begin{pmatrix} 0 & \gamma_{30} \\ \gamma_{30} & 0 \end{pmatrix} = -\Lambda_{051};\ \lambda_{015}^2 = I_8, \tag{B.122}$$

$$\Lambda_{023} = \begin{pmatrix} 0 & \gamma_{013} \\ \gamma_{013} & 0 \end{pmatrix} = -\Lambda_{032};\ \Lambda_{023}^2 = -I_8, \tag{B.123}$$

$$\Lambda_{024} = \begin{pmatrix} 0 & \gamma_{053} \\ \gamma_{053} & 0 \end{pmatrix} = -\Lambda_{042};\ \Lambda_{024}^2 = I_8, \tag{B.124}$$

$$\Lambda_{025} = \begin{pmatrix} 0 & \gamma_{015} \\ \gamma_{015} & 0 \end{pmatrix} = -\Lambda_{052};\ \lambda_{025}^2 = I_8, \tag{B.125}$$

$$\Lambda_{034} = \begin{pmatrix} 0 & \gamma_{15} \\ \gamma_{15} & 0 \end{pmatrix} = -\Lambda_{043};\ \Lambda_{034}^2 = I_8, \tag{B.126}$$

$$\Lambda_{035} = \begin{pmatrix} 0 & \gamma_{35} \\ \gamma_{35} & 0 \end{pmatrix} = -\Lambda_{053};\ \Lambda_{035}^2 = I_8, \tag{B.127}$$

$$\Lambda_{045} = \begin{pmatrix} 0 & \gamma_{31} \\ \gamma_{31} & 0 \end{pmatrix} = -\Lambda_{054};\ \Lambda_{045}^2 = -I_8, \tag{B.128}$$

$$\Lambda_{123} = \begin{pmatrix} 0 & \gamma_{13} \\ \gamma_{31} & 0 \end{pmatrix} = -\Lambda_{132};\ \Lambda_{123}^2 = I_8, \tag{B.129}$$

$$\Lambda_{124} = \begin{pmatrix} 0 & \gamma_{53} \\ \gamma_{35} & 0 \end{pmatrix} = -\Lambda_{142};\ \Lambda_{124}^2 = -I_8, \tag{B.130}$$

$$\Lambda_{125} = \begin{pmatrix} 0 & \gamma_{15} \\ \gamma_{51} & 0 \end{pmatrix} = -\Lambda_{152};\ \lambda_{125}^2 = -I_8, \tag{B.131}$$

$$\Lambda_{134} = \begin{pmatrix} 0 & \gamma_{501} \\ \gamma_{510} & 0 \end{pmatrix} = -\Lambda_{143};\ \Lambda_{134}^2 = -I_8, \tag{B.132}$$

$$\Lambda_{135} = \begin{pmatrix} 0 & \gamma_{503} \\ \gamma_{530} & 0 \end{pmatrix} = -\Lambda_{153};\ \lambda_{135}^2 = -I_8, \tag{B.133}$$

$$\Lambda_{145} = \begin{pmatrix} 0 & \gamma_{310} \\ \gamma_{301} & 0 \end{pmatrix} = -\Lambda_{154};\ \lambda_{145}^2 = I_8, \tag{B.134}$$

$$\Lambda_{234} = \begin{pmatrix} 0 & \gamma_{03} \\ \gamma_{30} & 0 \end{pmatrix} = -\Lambda_{243};\ \Lambda_{234}^2 = -I_8, \tag{B.135}$$

$$\Lambda_{235} = \begin{pmatrix} 0 & \gamma_{10} \\ \gamma_{01} & 0 \end{pmatrix} = -\Lambda_{253};\ \Lambda_{235}^2 = -I_8, \tag{B.136}$$

$$\Lambda_{245} = \begin{pmatrix} 0 & \gamma_{50} \\ \gamma_{05} & 0 \end{pmatrix} = -\Lambda_{254};\ \Lambda_{245}^2 = I_8, \tag{B.137}$$

$$\Lambda_{345} = \begin{pmatrix} 0 & \gamma_{153} \\ \gamma_{135} & 0 \end{pmatrix} = -\Lambda_{354};\ \Lambda_{345}^2 = I_8, \tag{B.138}$$

which means ten 3-vectors with square I_8 and ten 3-vectors with square $-I_8$. For the 4-vecteurs, we have:

$$\Lambda_{0123}^2 = \begin{pmatrix} \gamma_{31} & 0 \\ 0 & \gamma_{13} \end{pmatrix}^2 = -I_8;\ \Lambda_{0124}^2 = \begin{pmatrix} \gamma_{35} & 0 \\ 0 & \gamma_{53} \end{pmatrix}^2 = I_8, \tag{B.139}$$

$$\Lambda_{0125}^2 = \begin{pmatrix} \gamma_{51} & 0 \\ 0 & \gamma_{15} \end{pmatrix}^2 = I_8;\ \Lambda_{0134}^2 = \begin{pmatrix} \gamma_{051} & 0 \\ 0 & -\gamma_{051} \end{pmatrix}^2 = I_8, \tag{B.140}$$

$$\Lambda_{0135}^2 = \begin{pmatrix} -\gamma_{035} & 0 \\ 0 & \gamma_{035} \end{pmatrix}^2 = I_8;\ \Lambda_{0145}^2 = \begin{pmatrix} -\gamma_{031} & 0 \\ 0 & \gamma_{031} \end{pmatrix}^2 = -I_8, \tag{B.141}$$

$$\Lambda_{0234}^2 = \begin{pmatrix} \gamma_{30} & 0 \\ 0 & \gamma_{03} \end{pmatrix}^2 = I_8;\ \Lambda_{0235}^2 = \begin{pmatrix} \gamma_{01} & 0 \\ 0 & \gamma_{10} \end{pmatrix}^2 = I_8, \tag{B.142}$$

$$\Lambda_{0245}^2 = \begin{pmatrix} \gamma_{05} & 0 \\ 0 & \gamma_{50} \end{pmatrix}^2 = -I_8;\ \Lambda_{0345}^2 = \begin{pmatrix} -\gamma_{531} & 0 \\ 0 & \gamma_{531} \end{pmatrix}^2 = -I_8, \tag{B.143}$$

$$\Lambda_{1234}^2 = \begin{pmatrix} -\gamma_{3} & 0 \\ 0 & -\gamma_{3} \end{pmatrix}^2 = -I_8;\ \Lambda_{1235}^2 = \begin{pmatrix} \gamma_{1} & 0 \\ 0 & \gamma_{1} \end{pmatrix}^2 = -I_8, \tag{B.144}$$

$$\Lambda_{1245}^2 = \begin{pmatrix} \gamma_{5} & 0 \\ 0 & \gamma_{5} \end{pmatrix}^2 = I_8;\ \Lambda_{1345}^2 = \begin{pmatrix} \gamma_{0135} & 0 \\ 0 & \gamma_{0135} \end{pmatrix}^2 = I_8, \tag{B.145}$$

$$\Lambda_{2345}^2 = \begin{pmatrix} \gamma_{0} & 0 \\ 0 & \gamma_{0} \end{pmatrix}^2 = I_8, \tag{B.146}$$

which means nine 4-vectors with square I_8, but only six 4-vectors with square $-I_8$. For the 5-vectors, usually called pseudo-vectors, we have:

$$\Lambda_{01234}^2 = \begin{pmatrix} 0 & -\gamma_{3} \\ -\gamma_{3} & 0 \end{pmatrix}^2 = -I_8;\ \Lambda_{01235}^2 = \begin{pmatrix} 0 & \gamma_{1} \\ \gamma_{1} & 0 \end{pmatrix}^2 = -I_8, \tag{B.147}$$

$$\Lambda_{01245}^2 = \begin{pmatrix} 0 & \gamma_{5} \\ \gamma_{5} & 0 \end{pmatrix}^2 = I_8;\ \Lambda_{01345}^2 = \begin{pmatrix} 0 & \gamma_{0135} \\ \gamma_{0135} & \end{pmatrix}^2 = I_8, \tag{B.148}$$

$$\Lambda_{02345}^2 = \begin{pmatrix} 0 & \gamma_{0} \\ \gamma_{0} & 0 \end{pmatrix}^2 = I_8;\ \Lambda_{12345}^2 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}^2 = -I_8, \tag{B.149}$$

which means three 5-vectors with square I_8 and three 5-vectors with square $-I_8$. Finally for the pseudo-scalars we have only one generator:

$$\Lambda_{012345} = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix};\ \Lambda_{012345}^2 = I_8. \tag{B.150}$$

64 elements of a basis in $Cl_{3,3}$ split into $28 = 8 \times 7/2$ matrices with square $-I_8$, and $36 = 8 \times 9/2$ matrices with square I_8. Those 28 terms with square $-I_8$ generate the Lie algebra of the Lie group $SO(8)$. And the 36 terms with square I_8 are auto-adjoint and allows us the definition of 36 tensor densities associated to the relativistic wave of the electron.

The algebra generated by the Λ_a matrices contains space-time in the part generated by the Λ_μ, $\mu = 0, 1, 2, 3$. Similarly the algebra generated by Γ_a matrices contains space-time in the part generated by the Γ_μ, $\mu = 0, 1, 2, 3$. We thus have here two ways to describe space-time which have many common properties, for instance:

$$\begin{aligned} \Gamma_0^2 = I_8 = \Lambda_0^2;\ \Gamma_j^2 = -I_8 = \Lambda_j^2,\ j = 1, 2, 3 \\ \Gamma_{012345} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} = \Lambda_{012345}. \end{aligned} \tag{B.151}$$

But these two manners to describe space-time have nevertheless different properties since we have:

$$\sigma_1\sigma_2\sigma_3 = iI_2, \tag{B.152}$$

$$\Lambda_1\Lambda_2\Lambda_3 = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & -i\sigma_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_{31} \\ \gamma_{31} & 0 \end{pmatrix}, \tag{B.153}$$

while we have:

$$\Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -iI_2 \\ 0 & 0 & iI_2 & 0 \\ 0 & -iI_2 & 0 & 0 \\ iI_2 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_{3105} \\ \gamma_{3105} & 0 \end{pmatrix}. \tag{B.154}$$

B.2.2 $Cl_{3,3}$ with real 16×16 matrices

The Clifford algebra generated by the Γ_a matrices whose terms Γ_2 and Γ_4 contains i may also be described with the replacement of i by $-i\sigma_2$ and the replacement of 1 by I_2. This replacement transforms:

$$I_2 \mapsto I_4;\ \sigma_1 \mapsto \gamma_0;\ \sigma_2 \mapsto \gamma_{3105};\ \sigma_3 \mapsto \gamma_5;\ iI_2 \mapsto \gamma_{31}. \tag{B.155}$$

Hence the Γ_a generators are replaced by:

$$\mathbf{\Gamma}_\mu := \begin{pmatrix} 0 & -\Lambda_\mu \\ \Lambda_\mu & 0 \end{pmatrix}, \; \mu = 0, 1, 2, 3, \tag{B.156}$$

$$\mathbf{\Gamma}_4 := \begin{pmatrix} 0 & \Lambda_{45} \\ \Lambda_{45} & 0 \end{pmatrix}; \; \mathbf{\Gamma}_5 := \begin{pmatrix} 0 & \Lambda_{012345} \\ -\Lambda_{012345} & 0 \end{pmatrix}. \tag{B.157}$$

Similarly the Λ_a generators, $a = 0, 1, 2, 3, 4, 5$ may be replaced by:

$$\mathbf{\Lambda}_a = \begin{pmatrix} 0 & -\Lambda_a \\ \Lambda_a & 0 \end{pmatrix}, \; a = 0, 1, 2, 3, 4, 5. \tag{B.158}$$

Remark that the space-time part, which generates the $Cl_{1,3}$ algebra, is the same: $\mathbf{\Gamma}_\mu = \mathbf{\Lambda}_\mu, \; \mu = 0, 1, 2, 3$.

Appendix C

The Hydrogen Atom

We present the resolution of our improved equation for the hydrogen atom. Our resolution uses a method separating the variables in spherical coordinates. Angular functions use Gegenbauer's differential equation and polynomial functions previously used in linear Dirac theory. Here we study new solutions, for an electron with both electric charge and left and right masses. And we start from the fully invariant form of our improved wave equation.

The hydrogen atom is the jewel of Dirac theory. The solutions of the Dirac equation calculated by C. G. Darwin [11] in 1928, which we may also find in newer reports [106], are eigenvectors of *ad hoc* operators, obtained from the nonrelativistic theory of the Pauli equation. These operators cannot be angular momentum operators. These solutions give the expected number of states, the true formula for the energy levels, and have the expected nonrelativistic approximations. That was considered very satisfactory. But most of Darwin's solutions suffer the disadvantage that they have a Yvon–Takabayasi angle that is not everywhere defined and small. Therefore they cannot be linear approximations of the solutions to our improved equation.

We previously obtained [13] [20] new solutions of the linear Dirac equation, which have, on the contrary, an Yvon–Takabayasi angle everywhere defined and small (a maximum the fine structure constant). They may thus be the linear approximations of the solutions that we seek here, for the improved wave equation.

C.1 Separating variables

To solve the Dirac equation or our improved equation in the case of the hydrogen atom, besides the method originally used by Darwin, another method exists, thanks to H. Krüger [86], a very fine classic method from the mathematical point of view for a partial differential equation, separating the variables in spherical coordinates:

$$x^1 =: r\sin\theta\cos\varphi;\quad x^2 =: r\sin\theta\sin\varphi;\quad x^3 =: r\cos\theta. \tag{C.1}$$

We use the following notations [1]:

$$i_1 = \sigma_{23} := i\sigma_1\ ;\quad i_2 = \sigma_{31} := i\sigma_2\ ;\quad i_3 = \sigma_{12} := i\sigma_3, \tag{C.2}$$
$$u_1 : = \sin\theta\cos\varphi;\quad u_2 := \sin\theta\sin\varphi;\quad u_3 = \cos\theta;\quad \vec{u} := u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3,$$
$$S := e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2}\ ;\quad \Omega = \widehat{\Omega} := r^{-1}(\sin\theta)^{-\frac{1}{2}}S, \tag{C.3}$$
$$\vec{\partial}' := \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi. \tag{C.4}$$

H. Krüger obtained the remarkable identity see A.3.10:

$$\vec{\partial} = \Omega\vec{\partial}'\Omega^{-1}, \tag{C.5}$$

that gives with:

$$\begin{aligned}\nabla' : &= \partial_0 - \vec{\partial}' = \partial_0 - \Big(\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big),\\ &= \begin{pmatrix}\partial_0 - \partial_r & -\frac{1}{r}\partial_\theta + \frac{i}{r\sin\theta}\partial_\varphi\\ -\frac{1}{r}\partial_\theta - \frac{i}{r\sin\theta}\partial_\varphi & \partial_0 + \partial_r\end{pmatrix},\end{aligned} \tag{C.6}$$

the identity:

$$\Omega^{-1}(\nabla\widehat{\phi}) = \nabla'(\Omega^{-1}\widehat{\phi}). \tag{C.7}$$

Here we solve the improved wave equation with a left wave and a right wave, on its fully invariant form:

$$\begin{aligned}0 &= \overline{\phi}\nabla\widehat{\phi}\sigma_{21} + \overline{\phi}qA\widehat{\phi} + \rho\mathbf{m},\\ \rho e^{i\beta} &= \phi\overline{\phi} = \overline{\phi}\phi = \det(\phi).\end{aligned} \tag{C.8}$$

To separate, on a side the temporal variable $x^0 = ct$ and the angular variable φ, and on the other side the radial variable r and the angular variable θ, we let:

$$\phi =: \Omega X e^{(\lambda\varphi - Ex^0)i_3}, \tag{C.9}$$

where X is a function, with value in the Pauli algebra, of only r and θ, $\hbar cE$ is the energy of the electron, and λ is a real constant which will be interpreted as the magnetic quantum number. We then have:

$$\widehat{\phi} = \Omega\widehat{X}e^{(\lambda\varphi - Ex^0)i_3};\quad \overline{\phi} = e^{-(\lambda\varphi - Ex^0)i_3}\overline{X}\,\overline{\Omega}, \tag{C.10}$$
$$\overline{\Omega} = r^{-1}(\sin\theta)^{-\frac{1}{2}}e^{\frac{\theta}{2}i_2}e^{\frac{\varphi}{2}i_3} = \frac{1}{r\sqrt{\sin\theta}}S^{-1}.$$

1. S has nothing to do with the tensor S_3, and Ω must not be confused with the relativistic invariant Ω_1 and Ω_2 studied in Chapter 1.

We also have:

$$\begin{aligned}\rho e^{i\beta} &= \det(\phi) = \det(\Omega)\det(X)\det[e^{(\lambda\varphi - Ex^0+\delta)i_3}],\\ \det(\Omega) &= r^{-2}(\sin\theta)^{-1}\ ;\ \ \det[e^{(\lambda\varphi - Ex^0+\delta)i_3}] = 1,\\ \rho e^{i\beta} &= \frac{\det(X)}{r^2\sin\theta}.\end{aligned} \tag{C.11}$$

Hence if we let:

$$\rho_X e^{i\beta_X} := \det(X), \tag{C.12}$$

we get:

$$\rho = \frac{\rho_X}{r^2\sin\theta}\ ;\ \ \beta = \beta_X. \tag{C.13}$$

Hence with the form (C.9) of the electron wave, the Yvon-Takabayasi angle depends neither on time nor on the φ angle. It depends only on r and θ. For the wave equation, we next seek the solution as follows:

$$\begin{aligned}\Omega^{-1}\widehat{\phi} &:= \widehat{X}e^{(\lambda\varphi - Ex^0)i_3};\ \ \widehat{\phi} = \Omega\widehat{X}e^{(\lambda\varphi - Ex^0)i_3}, \qquad \text{(C.14)}\\ X &:= \begin{pmatrix} e^{a(r)+u(\theta)} & -e^{\bar{b}(r)+v(\theta)}\\ e^{c(r)+v(\theta)} & e^{\bar{d}(r)+u(\theta)}\end{pmatrix};\ \widehat{X} = \begin{pmatrix} e^{d(r)+u(\theta)} & -e^{\bar{c}+v(\theta)}\\ e^{b(r)+v(\theta)} & e^{\bar{a}+u(\theta)}\end{pmatrix}.\end{aligned}$$

That means that we suppose the existence of two functions with real value of θ and of four functions with complex value of r:

$$\begin{aligned} U &:= e^{u(\theta)};\ \ u(\theta) = \ln(U);\ \ V := e^{v(\theta)};\ \ v(\theta) = \ln(V), \qquad \text{(C.15)}\\ A &:= e^{a(r)};\ \ a(r) = \ln(A);\ \ B := e^{b(r)};\ \ b(r) = \ln(B),\\ C &:= e^{c(r)};\ \ c(r) = \ln(C);\ \ D := e^{d(r)};\ \ d(r) = \ln(D).\end{aligned}$$

And we have:

$$\begin{aligned}\widehat{X}e^{(\lambda\varphi - Ex^0)i_3} &= \begin{pmatrix} e^{d+u+i\lambda\varphi - iE\mathrm{x}^0} & -e^{\bar{c}+v-i\lambda\varphi+iE\mathrm{x}^0}\\ e^{b+v+i\lambda\varphi - iE\mathrm{x}^0} & e^{\bar{a}+u-i\lambda\varphi+iE\mathrm{x}^0}\end{pmatrix} \qquad \text{(C.16)}\\ \overline{\Omega}\Omega &= r^{-2}(\sin\theta)^{-1}.\end{aligned}$$

For the hydrogen atom we have: [2]

$$qA = qA^0 = -\frac{\alpha}{r}\ ;\ \ \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \tag{C.17}$$

2. For the Schrödinger equation the same potential is used. But the motion is supposed to be around the center of gravity of the hydrogen atom. A simple correction is made, using quantum mechanics for the motion around the center of gravity. That gives a tiny correction between energy levels in the hydrogen case and in the case of deuterium. No such correction is made with the relativistic equation, where the center of gravity does not have the same properties. That is to say that the correction was just made.

where α is the fine structure constant. We have:

$$qA\widehat{\phi} = -\frac{\alpha}{r}\widehat{\phi} = -\frac{\alpha}{r}\Omega\widehat{X}e^{(\lambda\varphi - Ex^0)i_3},$$
$$= \Omega\Big(-\frac{\alpha}{r}\Big)\begin{pmatrix} e^{d+u+i\lambda\varphi-iE\mathrm{x}^0} & -e^{\bar{c}+v-i\lambda\varphi+iE\mathrm{x}^0} \\ e^{b+v+i\lambda\varphi-iE\mathrm{x}^0} & e^{\bar{a}+u-i\lambda\varphi+iE\mathrm{x}^0} \end{pmatrix}. \quad \text{(C.18)}$$

And we get for the differential term:

$$\nabla'\big(\widehat{X}e^{(\lambda\varphi-E\mathrm{x}^0)i_3}\big)\sigma_{21} = \begin{pmatrix} -iM_1 & iM_3 \\ -iM_2 & iM_4 \end{pmatrix}, \quad \text{(C.19)}$$
$$-iM_1 = \Big[(-E+id')e^{d+u} + \frac{i}{r}\Big(v' + \frac{\lambda}{\sin\theta}\Big)e^{v+b}\Big]e^{i(\lambda\varphi-E\mathrm{x}^0)},$$
$$-iM_2 = \Big[(-E-ib')e^{b+v} + \frac{i}{r}\Big(u' - \frac{\lambda}{\sin\theta}\Big)e^{u+d}\Big]e^{i(\lambda\varphi-E\mathrm{x}^0)},$$
$$iM_3 = \Big[(E+i\bar{c}')e^{\bar{c}+v} - \frac{i}{r}\Big(u' - \frac{\lambda}{\sin\theta}\Big)e^{u+\bar{a}}\Big]e^{-i(\lambda\varphi-E\mathrm{x}^0)},$$
$$iM_4 = \Big[(-E+i\bar{a}')e^{\bar{a}+u} - \frac{i}{r}\Big(v' + \frac{\lambda}{\sin\theta}\Big)e^{v+\bar{c}}\Big]e^{-i(\lambda\varphi-E\mathrm{x}^0)},$$

We introduce a κ constant such as:

$$\Big(v' + \frac{\lambda}{\sin\theta}\Big)e^v = \kappa e^u; \quad \Big(u' - \frac{\lambda}{\sin\theta}\Big)e^u = -\kappa e^v,$$
$$\Big(V' + \frac{\lambda}{\sin\theta}V\Big) = \kappa U; \quad \Big(U' - \frac{\lambda}{\sin\theta}U\Big) = -\kappa V. \quad \text{(C.20)}$$

That system is linear and its structure does not change when κ changes sign. Consequently the general solution of the system could be sum or difference of a solution using $|\kappa|$, noted with $+$ indices, and of a solution using $-|\kappa|$, noted with $-$ indices:

$$X = \begin{pmatrix} e^{a_+u_+} \pm e^{a_-u_-} & -[e^{\bar{b}_++v_+} \pm e^{\bar{b}_-+v_-}] \\ e^{c_++v_+} \pm e^{c_-+v_-} & e^{\bar{d}_++u_+} \pm e^{\bar{b}_-+v_-} \end{pmatrix} \quad \text{(C.21)}$$

But we will establish that a simple relation exists between the u_- and u_+ functions, and also between the v_- and v_+ functions.

C.1.1 Resolution of the angular system

With $C = C(\theta)$ and if $\lambda > 0$ we let:

$$U := \sin^\lambda\theta\Big[\sin\Big(\frac{\theta}{2}\Big)C' - \Big(\kappa + \frac{1}{2} - \lambda\Big)\cos\Big(\frac{\theta}{2}\Big)C\Big], \quad \text{(C.22)}$$
$$V := \sin^\lambda\theta\Big[\cos\Big(\frac{\theta}{2}\Big)C' + \Big(\kappa + \frac{1}{2} - \lambda\Big)\sin\Big(\frac{\theta}{2}\Big)C\Big], \quad \text{(C.23)}$$
$$U =: \sqrt{\sin\theta}\cos\frac{\theta}{2}U_1; \quad V =: \sqrt{\sin\theta}\sin\frac{\theta}{2}V_1. \quad \text{(C.24)}$$

and if $\lambda < 0$ we let:

$$U := \sin^{-\lambda}\theta\Big[\cos\Big(\frac{\theta}{2}\Big)C' + \Big(\kappa + \frac{1}{2} + \lambda\Big)\sin\Big(\frac{\theta}{2}\Big)C\Big], \tag{C.25}$$

$$V := \sin^{-\lambda}\theta\Big[-\sin\Big(\frac{\theta}{2}\Big)C' + \Big(\kappa + \frac{1}{2} + \lambda\Big)\cos\Big(\frac{\theta}{2}\Big)C\Big], \tag{C.26}$$

$$U =: \sqrt{\sin\theta}\sin\frac{\theta}{2}V_1; \quad V =: -\sqrt{\sin\theta}\cos\frac{\theta}{2}U_1. \tag{C.27}$$

We thus have in both cases:

$$U_1 = \sin^{|\lambda|-\frac{1}{2}}(\theta)\Big[\tan\Big(\frac{\theta}{2}\Big)C' - \Big(\kappa + \frac{1}{2} - |\lambda|\Big)C\Big], \tag{C.28}$$

$$V_1 = \sin^{|\lambda|-\frac{1}{2}}(\theta)\Big[\mathrm{cotan}\Big(\frac{\theta}{2}\Big)C' + \Big(\kappa + \frac{1}{2} - |\lambda|\Big)C\Big]. \tag{C.29}$$

The angular system (C.20) is then equivalent (see [12]), for the two signs of λ, with the differential equation:

$$0 = C'' + \frac{2|\lambda|}{\tan\theta}C' + \Big[\Big(\kappa + \frac{1}{2}\Big)^2 - \lambda^2\Big]C. \tag{C.30}$$

We change the variable:

$$z = \cos\theta\ ; \quad f(z) = C[\theta(z)], \tag{C.31}$$

we hence obtain the Gegenbauer's differential equation :

$$0 = f''(z) - \frac{1 + 2|\lambda|}{1 - z^2}zf'(z) + \frac{(\kappa + \frac{1}{2})^2 - \lambda^2}{1 - z^2}f(z). \tag{C.32}$$

And we get:

$$\frac{C(\theta)}{C(0)} = \sum_{n=0}^{\infty}\frac{(|\lambda| - \kappa - \frac{1}{2})_n(|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!}\sin^{2n}(\frac{\theta}{2}), \tag{C.33}$$

with:

$$(a)_0 := 1 \ \ (a)_1 := a, \qquad (a)_n := a(a+1)\dots(a+n-1). \tag{C.34}$$

The $C(0)$ term is a factor in the U and V functions, it may be put into the radial functions. We may thus set $C(0) := 1$, without loss of generality. To get an integrable wave function the sum giving C must be finite. Thus an integer n must exist such that, if $\kappa > 0$:

$$C(\theta) = \sum_{n=0}^{n=\frac{1}{2}+|\kappa|-|\lambda|}\frac{(|\lambda| - \kappa - \frac{1}{2})_n(|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!}\sin^{2n}(\frac{\theta}{2})\,, \tag{C.35}$$

or, if $\kappa < 0$:

$$C(\theta) = \sum_{n=0}^{n=-\frac{1}{2}+|\kappa|-|\lambda|} \frac{(|\lambda| - \kappa - \frac{1}{2})_n(|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}(\frac{\theta}{2})\,. \tag{C.36}$$

And we have:

$$\Big(\tan\frac{\theta}{2}\Big)C' = \sum_{j=0}^{\infty} \frac{(|\lambda| - \kappa - \frac{1}{2})_j(|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j j!} j \sin^{2j}(\frac{\theta}{2}). \tag{C.37}$$

Using, for any a and for any integer number j, the remarkable identity:

$$(a)_j(a + j) = a(a + 1)_j, \tag{C.38}$$

we obtain:

$$\Big(\tan\frac{\theta}{2}\Big)C' - \Big(\kappa + \frac{1}{2} - |\lambda|\Big)C$$

$$= \sum_{j=0}^{j=n} \Big(j - \kappa - \frac{1}{2} + |\lambda|\Big) \frac{(|\lambda| - \kappa - \frac{1}{2})_j(|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j j!} \sin^{2j}(\frac{\theta}{2}) \tag{C.39}$$

$$= (|\lambda| - \kappa - \frac{1}{2}\Big) \sum_{j=0}^{\infty} \frac{(|\lambda| + \frac{1}{2} - \kappa)_j(|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j j!} \sin^{2j}(\frac{\theta}{2}). \tag{C.40}$$

The sum:

$$S_1 := \sum_{j=0}^{j=n} \frac{(|\lambda| + \frac{1}{2} - \kappa)_j(|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j j!} \sin^{2j}(\frac{\theta}{2}) \tag{C.41}$$

must necessary be finite so that angular functions be integrable, which imposes the normalization of the wave. Since we go from a term to the following in the products contained in each $(a)_j$ by adding always 1, and since a product is zero only if one of the terms is zero, it is necessary, for getting an integrable sum S_1, that at least one term of the suit be zero, and then all following products are zero. As the j-ranked term of the suit of factors in $(a)_j$ is $a + j - 1$, that implies the existence of an integer number n such as, for $\kappa > 0$, we have $|\lambda| + \frac{1}{2} - \kappa + n - 1 = 0$, which means $\kappa = n + |\lambda| - \frac{1}{2}$; and for $\kappa < 0$, an integer number n must exist such as $|\lambda| + \frac{1}{2} + \kappa + n - 1 = 0$, which means $-\kappa = n + |\lambda| - \frac{1}{2}$. Actually, since adding 0 does not change a sum, the sum stops at the previous term, hence for:

$$|\lambda| + \frac{1}{2} - |\kappa| + n - 1 = -1; \quad n = |\kappa| - |\lambda| - \frac{1}{2}. \tag{C.42}$$

$|\lambda| + \frac{1}{2}$ being an integer number, that imposes to the κ constant to be an integer, and this integer cannot be zero, otherwise the sum S_1 should be infinite. Moreover the n number is an index, which minimal value is

0. Hence $|\kappa| - |\lambda| - \frac{1}{2}$ is an integer number, and $|\lambda|$ is a half-odd integer, strictly less than $|\kappa|$. And the S_1 sum is, with the n integer specified above:

$$S_1 = \sum_{j=0}^{j=n} \frac{(|\lambda| + \frac{1}{2} - |\kappa|)_j (|\lambda| + |\kappa| + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j j!} \sin^{2j}(\frac{\theta}{2}), \tag{C.43}$$

which gives:

$$U_1 = (|\lambda| - \kappa - \frac{1}{2}) \sin^{|\lambda| - \frac{1}{2}}(\theta) S_1. \tag{C.44}$$

For V_1, we have:

$$\begin{aligned} V_1(\sin\theta)^{\frac{1}{2}-|\lambda|} &= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} C' + (\kappa + \frac{1}{2} - |\lambda|) C \qquad \text{(C.45)} \\ &= \sum_{j=0}^{j=n} \frac{(|\lambda| - \kappa - \frac{1}{2})_j (|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j j!} j \sin^{2j-1}(\frac{\theta}{2}) \frac{\cos^2(\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\ &+ (\kappa + \frac{1}{2} - |\lambda|) \sum_{n=0}^{\infty} \frac{(|\lambda| - \kappa - \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}(\frac{\theta}{2}) \\ &= \sum_{j=1}^{j=n} \frac{(|\lambda| - \kappa - \frac{1}{2})_j (|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j (j-1)!} \sin^{2j-2}(\frac{\theta}{2}) \\ &- \sum_{n=0}^{j=n} (j - \kappa - \frac{1}{2} + |\lambda|) \frac{(|\lambda| - \kappa - \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}(\frac{\theta}{2}). \end{aligned}$$

In the first sum we change indices; in the second sum we use (C.38), that gives:

$$\begin{aligned} V_1(\sin\theta)^{\frac{1}{2}-|\lambda|} &= \sum_{j=0}^{\infty} \frac{(|\lambda| - \frac{1}{2} - \kappa)_{j+1} (|\lambda| + \kappa + \frac{1}{2})_{j+1}}{(|\lambda| + \frac{1}{2})_{j+1} j!} \sin^{2j}(\frac{\theta}{2}) \\ &- (|\lambda| - \kappa - \frac{1}{2}) \sum_{n=0}^{\infty} \frac{(|\lambda| + \frac{1}{2} - \kappa)_j (|\lambda| + \kappa + \frac{1}{2})_j}{(\frac{1}{2} + |\lambda|)_j j!} \sin^{2j}(\frac{\theta}{2}). \end{aligned}$$

Next we have:

$$\begin{aligned} &V_1(\sin\theta)^{\frac{1}{2}-|\lambda|} = (|\lambda| - \frac{1}{2} - \kappa) \qquad \text{(C.46)} \\ &\times \sum_{j=0}^{\infty} \frac{(|\lambda| + \frac{1}{2} - \kappa)_j (|\lambda| + \kappa + \frac{1}{2})_j}{(|\lambda| + \frac{1}{2})_j (|\lambda| + \frac{1}{2} + j) j!} (|\lambda| + \kappa + \frac{1}{2} + j) \sin^{2j}(\frac{\theta}{2}) \\ &- (|\lambda| - \kappa - \frac{1}{2}) \sum_{j=0}^{j=n} \frac{(|\lambda| + \frac{1}{2} - \kappa)_j (|\lambda| + \kappa + \frac{1}{2})_j}{(\frac{1}{2} + |\lambda|)_j j!} \sin^{2j}(\frac{\theta}{2}). \end{aligned}$$

With due regard to the suppress by subtraction of the term with maximal index, we define a second sum:

$$S_2 := \sum_{j=0}^{j=n-1} \frac{(|\lambda|+\frac{1}{2}-|\kappa|)_j(|\lambda|+\frac{1}{2}+|\kappa|)_j}{(|\lambda|+\frac{1}{2})_{j+1}j!}\sin^{2j}(\frac{\theta}{2}) \tag{C.47}$$

And we get:

$$\begin{aligned} V_1(\sin\theta)^{\frac{1}{2}-|\lambda|} &= (|\lambda|-\frac{1}{2}-\kappa)[S_1+\kappa S_2]+(-|\lambda|+\kappa+\frac{1}{2})S_1 \\ &= \kappa(|\lambda|-\frac{1}{2}-\kappa)S_2. \end{aligned} \tag{C.48}$$

The S_2 sum, contrarily to S_1, is defined only if the maximal index n does not cancel. S_1 and S_2 are unchanged if we replace κ by $-\kappa$, except that the maximal index n is different when κ changes sign. Hence U_1 and V_1, and thus also U and V, depend on the sign of κ. We note with a $+$ index the functions obtained with $\kappa > 0$ and with a $-$ index that obtained with $\kappa < 0$. We have:

$$U_1^+ = \Big(|\lambda|-\frac{1}{2}-|\kappa|\Big)S_1;\quad U_1^- = \Big(|\lambda|-\frac{1}{2}+|\kappa|\Big)S_1, \tag{C.49}$$

$$U_1^- = kU_1^+;\quad k := \frac{|\lambda|-\frac{1}{2}+|\kappa|}{|\lambda|-\frac{1}{2}-|\kappa|};\quad U^- = kU^+, \tag{C.50}$$

$$V_1^+ = |\kappa|\Big(|\lambda|-\frac{1}{2}-|\kappa|\Big)S_2;\quad V_1^- = -|\kappa|\Big(|\lambda|-\frac{1}{2}+|\kappa|\Big)S_2, \tag{C.51}$$

$$V_1^- = -kV_1^+;\quad V^- = -kV^+. \tag{C.52}$$

If we seek for a solution containing both functions of the U^+ kind and U^- kind we do not obtain a more general solution, because:

$$\begin{aligned} A_1U^+ + A_2U^- &= A_1U^+ + A_2kU^+ = (A_1+kA_2)U^+ = A(r)U^+, \\ B_1V^+ + B_2V^- &= B_1V^+ - kB_2V^+ = (B_1-kB_2)V^+ = B(r)V^+. \end{aligned} \tag{C.53}$$

The most general solution thus uses $\kappa > 0$ and $U = U^+$, $V = V^+$:

$$X = \begin{pmatrix} AU & -\bar{B}V \\ CV & \bar{D}U \end{pmatrix};\quad \overline{X} = \begin{pmatrix} \bar{D}U & \bar{B}V \\ -CV & AU \end{pmatrix};\quad \widehat{X} = \begin{pmatrix} DU & -\bar{C}V \\ BV & \bar{A}U \end{pmatrix}. \tag{C.54}$$

C.1.2 Structure of the wave

We use the projectors p_r and p_l studied in A.3.9, together with the unitary vector $\vec{u}$, which also gives two projectors, p^+ and p^-, such that:

$$\begin{aligned} & p_r = \frac{1+\sigma_3}{2};\quad p_l = \frac{1-\sigma_3}{2};\quad \vec{u} := \frac{\vec{\mathrm{x}}}{r} = u_1\sigma_1+u_2\sigma_2+u_3\sigma_3, \\ & p^+ := \frac{1+\vec{u}}{2};\quad p^- := \frac{1-\vec{u}}{2};\quad p^++p^- = 1;\quad p^+-p^- = \vec{u}, \\ & p^+p^+ = p^+;\quad p^-p^- = p^-;\quad p^+p^- = p^-p^+ = 0. \end{aligned} \tag{C.55}$$

To simplify, we also use:

$$p := \lambda\varphi - E\mathrm{x}^0;\quad \underline{c} := \cos\frac{\theta}{2};\quad \underline{s} = \sin\frac{\theta}{2}. \tag{C.56}$$

We then have:

$$\underline{c}^2 = \frac{1+\cos\theta}{2} = \frac{1+u_3}{2};\quad \underline{s}^2 = \frac{1-\cos\theta}{2} = \frac{1-u_3}{2};\quad \underline{s}\,\underline{c} = \frac{\sin\theta}{2}. \tag{C.57}$$

We distinguish the two possible signs of the magnetic quantum number magnétique λ: if $\lambda > 0$ we let:

$$r\mathbf{F}_1 := U_1 A;\quad r\mathbf{F}_2 = U_1\bar{D};\quad r\mathbf{F}_3 = V_1 C;\quad r\mathbf{F}_4 = V_1\bar{B}, \tag{C.58}$$

$$U_1 =: e^{\mathbf{u}};\quad V_1 =: e^{\mathbf{v}};\quad \frac{A}{r} =: e^{\mathbf{a}};\quad \frac{C}{r} =: e^{\mathbf{c}},$$

$$\frac{\bar{B}}{r} =: e^{\bar{\mathbf{b}}};\quad \frac{\bar{D}}{r} =: e^{\bar{\mathbf{d}}};\quad \mathbf{F}_j =: e^{\mathbf{f}_j},\ j = 1,2,3,4. \tag{C.59}$$

Always if $\lambda > 0$, and with one or the other sign for κ, and with (C.22) and (C.24), we obtain:

$$\frac{X}{r\sqrt{\sin\theta}} := \begin{pmatrix} \mathbf{F}_1\cos\frac{\theta}{2} & -\mathbf{F}_4\sin\frac{\theta}{2} \\ \mathbf{F}_3\sin\frac{\theta}{2} & \mathbf{F}_2\cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \underline{c}\mathbf{F}_1 & -\underline{s}\mathbf{F}_4 \\ \underline{s}\mathbf{F}_3 & \underline{c}\mathbf{F}_2 \end{pmatrix} \tag{C.60}$$

$$\phi_{(\lambda>0)} = \frac{S}{r\sqrt{\sin\theta}} X e^{pi_3} = S\begin{pmatrix} \underline{c}\mathbf{F}_1 e^{ip} & -\underline{s}\mathbf{F}_4 e^{-ip} \\ \underline{s}\mathbf{F}_3 e^{ip} & \underline{c}\mathbf{F}_2 e^{-ip} \end{pmatrix}. \tag{C.61}$$

And since $\det(S) = 1$, we deduce:

$$\rho e^{i\beta} = \det(\phi) = \begin{vmatrix} \underline{c}\mathbf{F}_1 e^{ip} & -\underline{s}\mathbf{F}_4 e^{-ip} \\ \underline{s}\mathbf{F}_3 e^{ip} & \underline{c}\mathbf{F}_2 e^{-ip} \end{vmatrix} = \underline{c}^2\mathbf{F}_1\mathbf{F}_2 + \underline{s}^2\mathbf{F}_3\mathbf{F}_4. \tag{C.62}$$

We also have:

$$\phi^\dagger = \begin{pmatrix} \underline{c}\bar{\mathbf{F}}_1 e^{-ip} & \underline{s}\bar{\mathbf{F}}_3 e^{-ip} \\ -\underline{s}\bar{\mathbf{F}}_4 e^{ip} & \underline{c}\bar{\mathbf{F}}_2 e^{ip} \end{pmatrix} S^{-1}, \tag{C.63}$$

$$\phi\phi^\dagger = S\begin{pmatrix} \underline{c}\mathbf{F}_1 e^{ip} & -\underline{s}\mathbf{F}_4 e^{-ip} \\ \underline{s}\mathbf{F}_3 e^{ip} & \underline{c}\mathbf{F}_2 e^{-ip} \end{pmatrix}\begin{pmatrix} \underline{c}\bar{\mathbf{F}}_1 e^{-ip} & \underline{s}\bar{\mathbf{F}}_3 e^{-ip} \\ -\underline{s}\bar{\mathbf{F}}_4 e^{ip} & \underline{c}\bar{\mathbf{F}}_2 e^{ip} \end{pmatrix} S^{-1} \tag{C.64}$$

We get:

$$\mathrm{D}_0 = \phi\phi^\dagger = \mathrm{D}_0^0 + \vec{\mathrm{D}_0} \tag{C.65}$$

$$= S\begin{pmatrix} \underline{c}^2|\mathrm{F}_1|^2 + \underline{s}^2|\mathrm{F}_4|^2 & \underline{s}\,\underline{c}(\mathbf{F}_1\bar{\mathbf{F}}_3 - \mathbf{F}_4\bar{\mathbf{F}}_2) \\ \underline{s}\,\underline{c}(\mathbf{F}_3\bar{\mathbf{F}}_1 - \mathbf{F}_2\bar{\mathbf{F}}_4) & \underline{c}^2|\mathrm{F}_2|^2 + \underline{s}^2|\mathrm{F}_3|^2 \end{pmatrix} S^{-1} \tag{C.66}$$

$$\mathrm{D}_0^0 = \frac{1}{2}\left[\underline{c}^2|\mathrm{F}_1|^2 + \underline{s}^2|\mathrm{F}_4|^2 + \underline{c}^2|\mathrm{F}_2|^2 + \underline{s}^2|\mathrm{F}_3|^2\right] \tag{C.67}$$

$$\vec{\mathrm{D}_0} = \mathrm{D}_0 - \mathrm{D}_0^0. \tag{C.68}$$

If the magnetic quantum number λ is negative we let:

$$r\mathbf{G}_1 := V_1 A; \quad r\mathbf{G}_2 := U_1 \bar{B}; \quad r\mathbf{G}_3 := U_1 C; \quad r\mathbf{G}_4 := V_1 \bar{D}. \tag{C.69}$$

We have:

$$\frac{U}{\sqrt{\sin\theta}} = V_1 \sin\frac{\theta}{2}; \quad \frac{V}{\sqrt{\sin\theta}} = -U_1 \cos\frac{\theta}{2}, \tag{C.70}$$

$$\frac{X}{r\sqrt{\sin\theta}} = \begin{pmatrix} \underline{s}\mathbf{G}_1 & \underline{c}\mathbf{G}_2 \\ -\underline{c}\mathbf{G}_3 & \underline{s}\mathbf{G}_4 \end{pmatrix} \tag{C.71}$$

$$\phi_{(\lambda<0)} = \frac{S}{r\sqrt{\sin\theta}} X e^{p i_3} = S \begin{pmatrix} \underline{s}e^{ip}\mathbf{G}_1 & \underline{c}e^{-ip}\mathbf{G}_2 \\ -\underline{c}e^{ip}\mathbf{G}_3 & \underline{s}e^{-ip}\mathbf{G}_4 \end{pmatrix}. \tag{C.72}$$

We deduce:

$$\rho e^{i\beta} = \det(\phi) = \begin{vmatrix} \underline{s}\mathbf{G}_1 & \underline{c}\mathbf{G}_2 \\ -\underline{c}\mathbf{G}_3 & \underline{s}\mathbf{G}_4 \end{vmatrix} = \underline{s}^2\mathbf{G}_1\mathbf{G}_4 + \underline{c}^2\mathbf{G}_2\mathbf{G}_3. \tag{C.73}$$

We get:

$$\begin{aligned} \mathrm{D}_0 = \phi\phi^\dagger &= S \begin{pmatrix} \underline{s}e^{ip}\mathbf{G}_1 & \underline{c}e^{-ip}\mathbf{G}_2 \\ -\underline{c}e^{ip}\mathbf{G}_3 & \underline{s}e^{-ip}\mathbf{G}_4 \end{pmatrix} \begin{pmatrix} \underline{s}e^{-ip}\bar{\mathbf{G}}_1 & -\underline{c}e^{-ip}\bar{\mathbf{G}}_3 \\ \underline{c}e^{ip}\bar{\mathbf{G}}_2 & \underline{s}e^{ip}\bar{\mathbf{G}}_4 \end{pmatrix} S^{-1} \\ &= S \begin{pmatrix} \underline{s}^2|\mathbf{G}_1|^2 + \underline{c}^2|\mathbf{G}_2|^2 & \underline{s}\,\underline{c}(-\mathbf{G}_1\bar{\mathbf{G}}_3 + \mathbf{G}_2\bar{\mathbf{G}}_4) \\ \underline{s}\,\underline{c}(-\mathbf{G}_3\bar{\mathbf{G}}_1 + \mathbf{G}_4\bar{\mathbf{G}}_2) & \underline{c}^2|\mathbf{G}_3|^2 + \underline{s}^2|\mathbf{G}_4|^2 \end{pmatrix} S^{-1}. \end{aligned} \tag{C.74}$$

Thus we get:

$$\mathrm{D}_0^0 = \frac{1}{2}\big[\underline{s}^2|\mathbf{G}_1|^2 + \underline{c}^2|\mathbf{G}_2|^2 + \underline{c}^2|\mathbf{G}_3|^2 + \underline{s}^2|\mathbf{G}_4|^2\big]. \tag{C.75}$$

C.1.3 Separation for the r and θ variables

The wave equation reads:

$$\overline{\phi}(\nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi}) = -\rho\mathbf{m}, \tag{C.76}$$

$$\overline{X\Omega}\Omega\Big[\nabla'\Big(\widehat{X}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}\Big) - \frac{\alpha}{r}\widehat{X}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}\Big] = -\frac{\rho_X}{r^2\sin\theta}\mathbf{m}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3},$$

Since $\overline{\Omega}\Omega = \frac{1}{r^2\sin\theta}$, the equation becomes simpler as:

$$\overline{X}\Big[\nabla'\Big(\widehat{X}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}\Big)\sigma_{21} - \frac{\alpha}{r}\widehat{X}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}\Big] = -\rho_X\mathbf{m}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}. \tag{C.77}$$

And we have:

$$\nabla' = \begin{pmatrix} \partial_0 - \partial_r & -\frac{1}{r}\partial_\theta + \frac{i}{r\sin\theta}\partial_\varphi \\ -\frac{1}{r}\partial_\theta - \frac{i}{r\sin\theta}\partial_\varphi & \partial_0 + \partial_r \end{pmatrix}, \tag{C.78}$$

$$\nabla'\big(\widehat{X}e^{(\lambda\varphi-E\mathrm{x}^0)i_3}\big)\sigma_{21} - \frac{\alpha}{r}\widehat{X}e^{(\lambda\varphi-E\mathrm{x}^0)i_3} = \frac{1}{r}\begin{pmatrix} \mathcal{D}U & -\mathcal{C}V \\ \mathcal{B}V & \mathcal{A}U \end{pmatrix} e^{(\lambda\varphi-E\mathrm{x}^0)i_3},$$

with:

$$\frac{\mathcal{D}U}{r} = -\Big(E + \frac{\alpha}{r}\Big)DU + iD'U + \frac{i\kappa}{r}BU \quad \text{(C.79)}$$
$$\frac{\mathcal{D}}{r} = -\Big(E + \frac{\alpha}{r}\Big)D + iD' + \frac{i\kappa}{r}B.$$

And similarly we have:

$$\frac{\mathcal{B}}{r} = -\Big(E + \frac{\alpha}{r}\Big)B - iB' - \frac{i\kappa}{r}D, \quad \text{(C.80)}$$
$$\frac{\mathcal{C}}{r} = -\Big(E + \frac{\alpha}{r}\Big)\bar{C} - i\bar{C}' - \frac{i\kappa}{r}\bar{A},$$
$$\frac{\mathcal{A}}{r} = -\Big(E + \frac{\alpha}{r}\Big)\bar{A} + i\bar{A}' + \frac{i\kappa}{r}\bar{C},$$

The wave equation $\overline{\phi}(\nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi}) = -\rho\mathbf{m}$ gives:

$$\overline{X\Omega}\Omega\Big[\nabla'\Big(\widehat{X}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}\Big) - \frac{\alpha}{r}\widehat{X}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3}\Big] = -\frac{\rho_X}{r\sin\theta}\mathbf{m}e^{(\lambda\widehat{\phi}-E\mathrm{x}^0)i_3},$$
$$\overline{X}\begin{pmatrix}\mathcal{D}U & -\mathcal{C}V\\ \mathcal{B}V & \mathcal{A}U\end{pmatrix} = -r\rho_X\mathbf{m}. \quad \text{(C.81)}$$

That gives:

$$\begin{pmatrix}\bar{D}\mathcal{D}U^2 + \bar{B}\mathcal{B}V^2 & (-\bar{D}\mathcal{C} + \bar{B}\mathcal{A})UV\\ (-C\mathcal{D} + A\mathcal{B})UV & A\mathcal{A}U^2 + C\mathcal{C}V^2\end{pmatrix} = -\begin{pmatrix}r\rho_X\mathbf{l} & 0\\ 0 & r\rho_X\mathbf{r}\end{pmatrix}. \quad \text{(C.82)}$$

That matrix equation is equivalent to the system:

$$\bar{D}\mathcal{D}U^2 + \bar{B}\mathcal{B}V^2 = -r\rho_X\mathbf{l}, \quad \text{(C.83)}$$
$$\bar{D}\mathcal{C} = \bar{B}\mathcal{A}, \quad \text{(C.84)}$$
$$C\mathcal{D} = A\mathcal{B}, \quad \text{(C.85)}$$
$$A\mathcal{A}U^2 + C\mathcal{C}V^2 = -r\rho_X\mathbf{r} \quad \text{(C.86)}$$

And we have:

$$\rho_x e^{i\beta} = \det(X) = A\bar{D}U^2 + C\bar{B}V^2;\quad \rho_X e^{-i\beta} = \bar{A}DU^2 + \bar{C}BV^2. \quad \text{(C.87)}$$

The first equation of the system, (C.83) hence gives:

$$(\bar{D}\mathcal{D}U^2 + \bar{B}\mathcal{B}V^2)(D\bar{\mathcal{D}}U^2 + B\bar{\mathcal{B}}V^2)$$
$$= r^2\mathbf{l}^2(A\bar{D}U^2 + C\bar{B}V^2)(\bar{A}DU^2 + \bar{C}BV^2). \quad \text{(C.88)}$$

We will get a separation of variables only if we identify the terms containing U^4, V^4 and U^2V^2:

$$\bar{D}\mathcal{D}D\bar{\mathcal{D}} = r^2\mathbf{l}^2A\bar{D}\bar{A}D, \quad \text{(C.89)}$$
$$\bar{D}\mathcal{D}B\bar{\mathcal{B}} + \bar{B}\mathcal{B}D\bar{\mathcal{D}} = r^2\mathbf{l}^2(A\bar{D}\bar{C}B + \bar{A}DC\bar{B}) \quad \text{(C.90)}$$
$$\bar{B}\mathcal{B}B\bar{\mathcal{B}} = r^2\mathbf{l}^2C\bar{B}\bar{C}B. \quad \text{(C.91)}$$

First and third equations simplify and give:

$$\mathcal{D}\bar{\mathcal{D}} = r^2\mathbf{l}^2 A\bar{A}; \quad |\mathcal{D}| = r\mathbf{l}|A|; \quad \mathcal{D} = e^{il_1} r\mathbf{l}A, \tag{C.92}$$

$$\mathcal{B}\bar{\mathcal{B}} = r^2\mathbf{l}^2 C\bar{C}; \quad |\mathcal{B}| = r\mathbf{l}|C|; \quad \mathcal{B} = e^{il_2} r\mathbf{l}C, \tag{C.93}$$

where l_1 and l_2 are real numbers. The equation (C.85) becomes:

$$Ce^{il_1}\mathbf{l}A = Ae^{il_2}\mathbf{l}C,$$
$$l_1 = l_2 \mod 2\pi \tag{C.94}$$

Equation (C.90) becomes:

$$e^{i(l_1-l_2)}\bar{D}AB\bar{C} + e^{i(l_2-l_1)}\bar{A}DC\bar{B} = A\bar{D}\bar{C}B + \bar{A}DC\bar{B},$$
$$\Re[e^{i(l_1-l_2+a_2-d_2-c_2+b_2)}] = \Re[e^{i(-a_2+d_2+c_2-b_2)}],$$
$$\cos(l_1 - l_2 + a_2 - d_2 - c_2 + b_2) = \cos(-a_2 + d_2 + c_2 - b_2),$$
$$l_1 - l_2 + a_2 - d_2 - c_2 + b_2 = \pm(-a_2 + d_2 + c_2 - b_2). \tag{C.95}$$

Hence the equation is satisfied since $l_2 = l_1$. The same work, for (C.86), gives:

$$(A\mathcal{A}U^2 + C\mathcal{C}V^2)(\bar{A}\bar{\mathcal{A}}U^2 + \bar{C}\bar{\mathcal{C}}V^2)$$
$$= r^2\mathbf{r}^2(A\bar{D}U^2 + C\bar{B}V^2)(\bar{A}DU^2 + \bar{C}BV^2). \tag{C.96}$$

We get a separation of variables only by identifying terms containing U^4, V^4 and U^2V^2 :

$$A\mathcal{A}\bar{A}\bar{\mathcal{A}} = r^2\mathbf{r}^2 A\bar{D}\bar{A}D, \tag{C.97}$$

$$A\mathcal{A}\bar{C}\bar{\mathcal{C}} + C\bar{\mathcal{C}}\bar{A}\mathcal{A} = r^2\mathbf{r}^2(A\bar{D}\bar{C}B + \bar{A}DC\bar{B}) \tag{C.98}$$

$$C\mathcal{C}\bar{C}\bar{\mathcal{C}} = r^2\mathbf{r}^2 C\bar{B}\bar{C}B. \tag{C.99}$$

First and third equations simplify as:

$$\mathcal{A}\bar{\mathcal{A}} = r^2\mathbf{r}^2\bar{D}D; \quad |\mathcal{A}| = r\mathbf{r}|\bar{D}|; \quad \mathcal{A} = re^{ir_1}\mathbf{r}\bar{D}, \tag{C.100}$$

$$\mathcal{C}\bar{\mathcal{C}} = r^2\mathbf{r}^2\bar{B}B; \quad |\mathcal{C}| = r\mathbf{r}|\bar{B}|; \quad \mathcal{C} = re^{ir_2}\mathbf{r}\bar{B}, \tag{C.101}$$

where r_1 and r_2 are real numbers. The equation (C.84) becomes:

$$\bar{D}e^{ir_2}\mathbf{r}\bar{B} = \bar{B}e^{ir_1}\mathbf{r}\bar{D},$$
$$r_2 = r_1 \mod 2\pi \tag{C.102}$$

Similarly (C.98) is satisfied since $r_2 = r_1$. We hence get separation of variables only if radial functions satisfy the radial system:

$$\left(E + \frac{\alpha}{r}\right)D - iD' - i\frac{\kappa}{r}B + \mathbf{l}e^{il_1}A = 0, \tag{C.103}$$

$$\left(E + \frac{\alpha}{r}\right)B + iB' + i\frac{\kappa}{r}D + \mathbf{l}e^{il_1}C = 0, \tag{C.104}$$

$$\left(E + \frac{\alpha}{r}\right)A + iA' + i\frac{\kappa}{r}C - \mathbf{r}e^{-ir_1}D = 0, \tag{C.105}$$

$$\left(E + \frac{\alpha}{r}\right)C - iC' - i\frac{\kappa}{r}A - \mathbf{r}e^{-ir_1}B = 0. \tag{C.106}$$

C.2 Radial system

The system of radial equations (C.103), C.104 (C.105) and (C.106) is identical to the linear system coming from the Dirac equation, when l_1 and r_1 cancel. The asymptotic behavior at infinity of the A, B, C and D functions is hence the same as with the Dirac equation. It is that asymptotic behavior which implies to consider only functions such that:

$$\begin{aligned}
A &= (a_0 r^s + a_1 r^{s+1} + ... + a_n r^{s+n})e^{-\Lambda m r},\\
B &= (b_0 r^s + b_1 r^{s+1} + ... + b_n r^{s+n})e^{-\Lambda m r},\\
C &= (c_0 r^s + c_1 r^{s+1} + ... + c_n r^{s+n})e^{-\Lambda m r},\\
D &= (d_0 r^s + d_1 r^{s+1} + ... + d_n r^{s+n})e^{-\Lambda m r},
\end{aligned} \tag{C.107}$$

where s and Λ are two positive real numbers. The n degree of those polynomials may cancel. In this case sums are reduced to their first term. [3] The equation (C.105), (C.104), (C.106) and (C.103) are respectively equivalent to:

$$\begin{aligned}
0 = -E(&\quad + a_0 r + \cdots + a_{n-1} r^n + a_n r^{n+1})\\
&-\alpha(a_0 + a_1 r + \cdots + a_n r^n)\\
+i\Lambda m(&\quad + a_0 r + \cdots + a_{n-1} r^n + a_n r^{n+1})\\
&-i[s a_0 + (s+1)a_1 r + \cdots + a_n(s+n) r^n]\\
&-i\kappa(c_0 + c_1 r + \cdots + c_n r^n)\\
+e^{-ir_1}\mathbf{r}(&\quad + d_0 r + \cdots + d_{n-1} r^n + d_n r^{n+1}).
\end{aligned} \tag{C.108}$$

$$\begin{aligned}
0 = -E(&\quad + b_0 r + \cdots + b_{n-1} r^n + b_n r^{n+1})\\
&-\alpha(b_0 + b_1 r + \cdots + b_n r^n)\\
+i\Lambda m(&\quad + b_0 r + \cdots + b_{n-1} r^n + b_n r^{n+1})\\
&-i[s b_0 + (s+1)b_1 r + \cdots + (s+n) b_n r^n]\\
&-i\kappa(d_0 + d_1 r + \cdots + d_n r^n)\\
-e^{il_1}\mathbf{l}(&\quad + c_0 r + \cdots + c_{n-1} r^n + c_n r^{n+1}).
\end{aligned} \tag{C.109}$$

$$\begin{aligned}
0 = -E(&\quad + c_0^r + \cdots + c_{n-1} r^n + c_n r^{n+1})\\
&-\alpha(c_0 + c_1 r + \cdots + c_n r^n)\\
-i\Lambda m(&\quad + c_0 r + \cdots + c_{n-1} r^n + c_n r^{n+1}]\\
&+i[s c_0 + (s+1)c_1 r + \cdots + (s+n) c_n r^n)\\
&+i\kappa(a_0 + a_1 r + \cdots + a_n r^n)\\
+e^{-ir_1}\mathbf{r}(&\quad + b_0 r + \cdots + b_{n-1} r^n + b_n r^{n+1}).
\end{aligned} \tag{C.110}$$

3. The n degree of the radial polynomials is linked to the principal quantum number $\mathbf{n}$ of the Bohr atom by $\mathbf{n} = |\kappa| + n$.

$$\begin{aligned}0 = -E(&\quad + d_0 r + \cdots + d_{n-1} r^n + d_n r^{n+1}) \\ -\alpha(&d_0 + d_1 r + \cdots + d_n r^n) \\ -i\Lambda m(&\quad + d_0 r + \cdots + d_{n-1} r^n + d_n r^{n+1}) \\ +i[&s d_0 + (s+1) d_1 r + \cdots + (s+n) d_n r^n] \\ +i\kappa(&b_0 + b_1 x + \cdots + b_n x^n) \\ -e^{il_1}\mathbf{l}(&\quad + a_0 r + \cdots + a_{n-1} r^n + a_n r^{n+1}).\end{aligned} \tag{C.111}$$

We hence get three kinds of numeric systems: with 0 index, with index between 0 and n, and with n index. For the 0 index the system depends only on α, κ and s :

$$\begin{aligned}0 &= (-\alpha - is)a_0 - i\kappa c_0 \ ; \quad 0 = (-\alpha - is)b_0 - i\kappa d_0, \\ 0 &= i\kappa a_0 + (-\alpha + is)c_0 \ ; \quad 0 = i\kappa b_0 + (-\alpha + is)d_0.\end{aligned} \tag{C.112}$$

This system is exactly that coming from the linear Dirac equation. It is made of two sub-systems. We get a non null solution only if the determinant of each sub-system is null and hence if s is such that:

$$0 = (-\alpha - is)(-\alpha + is) - \kappa^2; \ \kappa^2 = s^2 + \alpha^2; \ s = \sqrt{\kappa^2 - \alpha^2}. \tag{C.113}$$

It is from there that comes thecondition $\kappa \neq 0$. Let:

$$s + i\alpha := |\kappa| e^{i\gamma}; \ s - i\alpha = |\kappa| e^{-i\gamma}. \tag{C.114}$$

System (C.112) gives:

$$c_0 = \frac{i\alpha - s}{|\kappa|} a_0 = -e^{-i\gamma} a_0; \ d_0 = \frac{i\alpha - s}{|\kappa|} b_0 = -e^{-i\gamma} b_0. \tag{C.115}$$

With the n index we get the following system:

$$\begin{aligned}0 &= (-E + i\Lambda m)a_n + e^{-ir_1}\mathbf{r} d_n; \ 0 = (-E + i\Lambda m)b_n - e^{il_1}\mathbf{l} c_n, \\ 0 &= -(E + i\Lambda m)d_n - e^{il_1}\mathbf{l} a_n; \ 0 = -(E + i\Lambda m)c_n + e^{-ir_1}\mathbf{r} b_n.\end{aligned} \tag{C.116}$$

That system is also made of two sub-systems, different but with the same determinant. A non zero solution exists only if that determinant is null:

$$0 = D = \begin{vmatrix} -E + i\Lambda m & e^{-ir_1}\mathbf{r} \\ -e^{il_1}\mathbf{l} & -E - i\Lambda m \end{vmatrix} = E^2 + \Lambda^2 m^2 - e^{i(l_1 - r_1 + \pi)}\mathbf{lr}, \tag{C.117}$$

That implies:

$$l_1 - r_1 = \pi \mod 2\pi; \quad e^{ir_1} = -e^{il_1}. \tag{C.118}$$

Then we have:

$$\begin{aligned}0 = D &\Leftrightarrow \mathbf{lr} = E^2 + \Lambda^2 m^2, \\ 0 = D &\Leftrightarrow E^2 + \Lambda^2 m^2 = m_g^2.\end{aligned} \tag{C.119}$$

A δ number exists such that:

$$E + i\Lambda m := m_g e^{i\delta}; \; E - i\Lambda m = m_g e^{-i\delta}. \tag{C.120}$$

System (C.116) is then reduced to:

$$d_n = \frac{E - i\Lambda m}{e^{-ir_1}\mathbf{r}} a_n = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}} e^{i(r_1 - \delta)} a_n, \tag{C.121}$$

$$b_n = \frac{E + i\Lambda m}{e^{-ir_1}\mathbf{r}} c_n = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}} e^{i(r_1 + \delta)} c_n. \tag{C.122}$$

C.2.1 Constant radial polynomial case

If $n = 0$, which means if radial polynomials are constant, the radial system is reduced to (C.115), (C.121) and (C.122), and we thus have:

$$d_0 = \frac{E - i\Lambda m}{e^{-ir_1}\mathbf{r}} a_0 = \frac{i\alpha - s}{|\kappa|} b_0, \tag{C.123}$$

$$c_0 = \frac{E - i\Lambda m}{e^{ir_1}\mathbf{l}} b_0 = \frac{i\alpha - s}{|\kappa|} a_0. \tag{C.124}$$

We then get:

$$\frac{(E - i\Lambda m)^2}{\mathbf{lr}} a_0 b_0 = \frac{(i\alpha - s)^2}{\kappa^2} b_0 a_0; \; \frac{E - i\Lambda m}{m_g} = \mp \frac{s - i\alpha}{|\kappa|}. \tag{C.125}$$

And since E, s and m_g are positive, we have only one possibility:

$$\begin{aligned} \frac{E}{m_g} &= \frac{s}{|\kappa|}; \; \frac{\Lambda m}{m_g} = \frac{\alpha}{|\kappa|}; \quad \alpha E = s\Lambda m, \\ \alpha^2 E^2 &= s^2 (m_g^2 - E^2); \quad E = \frac{s m_g}{\sqrt{s^2 + \alpha^2}}, \\ E &= \frac{m_g}{\sqrt{1 + \frac{\alpha^2}{(s+0)^2}}}, \end{aligned} \tag{C.126}$$

which is Sommerfeld's formula in the $n = 0$ case.

C.2.2 Non constant radial polynomials

In this case at least one system contains two consecutive indices:

$$0 = (E - i\Lambda m) a_{n-1} + [\alpha + i(s+n)] a_n + i\kappa c_n - e^{-ir_1}\mathbf{r} d_{n-1}, \tag{C.127}$$

$$0 = (E - i\Lambda m) b_{n-1} + [\alpha + i(s+n)] b_n + i\kappa d_n - e^{ir_1}\mathbf{l} c_{n-1}, \tag{C.128}$$

$$0 = (E + i\Lambda m) c_{n-1} + [\alpha - i(s+n)] c_n - i\kappa a_n - e^{-ir_1}\mathbf{r} b_{n-1}, \tag{C.129}$$

$$0 = (E + i\Lambda m) d_{n-1} + [\alpha - i(s+n)] d_n - i\kappa b_n - e^{ir_1}\mathbf{l} a_{n-1}. \tag{C.130}$$

Multiplying (C.127) by $E+i\Lambda m$, (C.130) by $e^{-ir_1}\mathbf{r}$, and adding, terms with index $n-1$ are suppressed, which gives:

$$\begin{aligned}0 =&(E+i\Lambda m)[\alpha+i(s+n)]a_n - i\kappa e^{-ir_1}\mathbf{r}b_n\\ &+ i\kappa(E+i\Lambda m)c_n + e^{-ir_1}\mathbf{r}[\alpha - i(s+n)]d_n.\end{aligned} \tag{C.131}$$

When n is the maximal index, which means the degree of each radial polynomial, we may use (C.121) and (C.122), that gives:

$$\begin{aligned}0 =&(E+i\Lambda m)[\alpha+i(s+n)]a_n - i\kappa e^{-ir_1}\mathbf{r}\frac{e^{ir_1}\mathbf{l}}{E-i\Lambda m}c_n\\ &+ i\kappa(E+i\Lambda m)c_n + e^{-ir_1}\mathbf{r}[\alpha - i(s+n)]\frac{E-i\Lambda m}{e^{-ir_1}\mathbf{r}}a_n\\ &= \big[(E+i\Lambda m)[\alpha+i(s+n)] + [\alpha-i(s+n)](E-i\Lambda m)\big]a_n.\end{aligned} \tag{C.132}$$

That is possible, since $a_n \neq 0$, only if:

$$\begin{aligned}0 &= (E+i\Lambda m)[\alpha+i(s+n)] + [\alpha-i(s+n)](E-i\Lambda m),\\ \alpha E &= (s+n)\Lambda m,\\ \alpha E &= (s+n)\sqrt{m_g^2 - E^2}.\end{aligned} \tag{C.133}$$

We deduce:

$$\alpha^2E^2 = (s+n)^2(m_g^2 - E^2), \tag{C.134}$$

$$[\alpha^2 + (s+n)^2]E^2 = (s+n)^2 m_g^2, \tag{C.135}$$

$$E = \frac{(s+n)m_g}{\sqrt{(s+n)^2+\alpha^2}}, \tag{C.136}$$

$$E = \frac{m_g}{\sqrt{1+\dfrac{\alpha^2}{(s+n)^2}}}. \tag{C.137}$$

That formula, first obtained by Sommerfeld in the framework of the corpuscular theory of Bohr, is thus effective for any integer n. We deduce:

$$m_g e^{i\delta} = E + i\Lambda m = \frac{E}{s+n}(s+n+i\alpha), \tag{C.138}$$

$$E = \frac{(s+n)m_g}{\sqrt{(s+n)^2+\alpha^2}}; \quad e^{i\delta} = \frac{s+n+i\alpha}{\sqrt{\kappa^2+2sn+n^2}}. \tag{C.139}$$

With any middle index $j > 0$ the system (C.127) to (C.130) allows us to calculate terms with index j from terms with index $j-1$:

$$(s+j-i\alpha)a_j + \kappa c_j = (\Lambda m + iE)a_{j-1} - ie^{-ir_1}\mathbf{r}d_{j-1} \tag{C.140}$$

$$\kappa a_j + (s+j+i\alpha)c_j = (\Lambda m - iE)c_{j-1} + ie^{-ir_1}\mathbf{r}b_{j-1} \tag{C.141}$$

$$(s+j+i\alpha)d_j + \kappa b_j = (\Lambda m - iE)d_{j-1} + ie^{ir_1}\mathbf{l}a_{j-1} \tag{C.142}$$

$$\kappa d_j + (s+j-i\alpha)b_j = (\Lambda m + iE)b_{j-1} - ie^{ir_1}\mathbf{l}c_{j-1}. \tag{C.143}$$

That gives:

$$\begin{pmatrix} a_j \\ c_j \end{pmatrix} = \begin{pmatrix} \frac{s+j+i\alpha}{j(2s+j)} & \frac{-\kappa}{j(2s+j)} \\ \frac{-\kappa}{j(2s+j)} & \frac{s+j-i\alpha}{j(2s+j)} \end{pmatrix} \begin{pmatrix} (\Lambda m + iE)a_{j-1} - ie^{-ir_1}\mathbf{r}d_{j-1} \\ (\Lambda m - iE)c_{j-1} + ie^{-ir_1}\mathbf{r}b_{j-1} \end{pmatrix}, \quad \text{(C.144)}$$

$$\begin{pmatrix} b_j \\ d_j \end{pmatrix} = \begin{pmatrix} \frac{s+j+i\alpha}{j(2s+j)} & \frac{-\kappa}{j(2s+j)} \\ \frac{-\kappa}{j(2s+j)} & \frac{s+j-i\alpha}{j(2s+j)} \end{pmatrix} \begin{pmatrix} (\Lambda m + iE)b_{j-1} - ie^{ir_1}\mathbf{l}c_{j-1} \\ (\Lambda m - iE)d_{j-1} + ie^{ir_1}\mathbf{l}a_{j-1} \end{pmatrix}. \quad \text{(C.145)}$$

And we get the recurrence relations:

$$\begin{aligned} j(2s+j)a_j &= (s+j+i\alpha)[(\Lambda m + iE)a_{j-1} - ie^{-ir_1}\mathbf{r}d_{j-1}] \\ &\quad - \kappa[(\Lambda m - iE)c_{j-1} + ie^{-ir_1}\mathbf{r}b_{j-1}] & \text{(C.146)} \\ j(2s+j)c_j &= -\kappa[(\Lambda m + iE)a_{j-1} - ie^{-ir_1}\mathbf{r}d_{j-1}] \\ &\quad (s+j-i\alpha)[(\Lambda m - iE)c_{j-1} + ie^{-ir_1}\mathbf{r}b_{j-1} & \text{(C.147)} \\ j(2s+j)b_j &= (s+j+i\alpha)[(\Lambda m + iE)b_{j-1} - ie^{ir_1}\mathbf{l}c_{j-1}] \\ &\quad - \kappa[(\Lambda m - iE)d_{j-1} + ie^{ir_1}\mathbf{l}a_{j-1}] & \text{(C.148)} \\ j(2s+j)d_j &= -\kappa[(\Lambda m + iE)b_{j-1} - ie^{ir_1}\mathbf{l}c_{j-1}] \\ &\quad (s+j-i\alpha)[(\Lambda m - iE)d_{j-1} + ie^{ir_1}\mathbf{l}b_{j-1}. & \text{(C.149)} \end{aligned}$$

The resolution of the wave equation ends by the normalization, that needs the calculation of $\mathbf{J}^0$. We have:

$$\mathrm{D}_R = \phi p_r \phi^\dagger; \quad \mathrm{D}_L = \phi p_l \phi^\dagger, \quad \text{(C.150)}$$

$$1 = \iiint \mathrm{dv}\frac{\mathbf{J}^0}{\hbar c} = \frac{1}{k\hbar c}\iiint \mathrm{dv}\Big(\frac{m}{\mathbf{l}}\mathrm{D}_L^0 + \frac{m}{\mathbf{r}}\mathrm{D}_R^0\Big).$$

With spherical coordinates, that gives:

$$1 = l_P^{-3}\int_0^\infty r^2 \mathrm{dr}\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi}\Big(\frac{m}{\mathbf{l}}\mathrm{D}_L^0 + \frac{m}{\mathbf{r}}\mathrm{D}_R^0\Big)\mathrm{d}\varphi. \quad \text{(C.151)}$$

C.3 Case of little quantum numbers

The degree of U_1 and V_1 polynomials is $n = |\kappa| - 1/2 - |\lambda|$, hence the absolute value of λ is a half-odd integer strictly inferior to the absolute value of κ. For the principal quantum number $\mathbf{n} = 1$, and since the smallest value of $|\kappa|$ is 1, the n degree of radial polynomials is necessary 0: that are the states $\mathbf{1}s_{1/2}$: $n = 0$, $|\kappa| = 1$, $\lambda = \pm 1/2$. Next for the sum $|\kappa| + n$, if $\mathbf{n} = 2$, there are only two possibilities, because $2 = 2 + 0 = 1 + 1$. If $|\kappa| = 2$ then $n = 0$ and $|\lambda|$ is lower than 2, thus we have 4 states $\mathbf{2}p_{3/2}$: $\lambda = \pm 3/2$ and $\lambda = \pm 1/2$. The second possibility is $|\kappa| = 1$, with which $|\lambda|$ is lower than 1, thus can be only 1/2. That gives the two states $\mathbf{2}s_{1/2}$: $|\kappa| = 1$, $\lambda = \pm 1/2$, and we will see that they are double.

C.3.1 States 1s1/2

As $\kappa = 1$ and $\lambda = 1/2$, (C.44) and (C.48) give:

$$U_1 = -1;\ V_1 = -1;\ \mathbf{F}_1 = -\frac{A}{r};\ \mathbf{F}_2 = -\frac{\bar{D}}{r};\ \mathbf{F}_3 = -\frac{C}{r};\ \mathbf{F}_4 = -\frac{\bar{B}}{r}. \quad \text{(C.152)}$$

With (C.61) we then get;

$$\phi = -\frac{S}{r}\begin{pmatrix} \underline{c}A & -\underline{s}\bar{B} \\ \underline{s}C & \underline{c}\bar{D} \end{pmatrix} e^{pi_3};\ p = \frac{\varphi}{2} - E\mathrm{x}^0, \quad \text{(C.153)}$$

$$\rho e^{i\beta} = \frac{1}{r^2}(\underline{c}^2 A\bar{D} + \underline{s}^2 C\bar{B}), \quad \text{(C.154)}$$

$$\mathrm{D}_0 = \frac{S}{r^2}\begin{pmatrix} \underline{c}^2 A\bar{A} + \underline{s}^2 B\bar{B} & \underline{s}\,\underline{c}(A\bar{C} - D\bar{B}) \\ \underline{s}\,\underline{c}(C\bar{A} - B\bar{D}) & \underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C} \end{pmatrix} S^{-1}, \quad \text{(C.155)}$$

$$\mathrm{D}_3 = \phi\sigma_3\phi^\dagger = \frac{S}{r^2}\begin{pmatrix} \underline{c}^2 A\bar{A} - \underline{s}^2 B\bar{B} & \underline{s}\,\underline{c}(A\bar{C} + D\bar{B}) \\ \underline{s}\,\underline{c}(C\bar{A} + B\bar{D}) & -\underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C} \end{pmatrix} S^{-1}, \quad \text{(C.156)}$$

$$\mathrm{D}_0^0 = \frac{1}{2r^2}(\underline{c}^2 A\bar{A} + \underline{s}^2 B\bar{B} + \underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C}), \quad \text{(C.157)}$$

$$\mathrm{D}_3^0 = \frac{1}{2r^2}(\underline{c}^2 A\bar{A} - \underline{s}^2 B\bar{B} - \underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C}). \quad \text{(C.158)}$$

We let:

$$|a_0| := \alpha_1;\ a_0 := \alpha_1 e^{i\alpha_2};\ \ 2\zeta := \ln\mathbf{l} - \ln\mathbf{r}. \quad \text{(C.159)}$$

Next, since $n = 0$ and $\kappa = 1$, for the state 1s1/2 such that $\lambda = 1/2$, we have:

$$E = sm_g;\ \Lambda m = \alpha m_g;\ \frac{E - i\Lambda m}{\mathbf{r}} = e^{-i\gamma}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}} = e^{\zeta - i\gamma}, \quad \text{(C.160)}$$

$$\begin{aligned} &b_0 = -\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}a_0 = -\alpha_1 e^{\zeta + i\alpha_2};\ c_0 = -\alpha_1 e^{i(\alpha_2 - \gamma)};\ d_0 = \alpha_1 e^{\zeta + i(\alpha_2 - \gamma)}, \\ &A = \alpha_1 e^{-\Lambda mr + i\alpha_2} r^s;\ \bar{B} = -\alpha_1 r^s e^{\zeta - \Lambda mr - i\alpha_2} \\ &C = -\alpha_1 e^{i(\alpha_2 - \gamma)} r^s e^{-\Lambda mr}; \bar{D} = \alpha_1 e^{i(\gamma - \alpha_2)} r^s e^{\zeta - \Lambda mr}, \\ &A\bar{A} = C\bar{C} = \alpha_1^2 r^{2s} e^{-2\Lambda mr}, \quad \text{(C.161)} \\ &B\bar{B} = D\bar{D} = \alpha_1^2 \frac{\mathbf{l}}{\mathbf{r}} r^{2s} e^{-2\Lambda mr}. \end{aligned}$$

l_P being the Planck length, we thus get:

$$\mathbf{J} = \frac{1}{k}(\mathrm{D}_0 + \frac{d}{m_a}\mathrm{D}_3);\ \ \frac{\mathbf{J}^0}{\hbar c} = \frac{2\mathbf{l}}{l_P^3 m_a}\alpha_1^2 r^{2s-2} e^{-2\Lambda mr}. \quad \text{(C.162)}$$

The normalization of the wave allows us to calculate α_1, the phase α_2 remaining arbitrary since the wave equation is gauge invariant:

$$1 = \iiint \mathrm{dv}\frac{\mathbf{J}^0_{\pm}}{\hbar c} = \frac{\alpha_1^2}{l_P^3}\frac{2\mathbf{l}}{m_a}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\mathrm{d}\theta\int_0^{\infty}\mathrm{dr}e^{-2\Lambda mr}r^{2s}$$
$$= \frac{8\pi\alpha_1^2\mathbf{l}\Gamma(2s+1)}{l_P^3 m_a(2\Lambda m)^{2s+1}};\ \alpha_1 = \frac{m_a^{1/2}l_P^{3/2}(2\Lambda)^{s+1/2}}{[8\pi\mathbf{l}\Gamma(2s+1)]^{1/2}}, \tag{C.163}$$

Remark the unicity of the obtained solution, up to a phase α_2.

For the other state 1**s**1/2, where $\kappa = 1$ and $\lambda = -1/2$ we have:

$$V = -\sqrt{\sin\theta}\cos\frac{\theta}{2}U_1;\quad U_1 = -1;\quad U = \sqrt{\sin\theta}\sin\frac{\theta}{2}V_1;\quad V_1 = -1,$$
$$r\mathbf{G}_1 = -A;\quad r\mathbf{G}_3 = -C;\quad r\mathbf{G}_2 = -\bar{B};\quad r\mathbf{G}_4 = -\bar{D} \tag{C.164}$$
$$\phi_{(\lambda=-\frac{1}{2})} = \frac{S}{r}\begin{pmatrix} -\underline{s}A & -\underline{c}\bar{B} \\ \underline{c}C & -\underline{s}\bar{D}\end{pmatrix}e^{p\,i_3};\quad p = -\frac{\varphi}{2} - E\mathrm{x}^0, \tag{C.165}$$

and we have:

$$\rho e^{i\beta} = \det(\phi) = \underline{s}^2\mathbf{G}_1\mathbf{G}_4 + \underline{c}^2\mathbf{G}_2\mathbf{G}_3 = \frac{1}{r^2}(\underline{s}^2A\bar{D} + \underline{c}^2C\bar{B}), \tag{C.166}$$
$$\mathrm{D}_0 = \phi\phi^\dagger = \frac{S}{r^2}\begin{pmatrix} \underline{s}^2A\bar{A} + \underline{c}^2B\bar{B} & \underline{s}\,\underline{c}(-A\bar{C} + D\bar{B}) \\ \underline{s}\,\underline{c}(-C\bar{A} + B\bar{D}) & \underline{c}^2C\bar{C} + \underline{s}^2D\bar{D}\end{pmatrix}S^{-1}, \tag{C.167}$$
$$\mathrm{D}_3 = \phi\sigma_3\phi^\dagger = \frac{S}{r^2}\begin{pmatrix} \underline{s}^2A\bar{A} - \underline{c}^2B\bar{B} & \underline{s}\,\underline{c}(-A\bar{C} - D\bar{B}) \\ \underline{s}\,\underline{c}(-C\bar{A} - B\bar{D}) & \underline{c}^2C\bar{C} - \underline{s}^2D\bar{D}\end{pmatrix}S^{-1}, \tag{C.168}$$

We then have:

$$2r^2(\phi\mathbf{m}\phi^\dagger)^0 = \mathbf{l}(\underline{s}^2A\bar{A} + \underline{c}^2C\bar{C}) + \mathbf{r}(\underline{c}^2B\bar{B} + \underline{s}^2D\bar{D})$$
$$= \mathbf{l}A\bar{A} + \mathbf{r}B\bar{B}. \tag{C.169}$$

Hence the normalization presents exactly the same result as with the positive value of the magnetic quantum number λ.

C.3.2 Non constant radial polynomials

We then let:

$$A := r^s e^{-\Lambda mr}P_a;\ P_a := a_0 + a_1 r + \cdots + a_n r^n, \tag{C.170}$$
$$B := r^s e^{-\Lambda mr}P_b;\ P_b := b_0 + b_1 r + \cdots + b_n r^n, \tag{C.171}$$
$$C := r^s e^{-\Lambda mr}P_c;\ P_c := c_0 + c_1 r + \cdots + c_n r^n, \tag{C.172}$$
$$D := r^s e^{-\Lambda mr}P_d;\ P_d := d_0 + d_1 r + \cdots + d_n r^n. \tag{C.173}$$

If $\kappa = 1$ and $\lambda = 1/2$, $\mathbf{J}^0$ is given by (C.162) and $\rho e^{i\beta}$ by (C.154).
If $\kappa = 1$ and $\lambda = -1/2$ then $\mathbf{J}^0$ is given by (C.169) and $\rho e^{i\beta}$ by (C.166).

C.3.3 Case $\mathbf{n} = 2$ and $n = 1$: 2s1/2 states

In that particular case we have $|\kappa| = 1$, and we may choose $\kappa = 1$ which corresponds to the states 2**s**1/2. In those states we have $|\lambda| < 1$, thus λ may be only 1/2 and $-1/2$. Since $\kappa^2 = 1$, we have:

$$s^2 + \alpha^2 = 1; \; s = \sqrt{1-\alpha^2}; \; s + i\alpha =: e^{i\gamma}; \; s - i\alpha = e^{-i\gamma}; \; \gamma \approx \alpha, \quad \text{(C.174)}$$

$$c_0 = -\frac{e^{-i\gamma}}{|\kappa|}a_0 = -|\kappa|e^{-i\gamma}a_0; \quad d_0 = -\frac{e^{-i\gamma}}{|\kappa|}b_0 = -|\kappa|e^{-i\gamma}b_0, \quad \text{(C.175)}$$

$$e^{i\delta} = \frac{s+1+i\alpha}{\sqrt{2s+2}} = \frac{E + i\Lambda m}{m_g}; \quad e^{-i\delta} = \frac{s+1-i\alpha}{\sqrt{2s+2}} = \frac{E - i\Lambda m}{m_g}, \quad \text{(C.176)}$$

$$e^{2i\delta} = \frac{(s+1+i\alpha)^2}{2s+2} = \frac{[s^2+2s+1+2i(s+1)\alpha - \alpha^2}{2s+2}$$

$$= \frac{2s^2+2s+(2s+2)i\alpha}{2s+2} = s + i\alpha = e^{i\gamma}; \quad \gamma = 2\delta \bmod 2\pi. \quad \text{(C.177)}$$

$$e^{i\delta} = e^{i(\gamma-\delta)}; \quad e^{i(\gamma+\delta)} = e^{3i\delta}. \quad \text{(C.178)}$$

We also have:

$$d_1 = \frac{E - i\Lambda m}{e^{-ir_1}\mathbf{r}}a_1 = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i(r_1-\delta)}a_1, \quad \text{(C.179)}$$

$$b_1 = \frac{E + i\Lambda m}{e^{-ir_1}\mathbf{r}}c_1 = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i(r_1+\delta)}c_1. \quad \text{(C.180)}$$

$$aE = (s+1)\Lambda m; \quad \Lambda m = \frac{\alpha m_g}{\sqrt{2s+2}}, \quad \text{(C.181)}$$

$$(s+1+i\alpha)(\Lambda m + iE) = im_g\sqrt{2s+2},$$
$$(s+1-i\alpha)(\Lambda m - iE) = -im_g\sqrt{2s+2}. \quad \text{(C.182)}$$

Relations between coefficients with index 0 and coefficients with index 1 become:

$$(2s+1)a_1 = (s+1+i\alpha)[(\Lambda m + iE)a_0 - ie^{-ir_1}\mathbf{r}d_0]$$
$$- |\kappa|[(\Lambda m - iE)c_0 + ie^{-ir_1}\mathbf{r}b_0], \quad \text{(C.183)}$$

$$(2s+1)c_1 = -|\kappa|[(\Lambda m + iE)a_0 - ie^{-ir_1}\mathbf{r}d_0]$$
$$+ (s+1-i\alpha)[(\Lambda m - iE)c_0 + ie^{-ir_1}\mathbf{r}b_0] \quad \text{(C.184)}$$

$$(2s+1)b_1 = (s+1+i\alpha)[(\Lambda m + iE)b_0 - ie^{ir_1}\mathbf{l}c_0]$$
$$- |\kappa|[(\Lambda m - iE)d_0 + ie^{ir_1}\mathbf{l}a_0] \quad \text{(C.185)}$$

$$(2s+1)d_1 = -|\kappa|[(\Lambda m + iE)b_0 - ie^{ir_1}\mathbf{l}c_0]$$
$$+ (s+1-i\alpha)[(\Lambda m - iE)d_0 + ie^{ir_1}\mathbf{l}b_0]. \quad \text{(C.186)}$$

With (C.175), (C.183) becomes:

$$\begin{aligned}(2s+1)a_1 &= \sqrt{2s+2}e^{i\delta}[im_g e^{-i\delta}a_0 - ie^{-ir_1}\mathbf{r}(-|\kappa|e^{-i\gamma}b_0)] \\ &\quad - |\kappa|[-im_g e^{i\delta}(-\kappa)e^{-i\gamma}a_0 + ie^{-ir_1}\mathbf{r}b_0] \\ &= im_g e^{-i\delta}(\sqrt{2s+2}e^{i\delta} - 1)a_0 + i|\kappa|\mathbf{r}e^{-ir_1}(\sqrt{2s+2}e^{-i\delta} - 1)b_0 \\ &= im_g e^{i\delta}a_0 + i|\kappa|\mathbf{r}e^{-i(r_1+2\delta)}b_0. \end{aligned} \quad (C.187)$$

Similarly (C.184 becomes:

$$\begin{aligned}(2s+1)c_1 &= -|\kappa|[im_g e^{-i\delta}a_0 - ie^{-ir_1}\mathbf{r}(-|\kappa|)e^{-2i\delta}b_0] \\ &\quad + \sqrt{2s+2}e^{-i\delta}[-im_g e^{i\delta}(-|\kappa|e^{-2i\delta}a_0) + ie^{-ir_1}\mathbf{r}b_0] \\ &= i|\kappa|m_g e^{-i\delta}(\sqrt{2s+2}e^{-i\delta} - 1)a_0 + ie^{-ir_1}\mathbf{r}e^{-2i\delta}(\sqrt{2s+2}e^{i\delta} - 1)b_0 \\ &= i|\kappa|m_g e^{-3i\delta}a_0 + ie^{-ir_1}\mathbf{r}b_0. \end{aligned} \quad (C.188)$$

That gives, with (C.179) and (C.180):

$$a_1 = \frac{i}{2s+1}(m_g e^{i\delta}a_0 + |\kappa|\mathbf{r}e^{-i(2\delta+r_1)}b_0), \quad (C.189)$$

$$b_1 = \frac{i}{2s+1}(m_g e^{i\delta}b_0 + |\kappa|\mathbf{l}e^{i(r_1-2\delta)}a_0), \quad (C.190)$$

$$c_1 = \frac{i}{2s+1}(|\kappa|m_g e^{-3i\delta}a_0 + \mathbf{r}e^{-ir_1}b_0), \quad (C.191)$$

$$d_1 = \frac{i}{2s+1}(|\kappa|m_g e^{-3i\delta}b_0 + \mathbf{l}e^{ir_1}a_0). \quad (C.192)$$

We also have:

$$c_0 = -e^{-i\gamma}a_0; \quad d_0 = -e^{-i\gamma}b_0, \quad (C.193)$$

$$\begin{aligned} a_1 &= \frac{i}{2s+1}(m_g e^{i\delta}a_0 + \mathbf{r}e^{-i(\gamma+r_1)}b_0); \quad b_1 = \frac{i}{2s+1}(m_g e^{i\delta}b_0 + \mathbf{l}e^{i(r_1-\gamma)}a_0), \\ c_1 &= \frac{i}{2s+1}(m_g e^{-3i\delta}a_0 + \mathbf{r}e^{-ir_1}b_0); \quad d_1 = \frac{i}{2s+1}(m_g e^{-3i\delta}b_0 + \mathbf{l}e^{ir_1}a_0). \end{aligned} \quad (C.194)$$

We obtained for the state $2\mathbf{s}1/2$ with quantum numbers $|\kappa| = 1$ and $\lambda = +1/2$:

$$\begin{aligned} \phi &= -\frac{S}{r}\begin{pmatrix} \underline{c}A & -\underline{s}\bar{B} \\ \underline{s}C & \underline{c}\bar{D} \end{pmatrix} e^{pi_3}; \quad p = \frac{\varphi}{2} - E\mathrm{x}^0, \\ \rho e^{i\beta} &= \frac{1}{r^2}(\underline{c}^2 A\bar{D} + \underline{s}^2 C\bar{B}) = \frac{1}{r^2}[\frac{1+u_3}{2}A\bar{D} + \frac{1-u_3}{2}C\bar{B}], \\ \mathrm{D}_0 &= \frac{S}{r^2}\begin{pmatrix} \underline{c}^2 A\bar{A} + \underline{s}^2 B\bar{B} & \underline{s}\,\underline{c}(A\bar{C} - D\bar{B}) \\ \underline{s}\,\underline{c}(C\bar{A} - B\bar{D}) & \underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C} \end{pmatrix} S^{-1}, \\ \mathrm{D}_3 &= \phi\sigma_3\phi^\dagger = \frac{S}{r^2}\begin{pmatrix} \underline{c}^2 A\bar{A} - \underline{s}^2 B\bar{B} & \underline{s}\,\underline{c}(A\bar{C} + D\bar{B}) \\ \underline{s}\,\underline{c}(C\bar{A} + B\bar{D}) & -\underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C} \end{pmatrix} S^{-1} \end{aligned} \quad (C.195)$$

$$\mathrm{D}_0^0 = \frac{1}{2r^2}(\underline{c}^2 A\bar{A} + \underline{s}^2 B\bar{B} + \underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C}), \tag{C.196}$$
$$\mathrm{D}_3^0 = \frac{1}{2r^2}(\underline{c}^2 A\bar{A} - \underline{s}^2 B\bar{B} - \underline{c}^2 D\bar{D} + \underline{s}^2 C\bar{C}).$$

The invariant Ω_2 and the angle β depends only on r and u_3, and they cancel in the equatorial plane $u_3 = 0$. In that equatorial plane, if Ω_1 also cancels then ρ also cancels. But the resolution of the equation relies on the equivalence with the equation on the completely invariant form, thus on the hypothesis that the wave function takes value in Cl_3^*. We can see that the existence of values of r canceling ρ comes from the term containing $a_0 b_0^*$ in Ω_1. There is nevertheless a solution to that difficulty, which is the cancellation of a_0 or b_0. If b_0 is null we have:

$$c_0 = -e^{-i\gamma} a_0; \quad d_0 = b_0 = 0; \quad a_1 = \frac{i}{2s+1} m_g e^{i\delta} a_0,$$
$$b_1 = \frac{i}{2s+1}\mathbf{l} e^{i(r_1-\gamma)} a_0; \quad c_1 = \frac{i}{2s+1} m_g e^{-3i\delta} a_0; \quad d_1 = \frac{i}{2s+1}\mathbf{l} e^{ir_1} a_0. \tag{C.197}$$

$$\frac{A\bar{D}}{r^2} = r^{2s-2} e^{-2\Lambda mr}(a_0 + a_1 r)\bar{d}_1 r$$
$$= r^{2s-1} e^{-2\Lambda mr} \frac{\mathbf{l} a_0 \bar{a}_0}{2s+1} e^{-ir_1}\Big(-i + \frac{e^{i\delta}}{2s+1} m_g r\Big), \tag{C.198}$$
$$\frac{C\bar{B}}{r^2} = r^{2s+2} e^{-2\Lambda mr}(c_0 + c_1 r)\bar{b}_1 r$$
$$= r^{2s-1} e^{-2\Lambda mr} \frac{\mathbf{l} a_0 \bar{a}_0}{2s+1} e^{-ir_1}\Big(i + \frac{e^{-i\delta}}{2s+1} m_g r\Big). \tag{C.199}$$

And then (C.195) becomes:

$$\rho e^{i\beta} = r^{2s+2} e^{-2\Lambda mr} \frac{\mathbf{l} a_0 \bar{a}_0}{2s+1} e^{-ir_1}$$
$$\times \Big[\sqrt{\frac{s+1}{2}}\frac{m_g r}{2s+1} + iu_3\Big(-1 + \frac{\alpha}{\sqrt{2s+2}(2s+1)} m_g r\Big)\Big]. \tag{C.200}$$

Since $\sqrt{\frac{s+1}{2}\frac{m_g r}{2s+1}}$ is strictly positive for r non null, that is the same with ρ: ϕ is thus with value in Cl_3^*. It remains to determine the value of $|a_0|$, through the normalization. We have:

$$\frac{\mathbf{J}}{\hbar c} = \frac{1}{l_P^3}\Big(\frac{m}{\mathbf{l}}\mathrm{D}_L + \frac{m}{\mathbf{r}}\mathrm{D}_R\Big)\Big) = \frac{1}{l_P^3}\Big(\mathrm{D}_0 + \frac{d}{m_a}\mathrm{D}_3\Big), \tag{C.201}$$
$$A\bar{A} = r^{2s} e^{-2\Lambda mr}(a_0 + a_1 r)(\bar{a}_0 + \bar{a}_1 r) = C\bar{C}$$
$$= |a_0|^2 r^{2s} e^{-2\Lambda mr}\Big[1 - \frac{2\alpha m_g r}{\sqrt{2s+2}(2s+1)} + \frac{m_g^2 r^2}{(2s+1)^2}\Big], \tag{C.202}$$

$$B\bar{B} = r^{2s} e^{-2\Lambda mr}(a_1 r)(\bar{a}_1 r) = r^{2s+2} e^{-2\Lambda mr} \frac{|a_0|^2 \mathbf{l}^2}{(2s+1)^2} = D\bar{D}, \tag{C.203}$$

$$\frac{\mathbf{J}}{\hbar c} = \frac{1}{l_P^3 m_a}\frac{S}{r^2}\begin{pmatrix} \underline{c}^2\mathbf{l}A\bar{A} + \underline{s}^2\mathbf{r}B\bar{B} & \underline{s}\,\underline{c}(\mathbf{l}A\bar{C} - \mathbf{r}D\bar{B}) \\ \underline{s}\,\underline{c}(\mathbf{l}C\bar{A} - \mathbf{r}B\bar{D}) & \underline{c}^2\mathbf{r}D\bar{D} + \underline{s}^2\mathbf{l}C\bar{C} \end{pmatrix} S^{-1}. \tag{C.204}$$

The normalization is obtained through:

$$\begin{aligned} \frac{\mathbf{J}^0}{\hbar c} &= \frac{1}{2l_P^3 m_a r^2}[\underline{c}^2\mathbf{l}A\bar{A} + \underline{s}^2\mathbf{r}B\bar{B} + \underline{s}^2\mathbf{l}A\bar{A} + \underline{C}^2 B\bar{B}] \\ &= \frac{1}{2l_P^3 m_a r^2}[\mathbf{l}A\bar{A} + \mathbf{r}B\bar{B}] \end{aligned} \tag{C.205}$$

$$\begin{aligned} 1 &= \iiint \mathrm{dv}\frac{\mathbf{J}^0}{\hbar c} = \frac{\mathbf{l}\alpha_3^2}{l_P^3 m_a}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \sin\theta \\ &\times \int_0^{\infty} \mathrm{d}r\ r^{2s} e^{-2\Lambda m r}[1 - \frac{2\alpha m_g r}{\sqrt{2s+2}(2s+1)} + \frac{2m_g^2}{(2s+1)^2}r^2]. \end{aligned} \tag{C.206}$$

With:

$$I_n = \int_0^{\infty} \mathrm{d}r\ r^{2s+n} e^{-2\Lambda r m} = \frac{\Gamma(2s+1+n)}{(2\Lambda m)^{2s+1+n}};\quad \nu := \frac{m_g}{m} = \frac{m_a}{m_g}, \tag{C.207}$$

we get:

$$\begin{aligned} 1 &= \frac{2\pi\mathbf{l}|a_0|^2}{l_P^3 m_a}[I_0 - \frac{2\alpha m_g}{\sqrt{2s+2}(2s+1)}I_1 + \frac{2m_g^2}{(2s+1)^2}I_2], \\ &= \frac{2\pi\mathbf{l}|a_0|^2\Gamma(2s+1)S}{l_P^3 m_a (2\Lambda m)^{2s+1}};\quad S = 1 - \frac{\alpha\nu}{\Lambda\sqrt{2s+2}} + \frac{\nu^2(s+1)}{\Lambda^2(2s+1)}, \end{aligned} \tag{C.208}$$

$$|a_0|^2 = \frac{l_P^3 m_a (2\Lambda m)^{2s+1}}{2\pi\mathbf{l}S\Gamma(2s+1)};\quad |a_0| = \sqrt{\frac{l_P^3 m_a (2\Lambda m)^{2s+1}}{2\pi\mathbf{l}S\Gamma(2s+1)}}. \tag{C.209}$$

For the $2\mathbf{s}1/2$ state, with a magnetic quantum number $\lambda = -1/2$, we also study first the particular case $b_0 = 0$. Since $|\kappa| = 1$ and $\lambda = -1/2$ we have:

$$\begin{aligned} &V = -\sqrt{\sin\theta}\cos\frac{\theta}{2}U_1;\quad U_1 = -1;\quad U = \sqrt{\sin\theta}\sin\frac{\theta}{2}V_1;\quad V_1 = -1, \\ &r\mathbf{G}_1 = -A;\quad r\mathbf{G}_3 = -C;\quad r\mathbf{G}_2 = -\bar{B};\quad r\mathbf{G}_4 = -\bar{D} \end{aligned} \tag{C.210}$$

$$\phi_{(\lambda=-\frac{1}{2})} = \frac{S}{r}\begin{pmatrix} -\underline{s}A & -\underline{c}\bar{B} \\ \underline{c}C & -\underline{s}\bar{D} \end{pmatrix} e^{pi_3};\quad p = -\frac{\varphi}{2} - E\mathrm{x}^0, \tag{C.211}$$

And we get:

$$\overline{\phi} = e^{-pi_3}\begin{pmatrix} -\underline{s}\bar{D} & \underline{c}\bar{B} \\ -\underline{c}C & -\underline{s}A \end{pmatrix}\frac{S^{-1}}{r} \tag{C.212}$$

$$\phi^\dagger = e^{-pi_3}\begin{pmatrix} -\underline{s}\bar{A} & \underline{c}\bar{C} \\ -\underline{c}B & -\underline{s}D \end{pmatrix}\frac{S^{-1}}{r} \tag{C.213}$$

$$\phi\overline{\phi} = r^{-2}(\underline{s}^2 A\bar{D} + \underline{c}^2 C\bar{B}). \tag{C.214}$$

We study here the case $b_0 = 0$, which gives:

$$c_0 = -e^{-i\gamma}a_0;\quad d_0 = -e^{-i\gamma}b_0 = 0;\quad a_1 = \frac{im_g e^{i\delta}}{2s+1}a_0$$
$$b_1 = \frac{i\mathbf{l}e^{i(-\gamma+r_1)}}{2s+1}a_0;\quad c_1 = \frac{im_g e^{-3i\delta}}{2s+1}a_0;\quad d_1 = \frac{i\mathbf{l}e^{ir_1}}{2s+1}a_0, \tag{C.215}$$
$$(a_0 + a_1 r)\bar{d}_1 = |a_0|^2 e^{-ir_1}\mathbf{l}\Big(\frac{e^{i\delta}}{(2s+1)^2}m_g r - \frac{i}{2s+1}\Big), \tag{C.216}$$
$$(c_0 + c_1 r)\bar{b}_1 = |a_0|^2\mathbf{l}e^{-ir_1}\Big(\frac{e^{-i\delta}}{(2s+1)^2}m_g r + \frac{i}{2s+1}\Big), \tag{C.217}$$

We then have:

$$\phi\overline{\phi} = r^{2s-1}e^{-2\Lambda mr}\Big(\frac{1-u_3}{2}(a_0+a_1 r)\bar{d}_1 + \frac{1+u_3}{2}(c_0+c_1 r)\bar{b}_1\Big) \tag{C.218}$$
$$= |a_0|^2\mathbf{l}r^{2s}e^{-2\Lambda mr}\Big[\frac{\sqrt{2s+2}}{2(2s+1)^2}m_g r + \frac{iu_3}{2s+1}\Big(1 - \frac{\alpha m_g r}{(2s+1)\sqrt{2s+2}}\Big)\Big].$$

Thus the wave is actually with value on the Cl_3^* group, as previously. When λ changes sign the only change is the change of sign for $\beta + r_1$:

$$\rho e^{i(\beta+r_1)} = \Omega_1 + i\Omega_2;\quad \Omega_1 = |a_0|^2\mathbf{l}r^{2s}e^{-2\Lambda mr}\frac{m_g\sqrt{2s+2}}{(2s+1)^2}, \tag{C.219}$$
$$\Omega_2 = |a_0|^2\mathbf{l}r^{2s}e^{-2\Lambda mr}\frac{u_3}{2s+1}\Big(1 - \frac{\alpha m_g r}{(2s+1)\sqrt{2s+2}}\Big). \tag{C.220}$$

Then the normalization needs:

$$\frac{\mathbf{J}}{\hbar c} = \frac{1}{l_P^3 r^2}S\begin{pmatrix} \underline{s}^2\mathbf{l}A\bar{A} + \underline{c}^2\mathbf{r}B\bar{B} & \underline{s}\,\underline{c}(-\mathbf{l}A\bar{C} + \mathbf{r}D\bar{B}) \\ \underline{s}\,\underline{c}(-\mathbf{l}C\bar{A} + \mathbf{r}B\bar{D}) & \underline{c}^2\mathbf{l}C\bar{C} + \underline{s}^2\mathbf{r}D\bar{D}\end{pmatrix}S^{-1}. \tag{C.221}$$

The density of probability thus is:

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{1}{2l_P^3 r^2}[\underline{s}^2\mathbf{l}A\bar{A} + \underline{c}^2\mathbf{r}B\bar{B} + \underline{c}^2\mathbf{l}C\bar{C} + \underline{s}^2\mathbf{r}D\bar{D}]. \tag{C.222}$$

So the calculation is exactly similar to that for the other sign of λ.

C.3.4 Case $a_0 = 0$

With a magnetic quantum number $\lambda = 1/2$ (spin "up") we have:

$$a_1 = \frac{i}{2s+1}\mathbf{r}e^{-i(\gamma+r_1)}b_0;\quad a_0 = c_0 = 0;\quad b_1 = \frac{i}{2s+1}m_g e^{i\delta}b_0,$$
$$c_1 = \frac{i}{2s+1}\mathbf{r}e^{-ir_1}b_0;\quad d_1 = \frac{i}{2s+1}m_g e^{-3i\delta}b_0, \tag{C.223}$$
$$A = b_0 r^{s+1}e^{-\Lambda mr}\frac{i\mathbf{r}e^{-i(\gamma+r_1)}}{2s+1};\ B = b_0 r^s e^{-\Lambda mr}\Big(1 + \frac{im_g r e^{i\delta}}{2s+1}\Big), \tag{C.224}$$
$$C = b_0 r^{s+1}e^{-\Lambda mr}\frac{i\mathbf{r}}{2s+1}e^{-ir_1};\ D = b_0 r^s e^{-\Lambda mr}\Big(-e^{-i\gamma} + \frac{im_g r}{2s+1}e^{-3i\delta}\Big).$$

The invariant ρ and β satisfy:

$$\rho e^{i\beta} = \frac{1}{r^2}(\underline{c}^2 A\bar{D} + \underline{s}^2 C\bar{B}) = \frac{1}{2r^2}[A\bar{D} + C\bar{B} + u_3(A\bar{D} - C\bar{B})], \quad \text{(C.225)}$$

$$\frac{1}{2r^2}(A\bar{D} + C\bar{B}) = r^{2s-1}e^{-2\Lambda mr}|b_0|^2\Big[\frac{\mathbf{r}}{2s+1}e^{-ir_1}m_g r\cos\delta\Big] \quad \text{(C.226)}$$

$$\frac{1}{2r^2}(A\bar{D} - C\bar{B}) = ir^{2s-1}e^{-2\Lambda mr}|b_0|^2\frac{\mathbf{r}}{2s+1}e^{-ir_1}\Big(1 + \frac{m_g r}{2s+1}\sin\delta\Big) \quad \text{(C.227)}$$

We deduce that we have:

$$\rho e^{i(\beta+r_1)} = \phi\bar{\phi}e^{ir_1} =: \Omega_1 + i\Omega_2 \quad \text{(C.228)}$$

$$= \frac{|b_0|^2\mathbf{l}}{2s+1}r^{2s+1}e^{-2\Lambda mr}\Big[\frac{m_g r}{2s+1}\frac{\sqrt{2s+2}}{2} + iu_3\Big(1 + \frac{\alpha m_g r}{(2s+1)\sqrt{2s+2}}\Big)\Big].$$

$$\Omega_1 = \frac{|b_0|^2\mathbf{l}}{2s+1}r^{2s+1}e^{-2\Lambda mr}\frac{m_g r}{2s+1}\frac{\sqrt{2s+2}}{2} \quad \text{(C.229)}$$

$$\Omega_2 = \frac{|b_0|^2\mathbf{l}}{2s+1}r^{2s+1}e^{-2\Lambda mr}u_3\Big(1 + \frac{\alpha m_g r}{(2s+1)\sqrt{2s+2}}\Big). \quad \text{(C.230)}$$

We see that ρ is defined in any point and that the value of relativistic densities Ω_1 and Ω_2 are everywhere equal to that of the state $2\mathbf{s}1/2$ with same value of λ. We now establish that it is the same for the normalization of the wave. We have:

$$\frac{\mathbf{J}}{\hbar c} = \frac{1}{l_P^3 m_a r^2}S\begin{pmatrix} \underline{c}^2\mathbf{l}A\bar{A} + \underline{s}^2\mathbf{r}B\bar{B} & \underline{s}\,\underline{c}(-\mathbf{l}A\bar{C} + \mathbf{r}D\bar{B}) \\ \underline{s}\,\underline{c}(-\mathbf{l}C\bar{A} + \mathbf{r}B\bar{D} & \underline{s}^2\mathbf{l}C\bar{C} + \underline{c}^2\mathbf{r}D\bar{D} \end{pmatrix} S^{-1} \quad \text{(C.231)}$$

And we have:

$$B\bar{B} = |b_0|^2 r^{2s}e^{-2\Lambda mr}\Big[1 - \frac{2\alpha m_g r}{(2s+1)\sqrt{2s+2}} + \frac{m_g^2 r^2}{(2s+1)^2}\Big] = D\bar{D},$$

$$A\bar{A} = |b_0|^2 r^{2s}e^{-2\Lambda mr}\frac{\mathbf{r}^2 r^2}{(2s+1)^2} = C\bar{C}. \quad \text{(C.232)}$$

We thus get:

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{|b_0|^2\mathbf{r}}{l_P^3 m_a}r^{2s-2}e^{-2\Lambda mr}\Big[1 - \frac{2\alpha m_g r}{(2s+1)\sqrt{2s+2}} + \frac{2m_g^2 r^2}{(2s+1)^2}\Big]. \quad \text{(C.233)}$$

Hence the normalization of the wave when $\lambda = 1/2$ and $a_0 = 0$ is exactly the same as for the state with $\lambda = 1/2$ and $b_0 = 0$.

With $a_0 = 0$, we then get for the state with magnetic quantum number

$\lambda = -1/2$:

$$V = -\sqrt{\sin\theta}\cos\frac{\theta}{2}U_1;\quad U_1 = -1;\quad U = \sqrt{\sin\theta}\sin\frac{\theta}{2}V_1;\quad V_1 = -1,$$

$$r\mathbf{G}_1 = -A;\quad r\mathbf{G}_3 = -C;\quad r\mathbf{G}_2 = -\bar{B};\quad r\mathbf{G}_4 = -\bar{D} \tag{C.234}$$

$$\phi_{(\lambda=-\frac{1}{2})} = \frac{S}{r}\begin{pmatrix} -\underline{s}A & -\underline{c}\bar{B} \\ \underline{c}C & -\underline{s}\bar{D} \end{pmatrix} e^{pi_3};\quad p = -\frac{\varphi}{2} - E\mathrm{x}^0, \tag{C.235}$$

And we get:

$$\overline{\phi} = e^{-pi_3}\begin{pmatrix} -\underline{s}\bar{D} & \underline{c}\bar{B} \\ -\underline{c}C & -\underline{s}A \end{pmatrix}\frac{S^{-1}}{r} \tag{C.236}$$

$$\phi^\dagger = e^{-pi_3}\begin{pmatrix} -\underline{s}\bar{A} & \underline{c}\bar{C} \\ -\underline{c}B & -\underline{s}D \end{pmatrix}\frac{S^{-1}}{r} \tag{C.237}$$

$$\phi\overline{\phi} = r^{-2}(\underline{s}^2 A\bar{D} + \underline{c}^2 C\bar{B}). \tag{C.238}$$

We choose here the case $a_0 = 0$, which gives:

$$d_0 = -e^{-i\gamma}b_0;\quad c_0 = -e^{-i\gamma}a_0 = 0;\ b_1 = \frac{im_g e^{i\delta}}{2s+1}b_0$$

$$a_1 = \frac{i\mathbf{r}e^{i(-\gamma-r_1)}}{2s+1}b_0;\quad d_1 = \frac{im_g e^{-3i\delta}}{2s+1}b_0;\quad c_1 = \frac{i\mathbf{r}e^{-ir_1}}{2s+1}b_0, \tag{C.239}$$

$$a_1 r(\bar{d}_0 + \bar{d}_1 r) = |b_0|^2 e^{-ir_1}\mathbf{r}r\Big(\frac{e^{i\delta}}{(2s+1)^2}m_g r - \frac{i}{2s+1}\Big), \tag{C.240}$$

$$c_1 r(\bar{b}_0\bar{b}_1 r) = |b_0|^2\mathbf{r}re^{-ir_1}\Big(\frac{e^{-i\delta}}{(2s+1)^2}m_g r + \frac{i}{2s+1}\Big), \tag{C.241}$$

We then have:

$$\phi\overline{\phi} = r^{2s-1}e^{-2\Lambda mr}\Big(\frac{1-u_3}{2}a_1 r(\bar{b}_0 + \bar{d}_1 r) + \frac{1+u_3}{2}c_1 r(\bar{b}_0 + \bar{b}_1 r)\Big) \tag{C.242}$$

$$= |b_0|^2\mathbf{r}r^{2s}e^{-2\Lambda mr}\Big[\frac{\sqrt{2s+2}}{2(2s+1)^2}m_g r + \frac{iu_3}{2s+1}\Big(1 - \frac{\alpha m_g r}{(2s+1)\sqrt{2s+2}}\Big)\Big].$$

Hence the wave has well value on the Cl_3^* group, as previously. We have, when λ changes sign, only a change of sign for $\beta + r_1$:

$$\rho e^{i(\beta+r_1)} = \Omega_1 + i\Omega_2;\quad \Omega_1 = |b_0|^2\mathbf{r}r^{2s}e^{-2\Lambda mr}\frac{m_g\sqrt{2s+2}}{(2s+1)^2}, \tag{C.243}$$

$$\Omega_2 = |b_0|^2\mathbf{r}r^{2s}e^{-2\Lambda mr}\frac{u_3}{2s+1}\Big(1 - \frac{\alpha m_g r}{(2s+1)\sqrt{2s+2}}\Big). \tag{C.244}$$

Next the normalization needs:

$$\frac{\mathbf{J}}{\hbar c} = \frac{1}{l_P^3 r^2}S\begin{pmatrix} \underline{s}^2\mathbf{l}A\bar{A} + \underline{c}^2\mathbf{r}B\bar{B} & \underline{s}\,\underline{c}(-\mathbf{l}A\bar{C} + \mathbf{r}D\bar{B}) \\ \underline{s}\,\underline{c}(-\mathbf{l}C\bar{A} + \mathbf{r}B\bar{D}) & \underline{c}^2\mathbf{l}C\bar{C} + \underline{s}^2\mathbf{r}D\bar{D} \end{pmatrix} S^{-1}. \tag{C.245}$$

The density of probability is thus:

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{1}{2l_P^3 r^2}[\underline{s}^2\mathbf{l}A\bar{A} + \underline{c}^2\mathbf{r}B\bar{B} + \underline{c}^2\mathbf{l}C\bar{C} + \underline{s}^2\mathbf{r}D\bar{D}]. \tag{C.246}$$

Hence the calculation is the same as for the other sign of λ.

C.3.5 Case of $2p_{3/2}$ state

These states have a quantum number n null, which means constant radial polynomials. We have here:

$$\begin{aligned} \kappa = 2;\ \ \kappa^2 = 4 = s^2 + \alpha^2;\ \ s = \sqrt{4-\alpha^2};\ \ s + i\alpha =: 2e^{i\gamma}, \\ E + i\Lambda m = m_g e^{i\gamma};\ \ \alpha E = s\Lambda m;\ \ E = \frac{sm_g}{2}, \\ d_0 = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i(r_1-\gamma)}a_0;\ \ c_0 = -e^{-i\gamma}a_0;\ \ b_0 = -\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{ir_1}a_0. \end{aligned} \tag{C.247}$$

Radial functions thus satisfy:

$$\begin{aligned} &A = a_0 r^s e^{-\Lambda m r};\ \ C = -a_0 e^{-i\gamma} r^s e^{-\Lambda m r}, \\ &B = -\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{ir_1}a_0 r^s e^{-\Lambda m r};\ \ D = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i(r_1-\gamma)}a_0 r^s e^{-\Lambda m r}. \end{aligned} \tag{C.248}$$

The angular functions are given by:

$$U_1 = (|\lambda| - \kappa - \frac{1}{2})(\sin\theta)^{|\lambda|-\frac{1}{2}} \tag{C.249}$$
$$\times \sum_{j=0}^{j=|\kappa|-|\lambda|-\frac{1}{2}} \frac{(|\lambda|+\frac{1}{2}-|\kappa|)_j(|\lambda|+|\kappa|+\frac{1}{2})_j}{(|\lambda|+\frac{1}{2})_j j!}\sin^{2j}(\frac{\theta}{2}),$$

$$V_1 = \kappa(|\lambda| - \frac{1}{2} - \kappa)(\sin\theta)^{|\lambda|-\frac{1}{2}} \tag{C.250}$$
$$\times \sum_{j=0}^{j=|\kappa|-|\lambda|-\frac{1}{2}} \frac{(|\lambda|+\frac{1}{2}-|\kappa|)_j(|\lambda|+\frac{1}{2}+|\kappa|)_j}{(|\lambda|+\frac{1}{2})_{j+1} j!}\sin^{2j}(\frac{\theta}{2}).$$

State $2p_{3/2}$, $\lambda = 3/2$

The angular functions satisfy then:

$$\begin{aligned} U_1 &= (\frac{3}{2} - 2 - \frac{1}{2})(\sin\theta)^{\frac{3}{2}-\frac{1}{2}} \sum_{j=0}^{j=2-\frac{3}{2}-\frac{1}{2}} \frac{(\frac{3}{2}+\frac{1}{2}-2)_j(\frac{3}{2}+2+\frac{1}{2})_j}{(\frac{3}{2}+\frac{1}{2})_j j!}\sin^{2j}(\frac{\theta}{2}) \\ &= -\sin\theta, \end{aligned} \tag{C.251}$$

$$V_1 = 2(\frac{3}{2} - \frac{1}{2} - 2)(\sin\theta)^{\frac{3}{2}-\frac{1}{2}} \sum_{j=0}^{j=2-\frac{3}{2}-\frac{1}{2}} \frac{(\frac{3}{2}+\frac{1}{2}-2)_j(\frac{3}{2}+\frac{1}{2}+2)_j}{(\frac{3}{2}+\frac{1}{2})_{j+1}j!}\sin^{2j}(\frac{\theta}{2})$$
$$= 2(-\sin\theta)\frac{1}{2} = -\sin\theta \quad (C.252)$$

We thus have:

$$\mathbf{F}_1 = \frac{-\sin\theta}{r}A;\quad \mathbf{F}_2 = \frac{-\sin\theta}{r}\bar{D};\quad \mathbf{F}_3 = \frac{-\sin\theta}{r}C;\quad \mathbf{F}_4 = \frac{-\sin\theta}{r}\bar{B}. \quad (C.253)$$

$$\phi = S\begin{pmatrix} \underline{c}\mathbf{F}_1 & -\underline{s}\mathbf{F}_4 \\ \underline{s}\mathbf{F}_3 & \underline{c}\mathbf{F}_2 \end{pmatrix} e^{p\,i_3},\quad p = \frac{3\varphi}{2} - E\mathrm{x}^0, \quad (C.254)$$

$$\phi = -\frac{\sin\theta}{r}S\begin{pmatrix} \underline{c}A & -\underline{s}\bar{B} \\ \underline{s}C & \underline{c}\bar{D} \end{pmatrix} e^{p\,i_3} \quad (C.255)$$

That gives:

$$\rho e^{i\beta} = \frac{\sin^2\theta}{r^2}(\underline{c}^2 A\bar{D} + \underline{s}^2 C\bar{B})$$
$$= \frac{|a_0|^2}{2}\sin^2\theta r^{2s-2}e^{-2\Lambda mr}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{-ir_1}(s + iu_3\alpha), \quad (C.256)$$

$$\phi^\dagger = -\frac{\sin\theta}{r}e^{-pi_3}\begin{pmatrix} \underline{c}\bar{A} & \underline{s}\bar{C} \\ -\underline{s}B & \underline{c}D \end{pmatrix} S^{-1} \quad (C.257)$$

It results:

$$\frac{\mathbf{J}}{\hbar c} = \frac{\sin^2\theta}{r^2}S\begin{pmatrix} \underline{c}^2\mathbf{l}A\bar{A} + \underline{s}^2\mathbf{r}B\bar{B} & \underline{s}\,\underline{c}(\mathbf{l}A\bar{C} - \mathbf{r}D\bar{B}) \\ \underline{s}\,\underline{c}(\mathbf{l}C\bar{A} - \mathbf{r}B\bar{D} & \underline{s}^2\mathbf{l}C\bar{C} + \underline{c}^2\mathbf{r}D\bar{D} \end{pmatrix} S^{-1}, \quad (C.258)$$

$$1 = \iiint \mathrm{dv}\frac{\mathbf{J}^0}{\hbar c} = \frac{|a_0|^2\mathbf{l}}{l_P^3 m_a}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\sin^3\theta\int_0^{\infty}\mathrm{d}r\,r^{2s}e^{-2\Lambda mr}$$
$$= \frac{|a_0|^2\mathbf{l}}{l_P^3 m_a}(2\pi)\frac{4}{3}\frac{\Gamma(2s+1)}{(2\Lambda m)^{2s+1}}, \quad (C.259)$$

$$|a_0| = \sqrt{\frac{3l_p^3 m_a(2\Lambda m)^{2s+1}}{8\pi\mathbf{l}\Gamma(2s+1)}}. \quad (C.260)$$

State 2$p_{3/2}$**,** $\lambda = 1/2$

The angular functions U_1 and V_1 satisfy:

$$U_1 = -2\sum_{j=0}^{j=1}\frac{(-1)_j(3)_j}{(1)_j j!}(\sin\frac{\theta}{2})^{2j} = -1 + 3cos\theta \quad (C.261)$$

$$V_1 = 2(-2)\sum_{j=0}^{j=1}\frac{(-1)_j(3)_j}{(1)_{j+1}j!}(\sin\frac{\theta}{2})^{2j} = -1 - 3\cos\theta. \quad (C.262)$$

It results:

$$\mathbf{F}_1 = \frac{-1+3\cos\theta}{r}A;\quad \mathbf{F}_2 = \frac{-1+3\cos\theta}{r}\bar{D}$$
$$\mathbf{F}_3 = -\frac{1+3\cos\theta}{r};\quad \mathbf{F}_4 = -\frac{1+3\cos\theta}{r}\bar{B}, \tag{C.263}$$
$$\phi = S\begin{pmatrix} \underline{c}\frac{-1+3\cos\theta}{r}A & \underline{s}\frac{1+3\cos\theta}{r}\bar{B} \\ -\underline{s}\frac{1+3\cos\theta}{r}C & \underline{c}\frac{-1+3\cos\theta}{r}\bar{D}\end{pmatrix}e^{p\,i_3};\quad p = \frac{\lambda}{2} - E\mathrm{x}^0. \tag{C.264}$$

With $\underline{c}^2 = \frac{1+u_3}{2}$ and $\underline{s}^2 = \frac{1-u_3}{2}$ we have:

$$\phi\overline{\phi} = \underline{c}^2\Big(\frac{-1+3u_3}{r}\Big)^2 A\bar{D} + \underline{s}^2\Big(\frac{1+3u_3}{r}\Big)^2 C\bar{B}, \tag{C.265}$$
$$A\bar{D} = |a_0|r^{2s}e^{-2\Lambda mr}e^{-ir_1}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i\gamma};\quad C\bar{B} = |a_0|r^{2s}e^{-2\Lambda mr}e^{-ir_1}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{i\gamma}.$$

Thus we get:

$$\phi\overline{\phi} = |a_0|^2 r^{2s-2}e^{-2\Lambda mr}e^{-ir_1}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}[s(1+3u_3^2) + i\alpha(9u_3^3 - 5u_3)]. \tag{C.266}$$

Thus ρ is everywhere non null and ϕ is well with value in Cl_3^*. Next we have:

$$\frac{\mathbf{J}}{\hbar c} = \frac{S}{l_P^3 m_a}\begin{pmatrix} j_0+j_3 & j_1-ij_2 \\ j_1+ij_2 & j_0-j_3\end{pmatrix}S^{-1}, \tag{C.267}$$
$$j_0+j_3 = \underline{c}^2\frac{(1-3u_3)^2}{r^2}\mathbf{l}A\bar{A} + \underline{s}^2\frac{(1+3u_3)^2}{r^2}\mathbf{r}B\bar{B}, \tag{C.268}$$
$$j_0-j_3 = \underline{s}^2\frac{(1+3u_3)^2}{r^2}\mathbf{l}C\bar{C} + \underline{c}^2\frac{(1-3u_3)^2}{r^2}\mathbf{r}D\bar{D}, \tag{C.269}$$
$$j_1+ij_2 = \underline{s}\,\underline{c}\frac{1-9u_3^2}{r^2}(\mathbf{l}C\bar{A} - \mathbf{r}B\bar{D}), \tag{C.270}$$

because $u_3 = \cos\theta$. That gives:

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{\mathbf{l}|a_0|^2}{l_P^3 m_a}(1+3u_3^2)r^{2s-2}e^{-2\Lambda mr}, \tag{C.271}$$
$$1 = \iiint \mathrm{dv}\frac{\mathbf{J}^0}{\hbar c}$$
$$= \frac{|a_0|^2\mathbf{l}}{l_P^3 m_a}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\sin\theta(1+3\cos^2\theta)\int_0^{\infty}\mathrm{d}r\,r^{2s}e^{-2\Lambda mr}$$
$$= \frac{|a_0|^2\mathbf{l}}{l_P^3 m_a}\frac{8\pi\Gamma(2s+1)}{(2\Lambda m)^{2s+1}};\quad |a_0| = \sqrt{\frac{l_P^3 m_a(2\Lambda m)^{2s+1}}{8\pi\mathbf{l}\Gamma(2s+1)}}. \tag{C.272}$$

State 2$p_{3/2}$, $\lambda = -1/2$

The angular functions satisfy then:

$$U_1 = (|\lambda| - \kappa - \frac{1}{2}) \sum_{n=0}^{n=\kappa-|\lambda|-1/2} \frac{(|\lambda| - \kappa + \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n} \frac{\theta}{2},$$
$$= -2 \sum_{n=0}^{n=1} \frac{(-1)_n (3)_n}{(1)_n n!} \sin^{2n} \frac{\theta}{2} = 1 - 3\cos\theta, \quad \text{(C.273)}$$

$$V_1 = \kappa(|\lambda| - \kappa - \frac{1}{2}) \sum_{n=0}^{n=\kappa-|\lambda|-1/2} \frac{(|\lambda| - \kappa + \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_{1+n} n!} \sin^{2n} \frac{\theta}{2},$$
$$= -1 - 3\cos\theta, \quad \text{(C.274)}$$

We have:

$$\phi\overline{\phi} = \underline{c}^2 \Big(\frac{-1 + 3u_3}{r}\Big)^2 A\bar{D} + \underline{s}^2 \Big(\frac{1 + 3u_3}{r}\Big)^2 C\bar{B}, \quad \text{(C.275)}$$
$$A\bar{D} = |a_0| r^{2s} e^{-2\Lambda mr} e^{-ir_1} \sqrt{\frac{\mathbf{l}}{\mathbf{r}}} e^{i\gamma}; \quad C\bar{B} = |a_0| r^{2s} e^{-2\Lambda mr} e^{-ir_1} \sqrt{\frac{\mathbf{l}}{\mathbf{r}}} e^{i\gamma}.$$

Thus we get:

$$\phi\overline{\phi} = |a_0|^2 r^{2s-2} e^{-2\Lambda mr} e^{-ir_1} \sqrt{\frac{\mathbf{l}}{\mathbf{r}}} [s(1 + 3u_3^2) + i\alpha(9u_3^3 - 5u_3)]. \quad \text{(C.276)}$$

Thus the ϕ wave has well value in Cl_3^*. Next we have:

$$\frac{\mathbf{J}}{\hbar c} = \frac{S}{l_P^3 m_a} \begin{pmatrix} j_0 + j_3 & j_1 - ij_2 \\ j_1 + ij_2 & j_0 - j_3 \end{pmatrix} S^{-1}, \quad \text{(C.277)}$$
$$j_0 + j_3 = \underline{c}^2 \frac{(1 - 3u_3)^2}{r^2} \mathbf{l} A\bar{A} + \underline{s}^2 \frac{(1 + 3u_3)^2}{r^2} \mathbf{r} B\bar{B}, \quad \text{(C.278)}$$
$$j_0 - j_3 = \underline{s}^2 \frac{(1 + 3u_3)^2}{r^2} \mathbf{l} C\bar{C} + \underline{c}^2 \frac{(1 - 3u_3)^2}{r^2} \mathbf{r} D\bar{D}, \quad \text{(C.279)}$$
$$j_1 + ij_2 = \underline{s}\,\underline{c} \frac{1 - 9u_3^2}{r^2} (\mathbf{l} C\bar{A} - \mathbf{r} B\bar{D}) \quad \text{(C.280)}$$

Thus we get:

$$\frac{\mathbf{J}^0}{\hbar c} = \frac{1}{l_P^3 m_a} (\phi \mathbf{m} \phi^\dagger)^0 = \frac{|a_0|^2 \mathbf{l} r^{2s} e^{-2\Lambda mr}}{l_P^3 m_a r^2} (1 + 9\cos^2\theta - 6u_3 \cos\theta).$$
$$= \frac{|a_0|^2 \mathbf{l} r^{2s} e^{-2\Lambda mr}}{l_P^3 m_a r^2} (1 + 3\cos^2\theta)), \quad \text{(C.281)}$$

And hence the normalization is identical to that of the previous case:

$$|a_0| = \sqrt{\frac{l_P^3 m_a (2\Lambda m)^{2s+1}}{8\pi \mathbf{l} \Gamma(2s + 1)}}. \quad \text{(C.282)}$$

State $\mathbf{2}p_{3/2}$, $\lambda = -3/2$

The angular functions now satisfy:

$$U_1 = V_1 = -\sin\theta, \tag{C.283}$$

thus the results are the same as for the $\mathbf{2}p_{3/2}$ state, with $\lambda = 3/2$:

$$|a_0| = \sqrt{\frac{3l_P^3 m_a (2\Lambda m)^{2s+1}}{8\pi \mathbf{l}\Gamma(2s+1)}}, \tag{C.284}$$

$$\rho e^{i\beta} = \frac{|a_0|^2}{2}\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\sin^2(\theta) r^{2s-2} e^{-2\Lambda m r}(s + i\alpha u_3). \tag{C.285}$$

Appendix D

Miscellaneous

D.1 Gauge invariance SU(2) of the quarks

D.1.1 Group generated by $\underline{P}_1$

We have in this case

$$\Psi_L = \underline{P}_+(\Psi);\ \underline{P}_1(\Psi_L) = \Gamma_{0123}\Psi_L\Gamma_{35};\ C = \cos(\theta);\ S = \sin(\theta), \quad \text{(D.1)}$$

$$\Psi'_L = [\exp(\theta\underline{P}_1)](\Psi_L) = C\Psi_L + S\Gamma_{0123}\Psi_L\Gamma_{35}, \quad \text{(D.2)}$$

$$W'^1_\mu = W^1_\mu - \frac{2}{g_2}\partial_\mu\theta. \quad \text{(D.3)}$$

This gives:

$$\begin{pmatrix} L'^n & \widetilde{L}'^{3+n} \\ \overline{L}'^{3+n} & -\widehat{L}'^n \end{pmatrix} = C\begin{pmatrix} L^n & \widetilde{L}^{3+n} \\ \overline{L}^{3+n} & -\widehat{L}^n \end{pmatrix} + S\begin{pmatrix} i\widetilde{L}^{3+n} & iL^n \\ -i\overline{L}^{3+n} & i\widehat{L}^n \end{pmatrix}.\ n = 2, 3, 4, \quad \text{(D.4)}$$

$$L'^n = CL^n + iS\widetilde{L}^{3+n}, \quad \text{(D.5)}$$

$$\widetilde{L}'^{3+n} = C\widetilde{L}^{3+n} + iSL^n. \quad \text{(D.6)}$$

We now let:

$$2L^nL^{3+n} = D_L^{n\,3+n} - id_L^{n\,3+n}. \quad \text{(D.7)}$$

We deduce for the left currents:

$$D_L^{n\,3+n} = L^nL^{3+n} + \widetilde{L}^{3+n}\widetilde{L}^n;\ d_L^{n\,3+n} = iL^nL^{3+n} - i\widetilde{L}^{3+n}\widetilde{L}^n, \quad \text{(D.8)}$$

$$2D'^n_L = D_L^n + D_L^{3+n} + \cos(2\theta)(D_L^n - D_L^{3+n}) - \sin(2\theta)d_L^{n\,3+n}, \quad \text{(D.9)}$$

$$2D'^{3+n} = D_L^n + D_L^{3+n} - \cos(2\theta)(D_L^n - D_L^{3+n}) + \sin(2\theta)d_L^{n\,3+n}. \quad \text{(D.10)}$$

Adding and subtracting these equations we get:

$$D_L'^n + D_L'^{3+n} = D_L^n + D_L^{3+n}, \tag{D.11}$$
$$D_L'^{3+n} - D_L'^n = \cos(2\theta)(D_L^{3+n} - D_L^n) + \sin(2\theta) d_L^{n\,3+n}. \tag{D.12}$$

From these equations, we obtain the conservation of the total current J_q, and also the difference between the left currents and the right currents. By bringing together these equations and (2.137) we may see that they are compatible with:

$$W^2 = d_L^{n\,3+n};\ W^3 = D_L^{3+n} - D_L^n. \tag{D.13}$$

D.1.2 Groups generated by $\underline{P}_2$ and $\underline{P}_3$

The calculation is completely similar to the previous section. In both cases we obtain the value of W^1 as only additional result. And as before, this value depends on the integer n. Also we are going to use a double system of indices: we will denote as W_n^j the potentials previously denoted as W^j acting on L^n. We will then have:

$$W_n^1 = D_L^{n\,3+n};\ W_n^2 = d_L^{n\,3+n};\ W_n^3 = D_L^{3+n} - D_L^n. \tag{D.14}$$

As in the previous section the sum of the left currents and the sum of the rights currents are invariant, which implies the conservation of the J_q current, and thus of $m\mathrm{v}_q$. This makes the system formed by the twelve wave equations (6 right and 6 left) invariant under the $SU(2)$ group.

D.2 Gauge invariance under SU(3)

We use the following transformation:

$$\Psi' = [\exp(\theta\Lambda_1)](\Psi),$$
$$\Psi'^2 = C\Psi^2 + S\mathbf{i}\Psi^3;\ C = \cos(\theta);\ S = \sin(\theta), \tag{D.15}$$
$$\Psi'^3 = C\Psi^3 + S\mathbf{i}\Psi^2;\ \Psi'^1 = \Psi^1;\ \Psi'^4 = \Psi^4. \tag{D.16}$$

Here we can forget about Ψ^1 and Ψ^4 which do not vary. The gauge invariance means that the system:

$$\boldsymbol{\partial}\Psi^2 = -\frac{g_3}{2}\mathbf{G}^1\mathbf{i}\Psi^3 + m\mathbf{v}_q\Psi^2\gamma_{12},$$
$$\boldsymbol{\partial}\Psi^3 = -\frac{g_3}{2}\mathbf{G}^1\mathbf{i}\Psi^2 + m\mathbf{v}_q\Psi^3\gamma_{12}. \tag{D.17}$$

must be equivalent to the system:

$$\boldsymbol{\partial}\Psi'^2 = -\frac{g_3}{2}\mathbf{G}'^1\mathbf{i}\Psi'^3 + m\mathbf{v}_q\Psi'^2\gamma_{12},$$
$$\boldsymbol{\partial}\Psi'^3 = -\frac{g_3}{2}\mathbf{G}'^1\mathbf{i}\Psi'^2 + m\mathbf{v}_q\Psi'^3\gamma_{12}. \tag{D.18}$$

Using (D.15) and (D.16) the system (D.17) is equivalent to (D.18) if and only if:

$$\mathbf{G}'^1 = \mathbf{G}^1 - \frac{2}{g_3}\boldsymbol{\partial}\theta;\ h_1'^1 = h_1^1 - \boldsymbol{\partial}\theta. \tag{D.19}$$

The equality (D.15) is equivalent to the following system:

$$L'^2 = CL^2 + iSL^3;\ \widetilde{L}'^5 = C\widetilde{L}^5 + iS\widetilde{L}^6, \tag{D.20}$$

$$R'^2 = CR^2 + iSR^3;\ \widetilde{R}'^5 = C\widetilde{R}^5 + iS\widetilde{R}^6. \tag{D.21}$$

Meanwhile the equality (D.16) is equivalent to the system:

$$L'^3 = CL^3 + iSL^2;\ \widetilde{L}'^6 = C\widetilde{L}^6 + iS\widetilde{L}^5, \tag{D.22}$$

$$R'^3 = CR^3 + iSR^2;\ \widetilde{R}'^6 = C\widetilde{R}^6 + iS\widetilde{R}^5. \tag{D.23}$$

These systems may be brought together: we obtain four systems with the same structure:

$$L'^2 = CL^2 + iSL^3;\ L'^3 = CL^3 + iSL^2, \tag{D.24}$$

$$R'^2 = CR^2 + iSR^3;\ R'^3 = CR^3 + iSR^2, \tag{D.25}$$

$$\widetilde{L}'^5 = C\widetilde{L}^5 + iS\widetilde{L}^6;\ \widetilde{L}'^6 = C\widetilde{L}^6 + iS\widetilde{L}^5, \tag{D.26}$$

$$\widetilde{R}'^6 = C\widetilde{R}^6 + iS\widetilde{R}^5;\ \widetilde{R}'^6 = C\widetilde{R}^6 + iS\widetilde{R}^5. \tag{D.27}$$

These systems have the same form as those of the left waves for the weak interaction. We can hence carry out similar calculations as in D.1.1. For the left waves of the d quark with color r or g, we consider the currents:

$$D_L^2 = L^2\widetilde{L}^2;\ D_L^3 = L^3\widetilde{L}^3;\ D_L^{2\,3} - id_L^{2\,3} = 2L^2\widetilde{L}^3, \tag{D.28}$$

$$D_L^{2\,3} + id_L^{2\,3} = 2L^3\widetilde{L}^2;\ D_L^{2\,3} = L^2\widetilde{L}^3 + L^3\widetilde{L}^2;\ d_L^{2\,3} = iL^2\widetilde{L}^3 - iL^3\widetilde{L}^2. \tag{D.29}$$

We then get:

$$D_L'^{2\,3} = D_L^{2\,3};\ D_L'^2 + D_L'^3 = D_L^2 + D_L^3, \tag{D.30}$$

$$d_L'^{2\,3} = \cos(2\theta)d_L^{2\,3} - \sin(2\theta)(D_L^3 - D_L^2), \tag{D.31}$$

$$D_L'^3 - D_L'^2 = \cos(2\theta)(D_L^3 - D_L^2) + \sin(2\theta)d_L^{2\,3}. \tag{D.32}$$

A comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}_1^1 = \frac{g_3}{2}D_L^{2\,3};\ \mathrm{h}_1^2 = \frac{g_3}{2}d_L^{2\,3};\ \mathrm{h}_1^3 = \frac{g_3}{2}(D_L^3 - D_L^2). \tag{D.33}$$

For the right waves of the d quark with color r or g we consider the currents:

$$D_R^2 = R^2\widetilde{R}^2;\ D_R^3 = R^3\widetilde{R}^3;\ D_R^{2\,3} - id_R^{2\,3} = 2R^2\widetilde{R}^3, \tag{D.34}$$

$$D_R^{2\,3} + id_R^{2\,3} = 2R^3\widetilde{R}^2;\ D_R^{2\,3} = R^2\widetilde{R}^3 + R^3\widetilde{R}^2;\ d_R^{2\,3} = iR^2\widetilde{R}^3 - iR^3\widetilde{R}^2. \tag{D.35}$$

We thus get :

$$D'^{2\,3}_R = D^{2\,3}_R;\ D'^2_R + D'^3_R = D^2_R + D^3_R, \tag{D.36}$$

$$d'^{2\,3}_R = \cos(2\theta) d^{2\,3}_R - \sin(2\theta)(D^3_R - D^2_R), \tag{D.37}$$

$$D'^3_R - D'^2_R = \cos(2\theta)(D^3_R - D^2_R) + \sin(2\theta) d^{2\,3}_R. \tag{D.38}$$

A comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^1_1 = \frac{g_3}{2} D^{2\,3}_R;\ \mathrm{h}^2_1 = \frac{g_3}{2} d^{2\,3}_R;\ \mathrm{h}^3_1 = \frac{g_3}{2}(D^3_R - D^2_R). \tag{D.39}$$

Hence, here we again see the dependence of the potentials on the wave which they are acting on. We thus note:

$$\mathrm{h}^{d\,1}_{L1} = \frac{g_3}{2} D^{2\,3}_L;\ \mathrm{h}^{d\,2}_{L1} = \frac{g_3}{2} d^{2\,3}_L;\ \mathrm{h}^{d\,3}_{L1} = \frac{g_3}{2}(D^3_L - D^2_L), \tag{D.40}$$

$$\mathrm{h}^{d\,1}_{R1} = \frac{g_3}{2} D^{2\,3}_R;\ \mathrm{h}^{d\,2}_{R1} = \frac{g_3}{2} d^{2\,3}_R;\ \mathrm{h}^{d\,3}_{R1} = \frac{g_3}{2}(D^3_R - D^2_R). \tag{D.41}$$

For the left wave of the u quark with color r or g we consider the currents:

$$D^5_L = \widetilde{L}^5 L^5;\ D^6_L = \widetilde{L}^6 L^6;\ D^{5\,6}_L - i d^{5\,6}_L = 2\widetilde{L}^5 L^6, \tag{D.42}$$

$$D^{5\,6}_L + i d^{5\,6}_L = 2\widetilde{L}^6 L^5;\ D^{5\,6}_L = \widetilde{L}^5 L^6 + \widetilde{L}^6 L^5;\ d^{5\,6}_L = i\widetilde{L}^5 L^6 - i\widetilde{L}^6 L^5. \tag{D.43}$$

We thus obtain:

$$D'^{5\,6}_L = D^{5\,6}_L;\ D'^5_L + D'^6_L = D^5_L + D^6_L, \tag{D.44}$$

$$d'^{5\,6}_L = \cos(2\theta) d^{5\,6}_L - \sin(2\theta)(D^6_L - D^5_L), \tag{D.45}$$

$$D'^6_L - D'^5_L = \cos(2\theta)(D^6_L - D^5_L) + \sin(2\theta) d^{5\,6}_L. \tag{D.46}$$

A comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^{u\,1}_{L1} = \frac{g_3}{2} D^{5\,6}_L;\ \mathrm{h}^{u\,2}_{L1} = \frac{g_3}{2} d^{5\,6}_L;\ \mathrm{h}^{u\,3}_{L1} = \frac{g_3}{2}(D^6_L - D^5_L). \tag{D.47}$$

For the right wave of the u quark with color r or g we consider the currents:

$$D^5_R = \widetilde{R}^5 R^5;\ D^6_R = \widetilde{R}^6 R^6;\ D^{5\,6}_R - i d^{5\,6}_R = 2\widetilde{R}^5 R^6, \tag{D.48}$$

$$D^{5\,6}_R + i d^{5\,6}_R = 2\widetilde{R}^6 R^5;\ D^{5\,6}_R = \widetilde{R}^5 R^6 + \widetilde{R}^6 R^5;\ d^{5\,6}_R = i\widetilde{R}^5 R^6 - i\widetilde{R}^6 R^5. \tag{D.49}$$

We then get:

$$D'^{5\,6}_R = D^{5\,6}_R;\ D'^5_R + D'^6_R = D^5_R + D^6_R, \tag{D.50}$$

$$d'^{5\,6}_R = \cos(2\theta) d^{5\,6}_R - \sin(2\theta)(D^6_R - D^5_R), \tag{D.51}$$

$$D'^6_R - D'^5_R = \cos(2\theta)(D^6_R - D^5_R) + \sin(2\theta) d^{5\,6}_R. \tag{D.52}$$

Finally, a comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^{u\,1}_{R1} = \frac{g_3}{2} D^{5\,6}_R;\ \mathrm{h}^{u\,2}_{R1} = \frac{g_3}{2} d^{5\,6}_R;\ \mathrm{h}^{u\,3}_{R1} = \frac{g_3}{2}(D^6_R - D^5_R). \tag{D.53}$$

D.3 Simplification of the wave equations

With (D.7) and (D.14) we have:

$$(W_n^1 + iW_n^2)\overline{L}^{3+n} = (D_L^{n\,3+n} + id_L^{n\,3+n})\overline{L}^{3+n} = 2\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n}, \tag{D.54}$$

$$\widetilde{L}^n\overline{L}^{3+n} = 2\begin{pmatrix}0 & 0\\ -\eta_2^n & \eta_1^n\end{pmatrix}\begin{pmatrix}\eta_1^{3+n} & 0\\ \eta_2^{3+n} & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 2(-\eta_2^n\eta_1^{3+n} + \eta_1^n\eta_2^{3+n}) & 0\end{pmatrix}.$$

And given that:

$$2(-\eta_2^n\eta_1^{3+n} + \eta_1^n\eta_2^{3+n}) = 2\widehat{\eta}^{n\dagger}\eta^{3+n} = \overline{s}_2^{3+n\,n}, \tag{D.55}$$

$$\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n} = \sqrt{2}\begin{pmatrix}0 & \overline{\eta}_2^{3+n}\\ 0 & \overline{\eta}_1^{3+n}\end{pmatrix}\begin{pmatrix}0 & 0\\ \overline{s}_2^{3+n\,n} & 0\end{pmatrix} = \overline{s}_2^{3+n\,n}\widetilde{L}^{3+n}\sigma_1.$$

We also have:

$$W_n^3\widehat{L}^n = (D^{3+n} - D^n)\widehat{L}^n = \widetilde{L}^{3+n}L^{3+n}\widehat{L}^n - L^n\widetilde{L}^n\widehat{L}^n = \widetilde{L}^{3+n}L^{3+n}\widehat{L}^n, \tag{D.56}$$

$$L^{3+n}\widehat{L}^n = \overline{\widetilde{L}^n\overline{L}^{3+n}} = \overline{\overline{s}_2^{3+n\,n}\frac{\sigma_1 - i\sigma_2}{2}} = -\overline{s}_2^{3+n\,n}\frac{\sigma_1 - i\sigma_2}{2} = -\widetilde{L}^n\overline{L}^{3+n},$$

$$W_n^3\widehat{L}^n = \widetilde{L}^{3+n}L^{3+n}\widehat{L}^n = -\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n}. \tag{D.57}$$

Thus we obtain:

$$\begin{aligned}(W_n^1 + iW_n^2)\overline{L}^{3+n} - W_n^3\widehat{L}^n &= 2\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n} - (-\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n})\\ &= 3\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n} = -3W_n^3\widehat{L}^n.\end{aligned} \tag{D.58}$$

Furthermore:

$$(W_n^1 - iW_n^2)\widehat{L}^n = (D_L^{n\,3+n} - id_L^{n\,3+n})\widehat{L}^n = 2L^nL^{3+n}\widehat{L}^n, \tag{D.59}$$

$$L^{3+n}\widehat{L}^n = \overline{\widetilde{L}^n\overline{L}^{3+n}} = -\widetilde{L}^n\overline{L}^{3+n},$$

$$(W_n^1 - iW_n^2)\widehat{L}^n = 2L^n(-\widetilde{L}^n\overline{L}^{3+n}) = -2L^n\widetilde{L}^n\overline{L}^{3+n} = -2D_L^n\overline{L}^{3+n}, \tag{D.60}$$

$$W_n^3\overline{L}^{3+n} = (D_n^{3+n} - D_L^n)\overline{L}^{3+n} = -D_L^n\overline{L}^{3+n},$$

$$(W_n^1 - iW_n^2)\widehat{L}^n + W_n^3\overline{L}^{3+n} = -3D_L^n\overline{L}^{3+n} = 3W_n^3\overline{L}^{3+n}. \tag{D.61}$$

For the gauge group of chromodynamics we have the same sort of simplification, we will see this in detail for one of the four cases, that of the left wave of the d quark. With the gauge transformation generated by Γ_1 and with an angle θ we have:

$$(\mathrm{h}^{d\,1}_{L1} + i\mathrm{h}^{d\,2}_{L1})\widehat{L}^3 - \mathrm{h}^{d\,3}_{L1}\widehat{L}^2 = \frac{g_3}{2}[(D_L^{23} + id_L^{23})\widehat{L}^3 - (D_L^3 - D_L^2)\widehat{L}^2], \tag{D.62}$$

$$L'^2 = CL^2 + iS\widetilde{L}^3;\ C = \cos(\theta);\ S = \sin(\theta), \tag{D.63}$$

$$\widetilde{L}'^3 = C\widetilde{L}^3 + iSL^2. \tag{D.64}$$

Only the indices change in comparison with (D.5) and (D.6), where for $n = 2$ we have the same relations with indices 2 and 5 instead of the current indices 2 and 3. And so we can employ the same procedure that carried out for the weak interactions. We finally have:

$$\mathrm{h}^{d\,1}_{L1} = \frac{g_3}{2} D^{23}_L;\ \mathrm{h}^{d\,2}_{L1} = \frac{g_3}{2} d^{23}_L;\ \mathrm{h}^{d\,3}_{L1} = \frac{g_3}{2}(D^3_L - D^2_L), \tag{D.65}$$

$$(\mathrm{h}^{d\,1}_{L1} + i\mathrm{h}^{d\,2}_{L1})\widehat{L}^3 - \mathrm{h}^{d\,3}_{L1}\widehat{L}^2 = -3\mathrm{h}^{d\,3}_{L1}\widehat{L}^2 = -\frac{3g_3}{2} D^3_L \widehat{L}^2, \tag{D.66}$$

$$(\mathrm{h}^{d\,1}_{L1} - i\mathrm{h}^{d\,2}_{L1})\widehat{L}^2 + \mathrm{h}^{d\,3}_{L1}\widehat{L}^3 = 3\mathrm{h}^{d\,3}_{L1}\widehat{L}^3 = -\frac{3g_3}{2} D^2_L \widehat{L}^3. \tag{D.67}$$

We obtain similar equations for the other indices, for the right waves and for the u quark, this allows us to simplify the wave equations.

D.3.1 Gauge terms of the Lagrangian density

We consider the S part of the Lagrangian density $\mathcal{L}^+$ that gives the gauge terms acting on the waves of the quarks:

$$\begin{aligned} S &= \sum_{n=2}^{4} \begin{bmatrix} -i\frac{\mathrm{m}}{q_1}\eta^{n\dagger}\sigma^\mu(ig^1_{n\mu})\eta^n - i\frac{\mathrm{m}}{q_2}\xi^{n\dagger}\widehat{\sigma}^\mu(ig^2_{n\mu})\xi^n \\ -i\frac{\mathrm{m}}{q_3}\eta^{3+n\dagger}\sigma^\mu(ig^3_{n\mu})\eta^{3+n} - i\frac{\mathrm{m}}{q_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu(ig^4_{n\mu})\xi^{3+n} \end{bmatrix} \\ &= \sum_{n=2}^{4} \begin{bmatrix} \frac{\mathrm{m}}{q_1} g^1_{n\mu}\eta^{n\dagger}\sigma^\mu\eta^n + \frac{\mathrm{m}}{q_2} g^2_{n\mu}\xi^{n\dagger}\widehat{\sigma}^\mu\xi^n \\ +\frac{\mathrm{m}}{q_3} g^3_{n\mu}\eta^{3+n\dagger}\sigma^\mu\eta^{3+n} + \frac{\mathrm{m}}{q_4} g^4_{n\mu}\xi^{3+n\dagger}\widehat{\sigma}^\mu\xi^{3+n} \end{bmatrix}. \end{aligned} \tag{D.68}$$

We thus have:

$$\begin{aligned} &S = S^1 + S^2 + S^3 + S^4;\ S^1 = \frac{\mathrm{m}}{q_1}\sum_{n=2}^{4} g^1_{n\mu} D^{n\mu}_L, \\ &S^2 = \frac{\mathrm{m}}{q_2}\sum_{n=2}^{4} g^2_{n\mu} D^{n\mu}_R;\ S^3 = \frac{\mathrm{m}}{q_3}\sum_{n=2}^{4} g^3_{n\mu} D^{3+n\mu}_L;\ S^4 = \frac{\mathrm{m}}{q_4}\sum_{n=2}^{4} g^4_{n\mu} D^{3+n\mu}_R. \end{aligned} \tag{D.69}$$

With (**??**) the S^1 term becomes:

$$\begin{aligned} \frac{q_1}{\mathrm{m}} S^1 &= \Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{2\mu} - 3\mathrm{h}^{d3}_{L3\mu} + 3\mathrm{h}^{d3}_{L1\mu}\Big) D^{2\mu}_L \\ &+ \Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{3\mu} - 3\mathrm{h}^{d3}_{L1\mu} + 3\mathrm{h}^{d3}_{L2\mu}\Big) D^{3\mu}_L \\ &+ \Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{4\mu} - 3\mathrm{h}^{d3}_{L2\mu} + 3\mathrm{h}^{d3}_{L3\mu}\Big) D^{4\mu}_L. \end{aligned} \tag{D.70}$$

Grouping together similar terms we get:

$$\begin{aligned} \frac{q_1}{\mathrm{m}} S^1 &= -\frac{1}{3}\mathrm{b}\cdot(D^2_L + D^3_L + D^4_L) + 3(\mathrm{w}^3_2\cdot D^2_L + \mathrm{w}^3_3\cdot D^3_L + \mathrm{w}^3_4\cdot D^4_L) \\ &- 3(\mathrm{h}^{d3}_{L3}\cdot D^2_L + \mathrm{h}^{d3}_{L1}\cdot D^3_L + \mathrm{h}^{d3}_{L2}\cdot D^4_L) + 3(\mathrm{h}^{d3}_{L1}\cdot D^2_L + \mathrm{h}^{d3}_{L2}\cdot D^3_L + \mathrm{h}^{d3}_{L3}\cdot D^4_L). \end{aligned} \tag{D.71}$$

And with (3.128) we have:

$$\begin{aligned}&3(\mathrm{w}_2^3 \cdot D_L^2 + \mathrm{w}_3^3 \cdot D_L^3 + \mathrm{w}_4^3 \cdot D_L^4)\\&= \frac{g_2}{2}[(D_L^5 - D_L^2)\cdot D_L^2 + (D_L^6 - D_L^3)\cdot D_L^3 + (D_L^7 - D_L^4)\cdot D_L^4\\&= \frac{g_2}{2}(D_L^5 \cdot D_L^2 + D_L^6 \cdot D_L^3 + D_L^7 \cdot D_L^4),\end{aligned} \tag{D.72}$$

and since the chiral currents are on the light cone, their space-time length is null. Next with (D.40) we have, ${\mathrm{h}_{L1}^{d}}^{1} = \frac{g_3}{2}{D_L^2}^3$, ${\mathrm{h}_{L1}^{d}}^{2} = \frac{g_3}{2}{d_L^2}^3$ and ${\mathrm{h}_{L1}^{d}}^{3} = \frac{g_3}{2}(D_L^3 - D_L^2)$. We thus get:

$$\begin{aligned}&3(\mathrm{h}_{L1}^{d3} \cdot D_L^2 + \mathrm{h}_{L2}^{d3} \cdot D_L^3 + \mathrm{h}_{L3}^{d3} \cdot D_L^4)\\&= 3\frac{g_3}{2}[(D_L^3 - D_L^2)\cdot D_L^2 + (D_L^4 - D_L^3)\cdot D_L^3 + (D_L^2 - D_L^4)\cdot D_L^4]\\&= 3\frac{g_3}{2}(D_L^3 \cdot D_L^2 + D_L^4 \cdot D_L^3 + D_L^2 \cdot D_L^4).\end{aligned} \tag{D.73}$$

Similarly we have:

$$\begin{aligned}&-3(\mathrm{h}_{L3}^{d3} \cdot D_L^2 + \mathrm{h}_{L1}^{d3} \cdot D_L^3 + \mathrm{h}_{L2}^{d3} \cdot D_L^4)\\&= -3\frac{g_3}{2}[(D_L^2 - D_L^4)\cdot D_L^2 + (D_L^3 - D_L^2)\cdot D_L^3 + (D_L^4 - D_L^3)\cdot D_L^4]\\&= 3\frac{g_3}{2}(D_L^4 \cdot D_L^2 + D_L^2 \cdot D_L^3 + D_L^3 \cdot D_L^4)\\&= 3\frac{g_3}{2}(D_L^3 \cdot D_L^2 + D_L^4 \cdot D_L^3 + D_L^2 \cdot D_L^4).\end{aligned} \tag{D.74}$$

And if we use the sums of currents:

$$\begin{aligned}&S_L^{25} = D_L^2 + D_L^5;\ S_L^{36} = D_L^3 + D_L^6;\ S_L^{47} = D_L^4 + D_L^7,\\&S_L^d = D_L^2 + D_L^3 + D_L^4,\end{aligned} \tag{D.75}$$

we have:

$$(S_L^{25})^2 = (D_L^2)^2 + (D_L^5)^2 + 2D_L^2 \cdot D_L^5 = 2D_L^2 \cdot D_L^5, \tag{D.76}$$

$$\begin{aligned}(S_L^d)^2 &= (D_L^2)^2 + (D_L^3)^2 + (D_L^4)^2 + 2D_L^2 \cdot D_L^3 + 2D_L^3 \cdot D_L^4 + 2D_L^4 \cdot D_L^2\\&= 2(D_L^2 \cdot D_L^3 + D_L^3 \cdot D_L^4 + D_L^4 \cdot D_L^2).\end{aligned} \tag{D.77}$$

And we thus get:

$$S^1 = \frac{\mathrm{m}}{q_1}\Big[-\frac{g_1}{6}B \cdot S_L^d + \frac{g_2}{4}\big[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2\big] + \frac{3g_3}{2}(S_L^d)^2\Big]. \tag{D.78}$$

Next for the right waves of the d quark we have:

$$\begin{aligned}&\frac{q_2}{\mathrm{m}}S^2 = \frac{2}{3}\mathrm{b} \cdot (D_R^2 + D_R^3 + D_R^4)\\&+ 3(\mathrm{h}_{R3}^{d3} \cdot D_R^2 + \mathrm{h}_{R1}^{d3} \cdot D_R^3 + \mathrm{h}_{R2}^{d3} \cdot D_R^4) - 3(\mathrm{h}_{R1}^{d3} \cdot D_R^2 + \mathrm{h}_{R2}^{d3} \cdot D_R^3 + \mathrm{h}_{R3}^{d3} \cdot D_R^4).\end{aligned} \tag{D.79}$$

We now use the sum of the right currents:

$$S_R^d = D_R^2 + D_R^3 + D_R^4, \tag{D.80}$$

And we get:

$$S^2 = \frac{\mathrm{m}}{q_2}\Big[\frac{g_1}{3}B\cdot S_R^d - 3\frac{g_3}{2}(S_R^d)^2\Big]. \tag{D.81}$$

The calculation of the terms corresponding to the u quark is totally resemblant, and the same sums are introduced:

$$S_L^u = D_L^5 + D_L^6 + D_L^7;\ S_R^u = D_R^5 + D_R^6 + D_R^7. \tag{D.82}$$

For the left waves we have:

$$S^3 = \frac{\mathrm{m}}{q_3}\Big[-\frac{g_1}{6}B\cdot S_L^u + \frac{g_2}{4}[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2] + \frac{3g_3}{2}(S_L^u)^2\Big]. \tag{D.83}$$

And we obtain for the right waves of the u quark:

$$S^4 = \frac{\mathrm{m}}{q_4}\Big[-\frac{2g_1}{3}B\cdot S_R^u - 3\frac{g_3}{2}(S_R^u)^2\Big]. \tag{D.84}$$

This gives:

$$\begin{aligned} S = {} & \frac{\mathrm{m}}{q_1}\Big[-\frac{g_1}{6}B\cdot S_L^d + \frac{g_2}{4}\big[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2\big] + \frac{3g_3}{2}(S_L^d)^2\Big] \\ & + \frac{\mathrm{m}}{q_2}\Big[\frac{g_1}{3}B\cdot S_R^d - \frac{3g_3}{2}(S_R^d)^2\Big] \\ & + \frac{\mathrm{m}}{q_3}\Big[-\frac{g_1}{6}B\cdot S_L^u + \frac{g_2}{4}\big[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2\big] + \frac{3g_3}{2}(S_L^u)^2\Big] \\ & + \frac{\mathrm{m}}{q_4}\Big[-\frac{2g_1}{3}B\cdot S_R^u - \frac{3g_3}{2}(S_R^u)^2\Big]. \end{aligned} \tag{D.85}$$

D.4 Calculation of $\Gamma_{\mu\nu}^{\rho}$

D.4.1 Calculation of $\mathcal{S}_{(k)}$ and $\mathcal{A}_{(k)}$

The S_k are bivectors in Cl_3. So they read:

$$\begin{aligned} \phi\sigma_k\overline{\phi} = S_k &:= \vec{E}_k + i\vec{H}_k, \\ \vec{E}_k &:= E_k^1\sigma_1 + E_k^2\sigma_2 + E_k^3\sigma_3, \\ \vec{H}_k &:= H_k^1\sigma_1 + H_k^2\sigma_2 + H_k^3\sigma_3. \end{aligned} \tag{D.86}$$

We thus have:

$$E_k^1 = S_k^{23}\ ;\quad E_k^2 = S_k^{31}\ ;\quad E_k^3 = S_k^{12}\ ;\quad H_k^j = S_k^{j0}. \tag{D.87}$$

Next we obtain:

$$\begin{aligned}\nabla S_k^\dagger &= (\partial_0 - \vec{\partial})(\vec{E}_k - i\vec{H}_k)\\ &= -\vec{\partial}\cdot\vec{E}_k + (\partial_0\vec{E}_k - \vec{\partial}\times\vec{H}_k) + i(-\vec{\partial}\times\vec{E}_k - \partial_0\vec{H}_k) + i\vec{\partial}\cdot\vec{H}_k\\ &= j_k + ij'_k = j_k^0 + \vec{j}_k + i\vec{j_k}' + i{j'}_k^0,\end{aligned} \tag{D.88}$$

with

$$j_k^0 := -\vec{\partial}\cdot\vec{E}_k\ ;\quad \vec{j}_k := \partial_0\vec{E}_k - \vec{\partial}\times\vec{H}_k, \tag{D.89}$$

$${j'}_k^0 := \vec{\partial}\cdot\vec{H}_k\ ;\quad \vec{j_k}' := -\partial_0\vec{H}_k - \vec{\partial}\times\vec{E}_k. \tag{D.90}$$

With the electromagnetic potential A we have:

$$\begin{aligned}AS_k^\dagger &= (A^0 + \vec{A})(\vec{E}_k - i\vec{H}_k)\\ &= \vec{A}\cdot\vec{E}_k + (A^0\vec{E}_k + \vec{A}\times\vec{H}_k) + i(\vec{A}\times\vec{E}_k - A^0\vec{H}_k) - i\vec{A}\cdot\vec{H}_k\\ &= v_k + iv'_k = v_k^0 + \vec{v}_k + i\vec{v_k}' + i{v'}_k^0,\end{aligned} \tag{D.91}$$

with

$$v_k^0 := \vec{A}\cdot\vec{E}_k\ ;\quad \vec{v}_k := A^0\vec{E}_k + \vec{A}\times\vec{H}_k, \tag{D.92}$$

$${v'}_k^0 := -\vec{A}\cdot\vec{H}_k\ ;\quad \vec{v_k}' := -A^0\vec{H}_k + \vec{A}\times\vec{E}_k. \tag{D.93}$$

Similarly with v and S_k we have:

$$\begin{aligned}JS_k^\dagger &= \rho\mathrm{v}S_k^\dagger = \rho(\mathrm{v}^0 + \vec{\mathrm{v}})(\vec{E}_k - i\vec{H}_k)\\ &= \rho[\vec{\mathrm{v}}\cdot\vec{E}_k + (\mathrm{v}^0\vec{E}_k + \vec{\mathrm{v}}\times\vec{H}_k) + i(\vec{\mathrm{v}}\times\vec{E}_k - \mathrm{v}^0\vec{H}_k) - i\vec{\mathrm{v}}\cdot\vec{H}_k]\\ &= \phi\phi^\dagger(\phi\sigma_k\overline{\phi})^\dagger = \phi\phi^\dagger\widehat{\phi}\sigma_k\phi^\dagger = \phi\rho e^{-i\beta}\sigma_k\phi^\dagger = e^{-i\beta}\phi\sigma_k\phi^\dagger = (\Omega_1 - i\Omega_2)\mathrm{D}_k\\ &= \Omega_1\mathrm{D}_k^0 + \Omega_1\vec{\mathrm{D}}_k - i\Omega_2\vec{\mathrm{D}}_k - i\Omega_2\mathrm{D}_k^0\end{aligned} \tag{D.94}$$

Thus we obtain:

$$\frac{\Omega_1}{\rho}\mathrm{D}_k^0 = \vec{\mathrm{v}}\cdot\vec{E}_k\ ;\quad \frac{\Omega_1}{\rho}\vec{\mathrm{D}}_k = \mathrm{v}^0\vec{E}_k + \vec{\mathrm{v}}\times\vec{H}_k, \tag{D.95}$$

$$\frac{\Omega_2}{\rho}\mathrm{D}_k^0 = \vec{\mathrm{v}}\cdot\vec{H}_k\ ;\quad \frac{\Omega_2}{\rho}\vec{\mathrm{D}}_k = \mathrm{v}^0\vec{H}_k - \vec{\mathrm{v}}\times\vec{E}_k. \tag{D.96}$$

With the definition (4.30) we have:

$$\begin{aligned}\mathcal{S}_{(k)} + i\mathcal{S}'_{(k)} &= \frac{\nabla S_k^\dagger}{\det(\phi^\dagger)} = \frac{\Omega_1 + i\Omega_2}{\rho^2}(j_k + ij'_k)\\ &= \rho^{-2}[\Omega_1 j_k - \Omega_2 j'_k + i(\Omega_1 j'_k + \Omega_2 j_k)],\end{aligned} \tag{D.97}$$

$$\rho^2\mathcal{S}_{(k)} = \Omega_1 j_k - \Omega_2 j'_k, \tag{D.98}$$

$$\rho^2\mathcal{S}'_{(k)} = \Omega_1 j'_k + \Omega_2 j_k. \tag{D.99}$$

Similarly we have :

$$\begin{aligned}
\mathcal{A}_{(k)} + i\mathcal{A}'_{(k)} &= \frac{AS_k^\dagger}{\det(\phi^\dagger)} = \frac{\Omega_1 + i\Omega_2}{\rho^2}(v_k + iv'_k) \\
&= \rho^{-2}[\Omega_1 v_k - \Omega_2 v'_k + i(\Omega_1 v'_k + \Omega_2 v_k)], \qquad \text{(D.100)} \\
\rho^2\mathcal{A}_{(k)} &= \Omega_1 v_k - \Omega_2 v'_k, \qquad \text{(D.101)} \\
\rho^2\mathcal{A}'_{(k)} &= \Omega_1 v'_k + \Omega_2 v_k. \qquad \text{(D.102)}
\end{aligned}$$

D.4.2 Calculation of $\Gamma^\mu_{\mu\nu}$

We start from the definition of Durand's spin density (4.32), which gives:

$$\begin{aligned}
\tau &= \frac{1}{2}[(\nabla\widehat{\phi})\phi^\dagger - \sigma^\mu\widehat{\phi}\partial_\mu\phi^\dagger], \\
2\tau &= (\nabla\widehat{\phi})\phi^\dagger - \dot{\nabla}\widehat{\phi}\dot{\phi}^\dagger, \qquad \text{(D.103)}
\end{aligned}$$

where the dots indicate that which we derive. And we have:

$$\nabla(\widehat{\phi}\phi^\dagger) = (\nabla\widehat{\phi})\phi^\dagger + \dot{\nabla}\widehat{\phi}\dot{\phi}^\dagger. \qquad \text{(D.104)}$$

Hence by adding we get:

$$2\tau + \nabla(\Omega_1 - i\Omega_2) = 2(\nabla\widehat{\phi})\phi^\dagger. \qquad \text{(D.105)}$$

With our improved wave equation we have:

$$\begin{aligned}
(\nabla\widehat{\phi})\phi^\dagger &= qA\widehat{\phi}\sigma_{21}\phi^\dagger + e^{-i\beta}\phi\mathbf{m}\sigma_{21}\phi^\dagger, \qquad \text{(D.106)} \\
\phi\mathbf{m}\sigma_{21}\phi^\dagger &= -i(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L), \qquad \text{(D.107)} \\
(\nabla\widehat{\phi})\phi^\dagger &= -iqAS_3^\dagger - i\Big(\frac{\Omega_1}{\rho} - i\frac{\Omega_2}{\rho}\Big)(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L). \qquad \text{(D.108)}
\end{aligned}$$

We now let:

$$\tau = \tau_1 + i\tau_2 \ ; \quad \tau_1 = \frac{1}{2}(\tau + \tau^\dagger) \ ; \quad i\tau_2 = \frac{1}{2}(\tau - \tau^\dagger). \qquad \text{(D.109)}$$

With (D.105) and (D.108) we obtain:

$$\begin{aligned}
\nabla\Omega_1 - i\nabla\Omega_2 &= -2(\tau_1 + i\tau_2) - 2iqAS_3^\dagger - 2\Big(\frac{\Omega_2}{\rho} + i\frac{\Omega_1}{\rho}\Big)(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L) \\
&= -2(\tau_1 + i\tau_2) - 2iq(v_3 + iv'_3) - 2\Big(\frac{\Omega_2}{\rho} + i\frac{\Omega_1}{\rho}\Big)(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L). \qquad \text{(D.110)}
\end{aligned}$$

This gives:

$$\begin{aligned}
\nabla\Omega_1 &= -2\tau_1 + 2qv'_3 - 2\frac{\Omega_2}{\rho}(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L), \qquad \text{(D.111)} \\
\nabla\Omega_2 &= 2\tau_2 + 2qv_3 + 2\frac{\Omega_1}{\rho}(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L). \qquad \text{(D.112)}
\end{aligned}$$

Since $\rho^2 = \Omega_1^2 + \Omega_2^2$, we have:

$$\rho\nabla\rho = \Omega_1\nabla\Omega_1 + \Omega_2\nabla\Omega_2. \tag{D.113}$$

With our improved wave equation and with (D.111) and (D.112) we get:

$$\begin{aligned}\rho\nabla\rho &= \Omega_1[-2\tau_1 + 2qv_3' - 2\frac{\Omega_2}{\rho}(\mathbf{l}\mathrm{D}_R - \mathbf{r}\mathrm{D}_L)] \\ &+ \Omega_2[2\tau_2 + 2qv_3 + 2m\frac{\Omega_1}{\rho}(\mathbf{l}\mathrm{D}_R - \mathbf{r}\mathrm{D}_L)] \\ &= 2(-\Omega_1\tau_1 + \Omega_2\tau_2) + 2q(\Omega_1 v_3' + \Omega_2 v_3).\end{aligned} \tag{D.114}$$

And with (D.102) we have:

$$\rho\nabla\rho = 2(-\Omega_1\tau_1 + \Omega_2\tau_2) + 2q\rho^2\mathcal{A}'_{(3)}. \tag{D.115}$$

With (4.33) we get:

$$\rho^2(\mathcal{T}+i\mathcal{T}') = (\tau_1+i\tau_2)(\Omega_1+i\Omega_2) = (\tau_1\Omega_1-\tau_2\Omega_2)+i(\tau_1\Omega_2+\tau_2\Omega_1). \tag{D.116}$$

Dividing (D.115) by ρ^2 we finally have:

$$\nabla(\ln\rho) = -2\mathcal{T} + 2q\mathcal{A}'_{(3)}. \tag{D.117}$$

With (4.26) we have:

$$\Gamma^0_{0\nu} = D^\mu_\nu\partial_\mu(\ln\rho) = D_\nu\cdot\nabla(\ln\rho) \tag{D.118}$$

$$= D_\nu\cdot(-2\mathcal{T} + 2q\mathcal{A}'_{(3)}), \tag{D.119}$$

which is (4.40).

D.4.3 Calculation of $\Gamma^0_{j\nu}$ and $\Gamma^j_{0\nu}$, $j = 1,\ 2,\ 3$

We start from:

$$\begin{aligned}\Gamma^0_{j\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^\mu_j)\overline{D}^0_\mu = \rho^{-2}[(\boldsymbol{\partial}_\nu D^0_j)\overline{D}^0_0 + \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_j)\overline{D}^0_k] \\ &= \rho^{-2}[(\boldsymbol{\partial}_\nu D^0_j)D^0_0 - \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_j)D^k_0],\end{aligned} \tag{D.120}$$

and similarly:

$$\begin{aligned}\Gamma^j_{0\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^\mu_0)\overline{D}^j_\mu = \rho^{-2}[(\boldsymbol{\partial}_\nu D^0_0)\overline{D}^j_0 + \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_0)\overline{D}^j_k] \\ &= \rho^{-2}[-(\boldsymbol{\partial}_\nu D^0_0)D^0_j + \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_0)D^k_j].\end{aligned} \tag{D.121}$$

We thus have:

$$\Gamma^0_{j\nu} - \Gamma^j_{0\nu} = \rho^{-2}[\boldsymbol{\partial}_\nu(D^0_0 D^0_j) - \sum_{k=1}^{3} \boldsymbol{\partial}_\nu(D^k_0 D^k_j)] = \rho^{-2}\boldsymbol{\partial}_\nu(D_0 \cdot D_j) = 0,$$

$$\Gamma^0_{j\nu} = \Gamma^j_{0\nu}. \tag{D.122}$$

We also get:

$$\Gamma^0_{j\nu} = \rho^{-2}\begin{pmatrix} D^0_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^0_j) \\ -D^1_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^1_j) \\ -D^2_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^2_j) \\ -D^0_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^3_j) \end{pmatrix}$$

$$= \rho^{-2}\begin{pmatrix} D^0_\nu(D^0_0\partial_0 D^0_j - D^1_0\partial_0 D^1_j - D^2_0\partial_0 D^2_j - D^3_0\partial_0 D^3_j) \\ +D^1_\nu(D^0_0\partial_1 D^0_j - D^1_0\partial_1 D^1_j - D^2_0\partial_1 D^2_j - D^3_0\partial_1 D^3_j) \\ +D^2_\nu(D^0_0\partial_2 D^0_j - D^1_0\partial_2 D^1_j - D^2_0\partial_2 D^2_j - D^3_0\partial_2 D^3_j) \\ +D^3_\nu(D^0_0\partial_3 D^0_j - D^1_0\partial_3 D^1_j - D^2_0\partial_3 D^2_j - D^3_0\partial_3 D^3_j) \end{pmatrix}$$

$$= \rho^{-2}D^\mu_\nu(D^0_0\partial_\mu D^0_j - D^1_0\partial_\mu D^1_j - D^2_0\partial_\mu D^2_j - D^3_0\partial_\mu D^3_j). \tag{D.123}$$

Next we obtain:

$$\Gamma^0_{j\nu} = \rho^{-2}D^\mu_\nu[\partial_\mu(D^0_0 D^0_j) - \partial_\mu(D^1_0 D^1_j) - \partial_\mu(D^2_0 D^2_j) - \partial_\mu(D^3_0 D^3_j)]$$

$$+ \rho^{-2}D^\mu_\nu[-D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0] \tag{D.124}$$

$$= \rho^{-2}D^\mu_\nu[\partial_\mu(D_0 \cdot D_j) - D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0]$$

$$= \rho^{-2}D^\mu_\nu[-D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0]. \tag{D.125}$$

We let:

$$\Gamma^0_{j\nu} = \rho^{-2}D^\mu_\nu X_{j\mu}, \tag{D.126}$$

$$X_{j\mu} = -D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0. \tag{D.127}$$

D.4.4 Calculation of $\Gamma^0_{1\nu}$

We start from:

$$\Gamma^0_{1\nu} = \rho^{-2}D^\mu_\nu X_{1\mu}, \tag{D.128}$$

$$X_{1\mu} = -D^0_1\partial_\mu D^0_0 + D^1_1\partial_\mu D^1_0 + D^2_1\partial_\mu D^2_0 + D^3_1\partial_\mu D^3_0. \tag{D.129}$$

We have:

$$\begin{aligned} X_{1\mu} = &-(-\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*) \\ &+ (\xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*) \\ &+ i(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*) \\ &+ (-\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*). \end{aligned} \tag{D.130}$$

This gives:

$$X_{1\mu} = 2\begin{pmatrix} -\xi_2(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\xi_1 + \xi_1(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\xi_2 \\ +\eta_2(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\eta_1 - \eta_1(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\eta_2 \\ -\xi_2^*(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\xi_1^* + \xi_1^*(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\xi_2^* \\ +\eta_2^*(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\eta_1^* - \eta_1^*(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\eta_2^* \end{pmatrix}, \tag{D.131}$$

and with (A.176) we get:

$$X_{1\mu} = \begin{pmatrix} -\xi_2(\Omega_1 - i\Omega_2)\partial_\mu\xi_1 + \xi_1(\Omega_1 - i\Omega_2)\partial_\mu\xi_2 \\ +\eta_2(\Omega_1 + i\Omega_2)\partial_\mu\eta_1 - \eta_1(\Omega_1 + i\Omega_2)\partial_\mu\eta_2 \\ -\xi_2^*(\Omega_1 + i\Omega_2)\partial_\mu\xi_1^* + \xi_1^*(\Omega_1 + i\Omega_2)\partial_\mu\xi_2^* \\ +\eta_2^*(\Omega_1 - i\Omega_2)\partial_\mu\eta_1^* - \eta_1^*(\Omega_1 - i\Omega_2)\partial_\mu\eta_2^* \end{pmatrix}. \tag{D.132}$$

We can thus express it as follows:

$$X_{1\mu} = \Omega_1 Y_\mu + i\Omega_2 Z_\mu, \tag{D.133}$$

$$\begin{aligned} Y_\mu &= \xi_1\partial_\mu\xi_2 - \xi_2\partial_\mu\xi_1 - \eta_1\partial_\mu\eta_2 + \eta_2\partial_\mu\eta_1 \\ &\quad + \xi_1^*\partial_\mu\xi_2^* - \xi_2^*\partial_\mu\xi_1^* - \eta_1^*\partial_\mu\eta_2^* + \eta_2^*\partial_\mu\eta_1^*, \end{aligned} \tag{D.134}$$

$$\begin{aligned} Z_\mu &= -\xi_1\partial_\mu\xi_2 + \xi_2\partial_\mu\xi_1 - \eta_1\partial_\mu\eta_2 + \eta_2\partial_\mu\eta_1 \\ &\quad + \xi_1^*\partial_\mu\xi_2^* - \xi_2^*\partial_\mu\xi_1^* + \eta_1^*\partial_\mu\eta_2^* - \eta_2^*\partial_\mu\eta_1^*. \end{aligned} \tag{D.135}$$

Our improved wave equation is equivalent to the system:

$$\begin{aligned} 0 &= (\nabla + ia)\eta;\ a := qA + \mathbf{l}\mathrm{v}, \\ 0 &= (\widehat{\nabla} + i\widehat{b})\xi;\ b := qA + \mathbf{r}\mathrm{v}. \end{aligned} \tag{D.136}$$

This is equivalent to the system of partial differential equations:

$$\begin{aligned} 0 &= \partial_0\eta_1 - \partial_1\eta_2 + i\partial_2\eta_2 - \partial_3\eta_1 + i(a_0\eta_1 - a_1\eta_2 + ia_2\eta_2 - a_3\eta_1), \\ 0 &= \partial_0\eta_2 - \partial_1\eta_1 - i\partial_2\eta_1 + \partial_3\eta_2 + i(a_0\eta_2 - a_1\eta_1 - ia_2\eta_1 + a_3\eta_2), \\ 0 &= \partial_0\xi_1 + \partial_1\xi_2 - i\partial_2\xi_2 + \partial_3\xi_1 + i(b_0\xi_1 + b_1\xi_2 - ib_2\xi_2 + b_3\xi_1), \\ 0 &= \partial_0\xi_2 + \partial_1\xi_1 + i\partial_2\xi_1 - \partial_3\xi_2 + i(b_0\xi_2 + b_1\xi_1 + ib_2\xi_1 - b_3\xi_2). \end{aligned} \tag{D.137}$$

Given these systems of equations, after simplification we get:

$$\begin{aligned} Y_0 &= -\partial_1 S_1^{23} - \partial_2 S_1^{31} - \partial_3 S_1^{12} \\ &\quad + 2b_1 H_R^1 + 2b_2 H_R^2 + 2b_3 H_R^3 - 2a_1 H_L^1 - 2a_2 H_L^2 - 2a_3 H_L^3 \\ &= -\vec{\partial}\cdot\vec{E}_1 + 2(q\vec{A} + \mathbf{r}\vec{\mathrm{v}})\cdot\vec{H}_R - 2(q\vec{A} + \mathbf{l}\vec{\mathrm{v}})\cdot\vec{H}_L \\ &= -\vec{\partial}\cdot\vec{E}_1 + 2q\vec{A}\cdot\vec{E}_2 + 2m\vec{\mathrm{v}}\cdot\vec{E}_2 - 2d\vec{\mathrm{v}}\cdot\vec{H}_1 \\ &= j_1^0 + 2qv_2^0 + \frac{2m}{\rho}\Omega_1\mathrm{D}_2^0 - \frac{2d}{\rho}\Omega_2\mathrm{D}_1^0. \end{aligned} \tag{D.138}$$

Similarly we have:

$$\begin{aligned}
Z_0 &= i\partial_1 S_1^{10} + i\partial_2 S_1^{20} + i\partial_3 S_1^{30} \\
&\quad + 2i(b_1 E_R^1 + b_2 E_R^2 + b_3 E_R^3) - 2i(a_1 E_L^1 + a_2 E_L^2 + a_3 E_L^3) \\
&= i(\vec{\partial} \cdot \vec{H}_1 + 2\vec{b} \cdot \vec{E}_R - 2\vec{a} \cdot \vec{E}_L) \qquad \text{(D.139)} \\
&= i[\vec{\partial} \cdot \vec{H}_1 - 2q\vec{A} \cdot \vec{H}_2 - 2m\vec{\mathrm{v}} \cdot \vec{H}_2 - 2d\vec{\mathrm{v}} \cdot \vec{E}_1]. \\
&= i[{j'}_1^0 + 2q{v'}_2^0 - \frac{2m}{\rho}\Omega_2 \mathrm{D}_2^0 - \frac{2d}{\rho}\Omega_1 \mathrm{D}_1^0]. \qquad \text{(D.140)}
\end{aligned}$$

We hence obtain:

$$\begin{aligned}
X_{10} &= \Omega_1 Y_0 + i\Omega_2 Z_0 \\
&= \Omega_1(j_1^0 - 2qv_2^0 - 2m\frac{\Omega_1}{\rho}D_2^0 + \frac{2d}{\rho}\Omega_2 \mathrm{D}_1^0) + i\Omega_2(i{j'}_1^0 - 2iq{v'}_2^0 \\
&\quad + 2im\frac{\Omega_2}{\rho}D_2^0 - \frac{2d}{\rho}\Omega_1 \mathrm{D}_1^0) \\
&= \Omega_1 j_1^0 - \Omega_2 {j'}_1^0 + 2q(-\Omega_1 v_2^0 + \Omega_2 {v'}_2^0) - 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^0 \\
&= \rho^2(\mathcal{S}_{(1)}^0 - 2q\mathcal{A}_{(2)}^0) - 2m\rho D_2^0.
\end{aligned}$$

Again using (D.137) and simplifying, we get:

$$\begin{aligned}
Y_1 &= -\partial_0 S_1^{23} + \partial_2 S_1^{30} - \partial_3 S_1^{20} + 2q(A_0 E_2^1 - A_2 H_2^3 + A_3 H_2^2) \\
&\quad + 2m(\mathrm{v}_0 E_2^1 - \mathrm{v}_2 H_2^3 + \mathrm{v}_3 H_2^2) - 2d(\mathrm{v}_0 H_1^1 + \mathrm{v}_2 E_1^3 - \mathrm{v}_3 E_1^2) \\
&= -j_1^1 + 2qv_2^1 + 2m\frac{\Omega_1}{\rho}\mathrm{D}_2^1 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^1. \qquad \text{(D.141)}
\end{aligned}$$

Similarly, after simplification (D.137) gives:

$$\begin{aligned}
Z_1 &= i[\partial_0 H_1^1 + \partial_2 E_1^3 - \partial_3 E_1^2 - 2q(A^0 H_2^1 - A^3 E_2^2 + A^2 E_2^3) \\
&\quad - 2m(\mathrm{v}_0 H_2^1 - \mathrm{v}^2 E_2^3 + \mathrm{v}^3 E_2^2) - 2d(\mathrm{v}_0 E_1^1 + \mathrm{v}^2 H_1^3 - \mathrm{v}^3 H_1^2) \\
&= i\Big[-{j'}_1^1 + 2q{v'}_2^1 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^1 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^1\Big]. \qquad \text{(D.142)}
\end{aligned}$$

With (D.133) and (D.137) we have:

$$\begin{aligned}
X_{11} &= \Omega_1 Y_1 + i\Omega_2 Z_1 \\
&= \Omega_1(-j_1^1 + 2qv_2^1 + 2m\frac{\Omega_1}{\rho}D_2^1 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^1) \\
&\quad - \Omega_2(-{j'}_1^1 + 2q{v'}_2^1 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^1 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^1) \\
&= -\Omega_1 j_1^1 + \Omega_2 {j'}_1^1 + 2q(\Omega_1 v_2^1 - \Omega_2 {v'}_2^1) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^1 \\
&= \rho^2(-\mathcal{S}_{(1)}^1 + 2q\mathcal{A}_{(2)}^1) + 2m\rho D_2^1. \qquad \text{(D.143)}
\end{aligned}$$

Again using (D.137) and simplifying we have:

$$\begin{aligned} Y_2 &= -\partial_0 S_1^{31} + \partial_3 S_1^{10} - \partial_1 S_1^{30} + 2q(A_0 E_2^2 - A_3 H_2^1 + A_1 H_2^3) \\ &+ 2m(\mathrm{v}_0 E_2^2 - \mathrm{v}_3 H_2^1 + \mathrm{v}_1 H_2^3) - 2d(\mathrm{v}_0 H_1^2 + \mathrm{v}_3 E_1^1 - \mathrm{v}_1 E_1^3) \\ &= -j_1^2 + 2qv_2^2 + 2m\frac{\Omega_1}{\rho}\mathrm{D}_2^2 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^2. \end{aligned} \tag{D.144}$$

Similarly, after simplification (D.137) gives:

$$\begin{aligned} Z_2 &= i[\partial_0 H_1^2 + \partial_3 E_1^1 - \partial_1 E_1^3 - 2q(A^0 H_2^2 - A^1 E_2^3 + A^3 E_2^1) \\ &- 2m(\mathrm{v}_0 H_2^2 - \mathrm{v}^3 E_2^1 + \mathrm{v}^1 E_2^3) - 2d(\mathrm{v}_0 E_1^2 + \mathrm{v}^3 H_1^1 - \mathrm{v}^1 H_1^3) \\ &= i[-{j'}_1^2 + 2q{v'}_2^2 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^2 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^2]. \end{aligned} \tag{D.145}$$

With (D.133) and (D.137) we have:

$$\begin{aligned} X_{12} &= \Omega_1 Y_2 + i\Omega_2 Z_2 \\ &= \Omega_1(-j_1^2 + 2qv_2^2 + 2m\frac{\Omega_1}{\rho}D_2^2 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^2) \\ &- \Omega_2(-{j'}_1^2 + 2q{v'}_2^2 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^2 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^2) \\ &= -\Omega_1 j_1^2 + \Omega_2 {j'}_1^2 + 2q(\Omega_1 v_2^2 - \Omega_2 {v'}_2^2) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^2 \\ &= \rho^2(-\mathcal{S}_{(1)}^2 + 2q\mathcal{A}_{(2)}^2) + 2m\rho D_2^2. \end{aligned} \tag{D.146}$$

Again using (D.137), and simplifying we have:

$$\begin{aligned} Y_3 &= -\partial_0 S_1^{12} + \partial_1 S_1^{10} - \partial_2 S_1^{10} + 2q(A_0 E_2^3 - A_1 H_2^2 + A_2 H_2^1) \\ &+ 2m(\mathrm{v}_0 E_2^3 - \mathrm{v}_1 H_2^2 + \mathrm{v}_2 H_2^1) - 2d(\mathrm{v}_0 H_1^3 + \mathrm{v}_1 E_1^2 - \mathrm{v}_2 E_1^1) \\ &= -j_1^3 + 2qv_2^3 + 2m\frac{\Omega_1}{\rho}\mathrm{D}_2^3 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^3. \end{aligned} \tag{D.147}$$

Similarly, and after simplification, (D.137) gives:

$$\begin{aligned} Z_3 &= i[\partial_0 H_1^3 + \partial_1 E_1^2 - \partial_2 E_1^1 - 2q(A^0 H_2^3 - A^2 E_2^1 + A^1 E_2^2) \\ &- 2m(\mathrm{v}_0 H_2^3 - \mathrm{v}^1 E_2^2 + \mathrm{v}^2 E_2^1) - 2d(\mathrm{v}_0 E_1^3 + \mathrm{v}^1 H_1^2 - \mathrm{v}^2 H_1^1) \\ &= i[-{j'}_1^3 + 2q{v'}_2^3 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^3 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^3]. \end{aligned} \tag{D.148}$$

With (D.133) and (D.137) we have:

$$\begin{aligned}
X_{13} &= \Omega_1 Y_3 + i\Omega_2 Z_3 \\
&= \Omega_1(-j_1^3 + 2qv_2^3 + 2m\frac{\Omega_1}{\rho}D_2^3 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^3) \\
&- \Omega_2(-j'^3_1 + 2qv'^3_2 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^3 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^3) \\
&= -\Omega_1 j_1^3 + \Omega_2 j'^3_1 + 2q(\Omega_1 v_2^3 - \Omega_2 v'^3_2) + 2m\frac{\Omega_1^2+\Omega_2^2}{\rho}D_2^3 \\
&= \rho^2(-\mathcal{S}^3_{(1)} + 2q\mathcal{A}^3_{(2)}) + 2m\rho D_2^3. \qquad \text{(D.149)}
\end{aligned}$$

And we thus get:

$$\begin{aligned}
\Gamma^0_{1\nu} &= \rho^{-2}D^\mu_\nu X_{1\mu} = \rho^{-2}[D^0_\nu X_{10} + \sum_{k=1}^3 D^k_\nu X_{1k}] \\
&= \rho^{-2}[D^0_\nu(\rho^2\mathcal{S}^0_{(1)} - 2q\rho^2\mathcal{A}^0_{(2)} - 2m\rho D_2^0) \\
&+ \sum_{k=1}^3 D^k_\nu(-\rho^2\mathcal{S}^k_{(1)} + 2q\rho^2\mathcal{A}^k_{(2)} + 2m\rho D_2^k)] \\
&= D_\nu \cdot (\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)} - 2\frac{m}{\rho}D_2) \\
&= D_\nu \cdot (\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}) + 2m\rho\delta^2_\nu, \qquad \text{(D.150)}
\end{aligned}$$

which is (4.34)

D.4.5 Calculation of $\Gamma^0_{2\nu}$

We start from:

$$\Gamma^0_{2\nu} = \rho^{-2}D^\mu_\nu X_{2\mu}, \qquad \text{(D.151)}$$

$$X_{2\mu} = -D_2^0\partial_\mu D_0^0 + D_2^1\partial_\mu D_0^1 + D_2^2\partial_\mu D_0^2 + D_2^3\partial_\mu D_0^3. \qquad \text{(D.152)}$$

We have:

$$\begin{aligned}
X_{2\mu} &= -i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*) \\
&+ i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*) \\
&+ (\xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*) \\
&+ i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*), \qquad \text{(D.153)}
\end{aligned}$$

which with (A.176) gives:

$$X_{2\mu} = \begin{pmatrix} \xi_2(i\Omega_1+\Omega_2)\partial_\mu\xi_1 - \xi_1(i\Omega_1+\Omega_2)\partial_\mu\xi_2 \\ -\eta_2(i\Omega_1-\Omega_2)\partial_\mu\eta_1 + \eta_1(i\Omega_1-\Omega_2)\partial_\mu\eta_2 \\ -\xi_2^*(i\Omega_1-\Omega_2)\partial_\mu\xi_1^* + \xi_1^*(i\Omega_1-\Omega_2)\partial_\mu\xi_2^* \\ +\eta_2^*(i\Omega_1+\Omega_2)\partial_\mu\eta_1^* - \eta_1^*(i\Omega_1+\Omega_2)\partial_\mu\eta_2^* \end{pmatrix}. \qquad \text{(D.154)}$$

We may then express it as:

$$X_{2\mu} = i\Omega_1 L_\mu + \Omega_2 M_\mu, \tag{D.155}$$

$$\begin{aligned} L_\mu &= -\xi_1\partial_\mu\xi_2 + \xi_2\partial_\mu\xi_1 + \eta_1\partial_\mu\eta_2 - \eta_2\partial_\mu\eta_1 \\ &\quad + \xi_1^*\partial_\mu\xi_2^* - \xi_2^*\partial_\mu\xi_1^* - \eta_1^*\partial_\mu\eta_2^* + \eta_2^*\partial_\mu\eta_1^*, \end{aligned} \tag{D.156}$$

$$\begin{aligned} M_\mu &= -\xi_1\partial_\mu\xi_2 + \xi_2\partial_\mu\xi_1 - \eta_1\partial_\mu\eta_2 + \eta_2\partial_\mu\eta_1 \\ &\quad - \xi_1^*\partial_\mu\xi_2^* + \xi_2^*\partial_\mu\xi_1^* - \eta_1^*\partial_\mu\eta_2^* + \eta_2^*\partial_\mu\eta_1^*. \end{aligned} \tag{D.157}$$

Again with (D.137) and after simplification we get:

$$\begin{aligned} L_0 &= i[\partial_1 E_2^1 + \partial_2 E_2^2 + \partial_3 E_2^3 - 2\vec{b}\cdot\vec{E}_R - 2\vec{a}\cdot\vec{E}_L \\ &= i[\vec{\partial}\cdot\vec{E}_2 - 2(q\vec{A} + \mathbf{r}\vec{\mathrm{v}})\cdot\vec{E}_R - 2(q\vec{A} + \mathbf{l}\vec{\mathrm{v}})\cdot\vec{E}_L] \\ &= i(-j_2^0 - 2qv_1^0 - 2m\frac{\Omega_1}{\rho}D_1^0 - 2d\frac{\Omega_2}{\rho}D_2^0). \end{aligned} \tag{D.158}$$

After simplification the equation (D.157) yields:

$$\begin{aligned} M_0 &= -\partial_1 H_2^1 - \partial_2 H_2^2 - \partial_3 H_2^3 + 2\vec{b}\cdot\vec{H}_R + 2\vec{a}\cdot\vec{H}_L \\ &= -\vec{\partial}\cdot\vec{H}_2 + 2(q\vec{A} + \mathbf{r}\vec{\mathrm{v}})\cdot\vec{H}_R + 2(q\vec{A} + \mathbf{l}\vec{\mathrm{v}})\cdot\vec{H}_L] \\ &= -{j'}_2^0 - 2q{v'}_1^0 + 2m\frac{\Omega_2}{\rho}D_1^0 - 2d\frac{\Omega_1}{\rho}D_2^0). \end{aligned} \tag{D.159}$$

With (D.155) and (D.158) we obtain:

$$\begin{aligned} X_{20} &= i\Omega_1 L_0 + \Omega_2 M_0 \\ &= -\Omega_1(-j_2^0 - 2qv_1^0 - 2m\frac{\Omega_1}{\rho}D_1^0 - 2d\frac{\Omega_2}{\rho}D_2^0) \\ &\quad + \Omega_2(-{j'}_2^0 - 2q{v'}_1^0 + 2m\frac{\Omega_2}{\rho}D_1^0 - 2d\frac{\Omega_1}{\rho}D_2^0) \\ &= (\Omega_1 j_2^0 - \Omega_2 {j'}_2^0) + 2q(\Omega_1 v_1^0 - \Omega_2 {v'}_1^0) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_1^0 \\ &= \rho^2(\mathcal{S}_{(2)}^0 + 2q\mathcal{A}_{(1)}^0) + 2m\rho D_1^0. \end{aligned} \tag{D.160}$$

As usual with (D.137) and after simplification we have:

$$\begin{aligned} L_1 &= i[\partial_0(H_R^1 - H_L^1) + \partial_2(E_R^3 - E_L^3) - \partial_3(E_R^2 - E_L^2) \\ &\quad + 2b_0E_R^1 + 2a_0E_L^1 - 2b_2H_R^3 - 2a_2H_L^3 + 2b_3H_R^2 + 2a_3H_L^2] \\ &= i\Big(j_2^1 + 2qv_1^1 + 2m\frac{\Omega_1}{\rho}D_1^1 + 2d\frac{\Omega_2}{\rho}D_2^1\Big). \end{aligned} \tag{D.161}$$

With (D.137) and after simplification we get:

$$\begin{aligned} M_1 &= \partial_0(E_R^1 - E_L^1) + \partial_2(-H_R^3 + H_L^3) + \partial_3(H_R^2 - H_L^2) \\ &\quad + 2(-b_0H_R^1 - b_2E_R^3 + b_3E_R^2) + 2(-a_0H_L^1 - a_2E_L^3 + a_3E_L^2) \\ &= {j'}_2^1 + 2q{v'}_1^1 - 2m\frac{\Omega_2}{\rho}D_1^1 + 2d\frac{\Omega_1}{\rho}D_2^1. \end{aligned} \tag{D.162}$$

With (D.155), (D.161) and (D.162) we have:

$$
\begin{aligned}
X_{21} &= i\Omega_1 L_1 + \Omega_2 M_1 \\
&= -\Omega_1 (j_2^1 + 2qv_1^1 + 2m\frac{\Omega_1}{\rho} D_1^1 + 2d\frac{\Omega_2}{\rho} D_2^1) \\
&\quad + \Omega_2 ({j'}_2^1 + 2q{v'}_1^1 - 2m\frac{\Omega_2}{\rho} D_1^1 + 2d\frac{\Omega_1}{\rho} D_2^1) \\
&= -\Omega_1 j_2^1 + \Omega_2 {j'}_2^1 + 2q(-\Omega_1 v_1^1 + \Omega_2 {v'}_1^1) - 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho} D_1^1 \\
&= \rho^2(-\mathcal{S}_{(2)}^1 - 2q\mathcal{A}_{(1)}^1) + 2m\rho D_1^1.
\end{aligned} \tag{D.163}
$$

Again with (D.137) and after simplification we obtain:

$$
\begin{aligned}
L_2 &= i[\partial_0(H_R^2 - H_L^2) + \partial_3(E_R^1 - E_L^1) - \partial_1(E_R^3 - E_L^3) \\
&\quad + 2b_0 E_R^2 + 2a_0 E_L^2 - 2b_3 H_R^1 - 2a_3 H_L^1 + 2b_1 H_R^3 + 2a_1 H_L^3] \\
&= i(j_2^2 + 2qv_1^2 + 2m\frac{\Omega_1}{\rho} D_1^2 + 2d\frac{\Omega_2}{\rho} D_2^2).
\end{aligned} \tag{D.164}
$$

Similarly with (D.137) and simplifying we get:

$$
\begin{aligned}
M_2 &= \partial_0(E_R^2 - E_L^2) + \partial_3(-H_R^1 + H_L^1) + \partial_1(H_R^3 - H_L^3) \\
&\quad + 2(-b_0 H_R^2 - b_3 E_R^1 + b_1 E_R^3) + 2(-a_0 H_L^2 - a_3 E_L^1 + a_1 E_L^3) \\
&= {j'}_2^2 + 2q{v'}_1^2 - 2m\frac{\Omega_2}{\rho} D_1^2.
\end{aligned} \tag{D.165}
$$

With (D.155), (D.164) and (D.165) we have:

$$
\begin{aligned}
X_{22} &= i\Omega_1 L_2 + \Omega_2 M_2 \\
&= -\Omega_1 \Big(j_2^2 + 2qv_1^2 + 2m\frac{\Omega_1}{\rho} D_1^2 + 2d\frac{\Omega_2}{\rho} D_2^2\Big) \\
&\quad + \Omega_2 \Big({j'}_2^2 + 2q{v'}_1^2 - 2m\frac{\Omega_2}{\rho} D_1^2 + 2d\frac{\Omega_1}{\rho} D_2^2\Big) \\
&= -\Omega_1 j_2^2 + \Omega_2 {j'}_2^2 + 2q(-\Omega_1 v_1^2 + \Omega_2 {v'}_1^2) - 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho} D_1^2 \\
&= \rho^2(-\mathcal{S}_{(2)}^2 - 2q\mathcal{A}_{(1)}^2) + 2m\rho D_1^2.
\end{aligned} \tag{D.166}
$$

Again with (D.137) and after simplification we get:

$$
\begin{aligned}
L_3 &= i[\partial_0(H_R^3 - H_L^3) + \partial_2(E_R^2 - E_L^2) - \partial_2(E_R^1 - E_L^1) \\
&\quad + 2b_0 E_R^3 + 2a_0 E_L^3 - 2b_1 H_R^2 - 2a_1 H_L^2 + 2b_2 H_R^1 + 2a_2 H_L^1] \\
&= i\Big(j_2^3 + 2qv_1^3 + 2m\frac{\Omega_1}{\rho} D_1^3 + 2d\frac{\Omega_2}{\rho} D_2^3\Big).
\end{aligned} \tag{D.167}
$$

Similarly with (D.137) and simplifying we have:

$$\begin{aligned}M_3 &= \partial_0(E_R^3 - E_L^3) + \partial_1(-H_R^2 + H_L^2) + \partial_2(H_R^1 - H_L^1)\\ &\quad + 2(-b_0H_R^3 - b_1D_R^2 + b_2E_R^1) + 2(-a_0H_L^3 - a_1E_L^2 + a_2E_L^1)\\ &= {j'}_2^3 + 2q{v'}_1^3 - 2m\frac{\Omega_2}{\rho}D_1^3. \qquad \text{(D.168)}\end{aligned}$$

With (D.155), (D.167) and (D.168) we obtain:

$$\begin{aligned}X_{23} &= i\Omega_1 L_3 + \Omega_2 M_3\\ &= -\Omega_1\Big(j_2^3 + 2qv_1^3 + 2m\frac{\Omega_1}{\rho}D_1^3 + 2d\frac{\Omega_2}{\rho}D_2^3\Big)\\ &\quad + \Omega_2\Big({j'}_2^3 + 2q{v'}_1^3 - 2m\frac{\Omega_2}{\rho}D_1^3 + 2d\frac{\Omega_1}{\rho}D_2^3\Big)\\ &= -\Omega_1 j_2^3 + \Omega_2{j'}_2^3 + 2q(-\Omega_1 v_1^3 + \Omega_2{v'}_1^3) - 2m\frac{\Omega_1^2+\Omega_2^2}{\rho}D_1^3\\ &= \rho^2(-\mathcal{S}^3_{(2)} - 2q\mathcal{A}^3_{(1)}) + 2m\rho D_1^3. \qquad \text{(D.169)}\end{aligned}$$

And we thus have:

$$\begin{aligned}\Gamma^0_{2\nu} &= \rho^{-2}D^\mu_\nu X_{2\mu} = \rho^{-2}\Big[D^0_\nu X_{20} + \sum_{k=1}^{3} D^k_\nu X_{2k}\Big]\\ &= \rho^{-2}[D^0_\nu(\rho^2\mathcal{S}^0_{(2)} + 2q\rho^2\mathcal{A}^0_{(1)} + 2m\rho D^0_1)\\ &\quad + \sum_{k=1}^{3} D^k_\nu(-\rho^2\mathcal{S}^k_{(2)} - 2q\rho^2\mathcal{A}^k_{(1)} - 2m\rho D^k_1)]\\ &= D_\nu\cdot\Big(\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)} + 2\frac{m}{\rho}D_1\Big)\\ &= D_\nu\cdot(\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}) - 2m\rho\delta^1_\nu, \qquad \text{(D.170)}\end{aligned}$$

which is (4.35),

D.4.6 Calculation of $\Gamma^0_{3\nu}$

We begin with:

$$\Gamma^0_{3\nu} = \rho^{-2}D^\mu_\nu X_{3\mu}, \qquad \text{(D.171)}$$
$$X_{3\mu} = -D^0_3\partial_\mu D^0_0 + D^1_3\partial_\mu D^1_0 + D^2_3\partial_\mu D^2_0 + D^3_3\partial_\mu D^3_0. \qquad \text{(D.172)}$$

We have:

$$\begin{aligned}X_{3\mu} &= -(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*)\\ &\quad + (\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*)\\ &\quad + i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*)\\ &\quad + (\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*), \qquad \text{(D.173)}\end{aligned}$$

which by (A.176) gives:

$$X_{3\mu} = \begin{pmatrix} \eta_1^*(\Omega_1 - i\Omega_2)\partial_\mu\xi_1 + \eta_2^*(\Omega_1 - i\Omega_2)\partial_\mu\xi_2 \\ -\xi_1^*(\Omega_1 + i\Omega_2)\partial_\mu\eta_1 - \xi_2^*(\Omega_1 + i\Omega_2)\partial_\mu\eta_2 \\ +\eta_1(\Omega_1 + i\Omega_2)\partial_\mu\xi_1^* + \eta_2(\Omega_1 + i\Omega_2)\partial_\mu\xi_2^* \\ -\xi_1(\Omega_1 - i\Omega_2)\partial_\mu\eta_1^* - \xi_2(\Omega_1 - i\Omega_2)\partial_\mu\eta_2^* \end{pmatrix}. \tag{D.174}$$

We let:

$$X_{3\mu} = \Omega_1 N_\mu + i\Omega_2 P_\mu, \tag{D.175}$$

$$\begin{aligned} N_\mu &= \eta_1^*\partial_\mu\xi_1 + \eta_2^*\partial_\mu\xi_2 - \xi_1^*\partial_\mu\eta_1 - \xi_2^*\partial_\mu\eta_2 \\ &\quad + \eta_1\partial_\mu\xi_1^* + \eta_2\partial_\mu\xi_2^* - \xi_1\partial_\mu\eta_1^* - \xi_2\partial_\mu\eta_2^*, \end{aligned} \tag{D.176}$$

$$\begin{aligned} P_\mu &= -\eta_1^*\partial_\mu\xi_1 - \eta_2^*\partial_\mu\xi_2 - \xi_1^*\partial_\mu\eta_1 - \xi_2^*\partial_\mu\eta_2 \\ &\quad + \eta_1\partial_\mu\xi_1^* + \eta_2\partial_\mu\xi_2^* + \xi_1\partial_\mu\eta_1^* + \xi_2\partial_\mu\eta_2^*. \end{aligned} \tag{D.177}$$

With (D.137) and after simplification we get:

$$\begin{aligned} N_0 &= -\partial_1 E_3^1 - \partial_2 E_3^2 - \partial_3 E_3^3 + (b_0 + a_0)\Omega_2 \\ &\quad + (b_1 - a_1)H_3^1 + (b_2 - a_2)H_3^2 + (b_3 - a_3)H_3^3 \\ &= j_3^0 + 2\Omega_2\Big(qA^0 + m\mathrm{v}^0 + \frac{d}{\rho}\mathrm{D}_3^0\Big). \end{aligned} \tag{D.178}$$

Again with (D.137) and after simplification we get:

$$\begin{aligned} P_0 &= \partial_1(iH_3^1) + \partial_2(iH_3^2) + \partial_3(iH_3^3) + i(b_0 + a_0)\Omega_1 \\ &\quad + i(b_1 - a_1)E_3^1 + i(b_2 - a_2)E_3^2 + i(b_3 - a_3)E_3^3 \\ &= i\Big[{j'}_3^0 + 2\Omega_1\Big(qA_0 + m\mathrm{v}_0 + \frac{d}{\rho}D_3^0\Big)\Big]. \end{aligned} \tag{D.179}$$

In light of (D.175), (D.178) and (D.179) taken together we get:

$$\begin{aligned} X_{30} &= \Omega_1 N_0 + \Omega_2 i P_0 \\ &= \Omega_1\Big[j_3^0 + 2\Omega_2\Big(qA^0 + m\mathrm{v}^0 + \frac{d}{\rho}\mathrm{D}_3^0\Big)\Big] - \Omega_2\Big[{j'}_3^0 + 2\Omega_1\Big(qA_0 + m\mathrm{v}_0 + \frac{d}{\rho}D_3^0\Big)\Big] \\ &= \Omega_1 j_3^0 - \Omega_2 {j'}_3^0 \\ &= \rho^2\mathcal{S}_{(3)}^0. \end{aligned} \tag{D.180}$$

As always with (D.137) and after simplification we get:

$$\begin{aligned} N_1 &= -\partial_0 E_3^1 + \partial_2 H_3^3 - \partial_3 H_3^2 \\ &\quad + (b_1 + a_1)\Omega_2 + (b_0 - a_0)H_3^1 + (b_2 - a_2)E_3^3 + (b_3 - a_3)(-E_3^2) \\ &= -j_3^1 - 2\Omega_2\Big(qA^1 + \frac{m}{\rho}\mathrm{D}_0^1 + \frac{d}{\rho}\mathrm{D}_3^1\Big). \end{aligned} \tag{D.181}$$

With (D.137) and after simplification we get:

$$\begin{aligned} P_1 &= i(\partial_0 H_3^1 + \partial_2 E_3^3 - \partial_3 E_3^2 + (b_1 + a_1)E_3^1 \\ &+ (b_0 - a_0)E_3^1 - (b_2 - a_2)H_3^3 + (b_3 - a_3)H_3^2 \\ &= i\Big[- j'^1_3 - 2\Omega_1 \Big(qA^1 + \frac{m}{\rho} D_0^1 + \frac{d}{\rho} D_3^1 \Big)\Big]. \end{aligned} \tag{D.182}$$

With (D.175), (D.181) and (D.182) we get:

$$\begin{aligned} X_{31} &= \Omega_1 N_1 + \Omega_2 i P_1 \\ &= \Omega_1 \Big[- j_3^1 - 2\Omega_2 \Big(qA^1 + \frac{m}{\rho} D_0^1 + \frac{d}{\rho} D_3^1 \Big)\Big] \\ &- \Omega_2 \Big[- j'^1_3 - 2\Omega_1 \Big(qA^1 + \frac{m}{\rho} D_0^1 + \frac{d}{\rho} D_3^1 \Big)\Big] \\ &= -\Omega_1 j_3^1 + \Omega_2 j'^1_3 = -\rho^2 \mathcal{S}^1_{(3)}. \end{aligned} \tag{D.183}$$

Again with (D.137) we get:

$$\begin{aligned} N_2 &= -\partial_0 E_3^2 - \partial_1 H_3^3 + \partial_3 H_3^1 + (b_2 + a_2)\Omega_2 \\ &+ (b_0 - a_0)H_3^2 + (b_3 - a_3)E_3^1 - (b_1 - a_1)E_3^2 \\ &= -j_3^2 - 2\Omega_2 \Big(qA^2 + \frac{m}{\rho} D_0^2 + \frac{d}{\rho} D_3^2 \Big). \end{aligned} \tag{D.184}$$

As usual with (D.137) we obtain:

$$\begin{aligned} P_2 &= i(\partial_0 H_3^2 + \partial_3 E_3^1 - \partial_1 E_3^3 + (b_2 + a_2)E_3^2 \\ &+ (b_0 - a_0)E_3^2 - (b_3 - a_3)H_3^1 + (b_1 - a_1)H_3^3 \\ &= i\Big[- j'^2_3 - 2\Omega_1 \Big(qA^2 + \frac{m}{\rho} D_0^2 + \frac{d}{\rho} D_3^2 \Big)\Big]. \end{aligned} \tag{D.185}$$

With (D.175), (D.184) and (D.185) we have:

$$\begin{aligned} X_{32} &= \Omega_1 N_2 + \Omega_2 i P_2 \\ &= \Omega_1 \Big[- j_3^2 - 2\Omega_2 \Big(qA^2 + \frac{m}{\rho} D_0^2 + \frac{d}{\rho} D_3^2 \Big)\Big] \\ &- \Omega_2 \Big[- j'^2_3 - 2\Omega_1 \Big(qA^2 + \frac{m}{\rho} D_0^2 + \frac{d}{\rho} D_3^2 \Big)\Big] \\ &= -\Omega_1 j_3^2 + \Omega_2 j'^2_3 = -\rho^2 \mathcal{S}^2_{(3)}. \end{aligned} \tag{D.186}$$

Always with (D.137) we get:

$$\begin{aligned} N_3 &= -\partial_0 E_3^3 - \partial_2 H_3^1 + \partial_1 H_3^2 + (b_3 + a_3)\Omega_2 \\ &+ (b_0 - a_0)H_3^3 + (b_1 - a_1)E_3^2 - (b_2 - a_2)E_3^3 \\ &= -j_3^3 - 2\Omega_2 (qA^3 + \frac{m}{\rho} D_0^3 + \frac{d}{\rho} D_3^3). \end{aligned} \tag{D.187}$$

With (D.137) and after simplification we have:

$$\begin{aligned} P_3 &= i(\partial_0 H_3^3 + \partial_1 E_3^2 - \partial_2 E_3^1 + (b_3 + a_3)E_3^3 \\ &\quad + (b_0 - a_0)E_3^3 - (b_1 - a_1)H_3^2 + (b_2 - a_2)H_3^1 \\ &= i\Big[- {j'}_3^3 - 2\Omega_1 \Big(qA^3 + \frac{m}{\rho} D_0^3 + \frac{d}{\rho} D_3^3 \Big) \Big]. \end{aligned} \tag{D.188}$$

Hence with (D.175), (D.187) and (D.188) we get:

$$\begin{aligned} X_{33} &= \Omega_1 N_3 + \Omega_2 i P_3 \\ &= \Omega_1 \Big[- j_3^3 - 2\Omega_2 \Big(qA^3 + \frac{m}{\rho} D_0^3 + \frac{d}{\rho} D_3^3 \Big) \Big] \\ &\quad - \Omega_2 \Big[- {j'}_3^3 - 2\Omega_1 \Big(qA^3 + \frac{m}{\rho} D_0^3 + \frac{d}{\rho} D_3^3 \Big) \Big] \\ &= -\Omega_1 j_3^3 + \Omega_2 {j'}_3^3 = -\rho^2 \mathcal{S}_{(3)}^3. \end{aligned} \tag{D.189}$$

And we thus have:

$$\begin{aligned} \Gamma_{3\nu}^0 &= \rho^{-2} D_\nu^\mu X_{3\mu} = \rho^{-2} \Big[D_\nu^0 X_{30} + \sum_{k=1}^{3} D_\nu^k X_{3k} \Big] \\ &= \rho^{-2} \Big[D_\nu^0 \Big(\rho^2 \mathcal{S}_{(3)}^0 + \sum_{k=1}^{3} D_\nu^k (-\rho^2 \mathcal{S}_{(3)}^k) \Big) \Big] \\ &= D_\nu \cdot \mathcal{S}_{(3)}, \end{aligned} \tag{D.190}$$

which is (4.36).

D.4.7 Calculation of $\Gamma_{l\nu}^k$

We must calculate these symbols for $k = 1,\ 2,\ 3$; $l = 1,\ 2,\ 3$ and $l \neq k$. We start from:

$$\begin{aligned} \Gamma_{l\nu}^k &= \rho^{-2} (\boldsymbol{\partial}_\nu D_l^\mu) \overline{D}_\mu^k \\ &= \rho^{-2} \Big[(\boldsymbol{\partial}_\nu D_l^0) \overline{D}_0^k + \sum_{n=1}^{3} (\boldsymbol{\partial}_\nu D_l^n) \overline{D}_n^k \Big] \\ &= \rho^{-2} \Big[(\boldsymbol{\partial}_\nu D_l^0)(-D_k^0) + \sum_{n=1}^{3} (\boldsymbol{\partial}_\nu D_l^n) D_k^n \Big], \end{aligned} \tag{D.191}$$

and similarly:

$$\begin{aligned}\Gamma^{l}_{k\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^{\mu}_{k})\overline{D}^{l}_{\mu}\\ &= \rho^{-2}\Big[(\boldsymbol{\partial}_\nu D^{0}_{k})\overline{D}^{l}_{0} + \sum_{n=1}^{3}(\boldsymbol{\partial}_\nu D^{n}_{k})\overline{D}^{l}_{n}\Big]\\ &= \rho^{-2}\Big[(\boldsymbol{\partial}_\nu D^{0}_{k})(-D^{0}_{l}) + \sum_{n=1}^{3}(\boldsymbol{\partial}_\nu D^{n}_{k})D^{n}_{l}\Big]. \qquad \text{(D.192)}\end{aligned}$$

We thus get:

$$\begin{aligned}\Gamma^{k}_{l\nu} + \Gamma^{l}_{k\nu} &= \rho^{-2}\Big[-\boldsymbol{\partial}_\nu(D^{0}_{k}D^{0}_{l}) + \sum_{n=1}^{3}\boldsymbol{\partial}_\nu(D^{n}_{k}D^{n}_{l})\Big],\\ &= -\rho^{-2}\boldsymbol{\partial}_\nu(D_k\cdot D_l) = 0 \qquad \text{(D.193)}\\ \Gamma^{l}_{k\nu} &= -\Gamma^{k}_{l\nu}. \qquad \text{(D.194)}\end{aligned}$$

The calculation of $\Gamma^{1}_{2\nu}$, $\Gamma^{2}_{3\nu}$ and $\Gamma^{3}_{1\nu}$ is thus sufficient. Moreover we have:

$$\begin{aligned}\rho^2\Gamma^{k}_{l\nu} &= \begin{pmatrix}-D^{0}_{k}(D^{0}_{\nu}\partial_0 + D^{1}_{\nu}\partial_1 + D^{2}_{\nu}\partial_2 + D^{3}_{\nu}\partial_3)(D^{0}_{l})\\ +D^{1}_{k}(D^{0}_{\nu}\partial_0 + D^{1}_{\nu}\partial_1 + D^{2}_{\nu}\partial_2 + D^{3}_{\nu}\partial_3)(D^{1}_{l})\\ +D^{2}_{k}(D^{0}_{\nu}\partial_0 + D^{1}_{\nu}\partial_1 + D^{2}_{\nu}\partial_2 + D^{3}_{\nu}\partial_3)(D^{2}_{l})\\ +D^{3}_{k}(D^{0}_{\nu}\partial_0 + D^{1}_{\nu}\partial_1 + D^{2}_{\nu}\partial_2 + D^{3}_{\nu}\partial_3)(D^{3}_{l})\end{pmatrix}\\ &= \begin{pmatrix}D^{0}_{\nu}(-D^{0}_{k}\partial_0 D^{0}_{l} + D^{1}_{k}\partial_0 D^{1}_{l} + D^{2}_{k}\partial_0 D^{2}_{l} + D^{3}_{k}\partial_0 D^{3}_{l})\\ +D^{1}_{\nu}(-D^{0}_{k}\partial_1 D^{0}_{l} + D^{1}_{k}\partial_1 D^{1}_{l} + D^{2}_{k}\partial_1 D^{2}_{l} + D^{3}_{k}\partial_1 D^{3}_{l})\\ +D^{2}_{\nu}(-D^{0}_{k}\partial_2 D^{0}_{l} + D^{1}_{k}\partial_2 D^{1}_{l} + D^{2}_{k}\partial_2 D^{2}_{l} + D^{3}_{k}\partial_2 D^{3}_{l})\\ +D^{3}_{\nu}(-D^{0}_{k}\partial_3 D^{0}_{l} + D^{1}_{k}\partial_3 D^{1}_{l} + D^{2}_{k}\partial_3 D^{2}_{l} + D^{3}_{k}\partial_3 D^{3}_{l})\end{pmatrix}\\ &= D^{\mu}_{\nu}(-D^{0}_{k}\partial_\mu D^{0}_{l} + D^{1}_{k}\partial_\mu D^{1}_{l} - D^{2}_{k}\partial_\mu D^{2}_{l} + D^{3}_{k}\partial_\mu D^{3}_{l}). \qquad \text{(D.195)}\end{aligned}$$

Calculation of $\Gamma^{1}_{2\nu}$

Given that:

$$\begin{aligned}\rho^2\Gamma^{1}_{2\nu} &= D^{\mu}_{\nu}(iW_\mu),\\ iW_\mu &= -D^{0}_{1}\partial_\mu D^{0}_{2} + D^{1}_{1}\partial_\mu D^{1}_{2} - D^{2}_{1}\partial_\mu D^{2}_{2} + D^{3}_{1}\partial_\mu D^{3}_{2} \qquad \text{(D.196)}\\ &= -(-\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1)\\ &+ (\xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1)\\ &+ i(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu(\xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1)\\ &+ (-\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1).\end{aligned}$$

We thus have:

$$\begin{aligned}\frac{1}{2}W_\mu &= (\partial_\mu\xi_1)(-\eta_1^*)(\eta_2\xi_2^* + \eta_1\xi_1^*) + (\partial_\mu\xi_2)(-\eta_2^*)(\eta_1\xi_1^* + \eta_2\xi_2^*)\\ &+ (\partial_\mu\eta_1)(-\xi_1^*)(\xi_2\eta_2^* + \xi_1\eta_1^*) + (\partial_\mu\eta_2)(-\xi_2^*)(\xi_1\eta_1^* + \xi_2\eta_2^*)\\ &+ (\partial_\mu\xi_1^*)\eta_1(\xi_2\eta_2^* + \xi_1\eta_1^*) + (\partial_\mu\xi_2^*)\eta_2(\xi_1\eta_1^* + \xi_2\eta_2^*)\\ &+ (\partial_\mu\eta_1^*)\xi_1(\eta_2\xi_2^* + \eta_1\xi_1^*) + (\partial_\mu\eta_2^*)\xi_2(\eta_1\xi_1^* + \eta_2\xi_2^*),\end{aligned} \tag{D.197}$$

which gives with equations (D.176) and(D.177) calculating N_μ and P_μ:

$$iW_\mu = \Omega_1 iP_\mu - \Omega_2 N_\mu. \tag{D.198}$$

We may thus use the results of D.4.6 and we directly obtain:

$$\begin{aligned}\rho^2\Gamma^1_{2\nu} &= D^0_\nu\Big[-\rho^2\mathcal{S}'^0_{(3)} - 2\rho^2\Big(qA^0 + \frac{m}{\rho}\mathrm{D}^0_0 + \frac{d}{\rho}\mathrm{D}^0_3\Big)\Big]\\ &\quad + D^k_\nu\Big[\rho^2\mathcal{S}'^k_{(3)} + 2\rho^2\Big(qA^k + \frac{m}{\rho}\mathrm{D}^k_0 + \frac{d}{\rho}\mathrm{D}^k_3\Big)\Big]\\ \Gamma^1_{2\nu} &= -D_\nu\cdot(\mathcal{S}'_{(3)} + 2qA) - 2m\rho\delta^0_\nu + 2d\rho\delta^3_\nu.\end{aligned} \tag{D.199}$$

which gives (4.39).

Calculation of $\Gamma^2_{3\nu}$

Similarly we have:

$$\begin{aligned}\rho^2\Gamma^2_{3\nu} &= D^\mu_\nu R_\mu,\\ R_\mu &= -D^0_2\partial_\mu D^0_3 + D^1_2\partial_\mu D^1_3 + D^2_2\partial_\mu D^2_3 + D^3_2\partial_\mu D^3_3\end{aligned} \tag{D.200}$$

$$\begin{aligned} &= -i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*)\\ &+ i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*)\\ &+ (\xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*)\\ &+ i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*).\end{aligned} \tag{D.201}$$

Using always (D.137) we have:

$$R_\mu = i\begin{pmatrix}\xi_2(\partial_\mu\xi_1)(\Omega_1 - i\Omega_2) - \xi_1(\partial_\mu\xi_2)(\Omega_1 - i\Omega_2)\\ +\eta_2(\partial_\mu\eta_1)(\Omega_1 + i\Omega_2) - \eta_1(\partial_\mu\eta_2)(\Omega_1 + i\Omega_2)\\ -\xi_2^*(\partial_\mu\xi_1^*)(\Omega_1 + i\Omega_2) + \xi_1^*(\partial_\mu\xi_2^*)(\Omega_1 + i\Omega_2)\\ -\eta_2^*(\partial_\mu\eta_1^*)(\Omega_1 - i\Omega_2) + \eta_1^*(\partial_\mu\eta_2^*)(\Omega_1 - i\Omega_2)\end{pmatrix}. \tag{D.202}$$

With equations (D.134) and (D.135) calculating Y_μ and Z_μ we deduce:

$$R_\mu = -\Omega_2 Y_\mu + \Omega_1 iZ_\mu. \tag{D.203}$$

We may then use the results in D.4.4 and we directly obtain:

$$\begin{aligned}\rho^2\Gamma^2_{3\nu} &= \mathrm{D}^{\mu}_{\nu}[-(\Omega_2 j_{1\mu} + \Omega_1 j_{1\mu} - 2q(\Omega_2 v^{\mu}_2 + \Omega_1 {v'}^{\mu}_2) \\ &\quad - 2m\frac{\Omega_1\Omega_2 - \Omega_2\Omega_1}{\rho}\mathrm{D}^{\mu}_2 + 2d\frac{\Omega_1^2 + \Omega_2^2}{\rho}\mathrm{D}^{\mu}_1] \\ \Gamma^2_{3\nu} &= -\mathrm{D}_{\nu}\cdot(\mathcal{S}'_{(1)} + 2q\mathcal{A}'_{(2)}) - 2d\rho\delta^1_{\nu}. \end{aligned} \qquad \text{(D.204)}$$

This is (4.37).

Calculation of $\Gamma^3_{1\nu}$

We finally have:

$$\begin{aligned}\rho^2\Gamma^3_{1\nu} &= \mathrm{D}^{\mu}_{\nu}(Q_{\mu}), \\ Q_{\mu} &= -\mathrm{D}^0_3\partial_{\mu}\mathrm{D}^0_1 + \mathrm{D}^1_3\partial_{\mu}\mathrm{D}^1_1 + \mathrm{D}^2_3\partial_{\mu}\mathrm{D}^2_1 + \mathrm{D}^3_3\partial_{\mu}\mathrm{D}^3_1 \qquad \text{(D.205)} \\ &= -(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_{\mu}(-\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1) \\ &\quad + (\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*)\partial_{\mu}(\xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1) \\ &\quad + i^2(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*)\partial_{\mu}(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1) \\ &\quad + (\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_{\mu}(-\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1). \end{aligned} \qquad \text{(D.206)}$$

We get with (D.156) and (D.157) calculating L_{μ} and M_{μ}:

$$Q_{\mu} = \Omega_1 M_{\mu} - i\Omega_2 L_{\mu}. \qquad \text{(D.207)}$$

We may thus use the results of D.4.5 and we directly get:

$$\begin{aligned}\rho^2\Gamma^3_{1\nu} &= \mathrm{D}^0_{\nu}[\Omega_2(-j^0_2 - 2qv^0_1 - 2m\frac{\Omega_1}{\rho}\mathrm{D}^0_1 - 2d\frac{\Omega_2}{\rho}\mathrm{D}^0_2) \\ &\quad + \Omega_1(-{j'}^0_2 - 2q{v'}^0_1 + 2m\frac{\Omega_2}{\rho}\mathrm{D}^0_1 - 2d\frac{\Omega_1}{\rho}\mathrm{D}^0_2)] \\ &\quad + \sum_{k=1}^{3}\mathrm{D}^k_{\nu}[\Omega_2(j^k_2 + 2qv^k_1 + 2m\frac{\Omega_1}{\rho}\mathrm{D}^k_2 + 2d\frac{\Omega_2}{\rho}\mathrm{D}^k_2) \qquad \text{(D.208)} \\ &\quad + \Omega_1({j'}^k_2 + 2q{v'}^k_1 - 2m\frac{\Omega_2}{\rho}\mathrm{D}^k_1 + 2d\frac{\Omega_1}{\rho}\mathrm{D}^k_2))]. \end{aligned}$$

This gives:

$$\Gamma^3_{1\nu} = \mathrm{D}_{\nu}\cdot(-\mathcal{S}'_{(2)} - 2q\mathcal{A}'_{(1)}) + 2d\rho\delta^2_{\nu}, \qquad \text{(D.209)}$$

which is (4.38), end of this long and tedious calculation.

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